BIFURCATIONS OF PERIODIC SOLUTIONS SATISFYING THE ZERO-HAMILTONIAN CONSTRAINT IN REVERSIBLE DIFFERENTIAL EQUATIONS

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Abstract. This is a study of the existence of bifurcation branches for the problem of finding even, periodic solutions in fourth-order, reversible Hamiltonian systems such that the Hamiltonian evaluates to zero along each solution on the branch. The class considered here is a generalisation of both the Swift-Hohenberg and extended Fisher-Kolmogorov equations that have been studied in several recent papers. We obtain the existence of local bifurcations from a trivial solution under mild restrictions on the nonlinearity and obtain existence and disjointness results regarding the global nature of the resulting bifurcating continua for the case where the Hamiltonian has a *single-well* potential.

The local results rest on two abstract bifurcation theorems which also have applications to sixth-order problems and which show that the curves of zero-Hamiltonian solutions are contained within two-dimensional manifolds of solutions of both negative and positive Hamiltonian.

Key words. reversible Hamiltonian systems, Lyapunov Schmidt reduction

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1. Introduction. In [25, 7, 28, 29, 36, 20, 9] the authors find periodic solutions of systems of Hamiltonian differential equations with the property of having *prescribed, zero Hamiltonian*. In particular, existence theorems for even, periodic orbits satisfying the zero-Hamiltonian constraint in certain fourth-order Hamiltonian systems have been derived by Peletier *et al.* and van den Berg using shooting techniques [35, 36]. While such shooting methods rely heavily on the particular form of the nonlinearities in a given problem and thus suffer from a lack of generality, the techniques do provide a great deal of quantitative information about the solutions. The problem of finding periodic solutions of Hamiltonian systems with prescribed *non-zero energy* has been studied extensively (see [31, 30] and more recently [4]).

The main contribution of this paper is to view the problem of finding zero-Hamiltonian, periodic solutions of (1.1) as a one-parameter bifurcation problem from a zero solution, with either *period* or an external parameter playing the role of bifurcation parameter. To solve this bifurcation problem we formulate two abstract Hopf bifurcation theorems (Theorems 2.2 and 2.4) and deduce the existence of the desired solutions as a corollary. The abstract results apply to reversible, fourth-order Hamiltonian systems at 1:1 and m:n resonances, provided that the Hamiltonian is indefinite about the trivial solution. We call these *simple* and *double* bifurcations respectively as the theorems lead to either a single continuum or a pair of bifurcating continua of solutions. Furthermore, the proofs of the bifurcation theorems are easily modified to show that the zero-Hamiltonian solutions that we find actually lie within manifolds of solutions of positive and negative Hamiltonian.

The proofs of our abstract results are achieved using a Lyapunov-Schmidt reduction technique, as can be found in many texts [38, 2, 11], and the fact that we

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essentially have only one bifurcation parameter means that some of the global bifurcation results of [6] are applicable. Using arguments from the *configuration-space* formulation of fourth-order problems [34, 17, 29], we shall be able to find bifurcation invariants which demonstrate that the bifurcating continua form a countable collection of mutually disjoint sets. Subsequently, we shall be able to show that a simple bifurcation for fourth-order problems results in the existence of an unbounded (in a suitable sense) continuum of solutions, rather like the classical global Hopf bifurcation theorem described in [1]. The global aspect of the paper is peculiar to fourth-order equations and does not immediately apply to more general Hamiltonian systems (like the sixth-order problem [33, equation (2)] for which we also have local results). Consequently, we have what approaches a nonlinear Sturm-Liouville theory (which is well-known in the context of elliptic, two-point boundary-value problems [5]) for zero-Hamiltonian solutions of equation (1.1) given below.

A Lyapunov-Schmidt reduction procedure is specifically available for systems that are either reversible or Hamiltonian [37, 22], however we do not make use of these results in this paper. The reason for this is that it is not clear that studying the problem in a space of reduced dimension helps to elucidate the rôle played by the Hamiltonian constraint and consequently we approach the problem *ab initio*.

So, consider the class of fourth-order differential equations

$$u'''' + pu'' + F_u(u) = 0, \tag{1.1}$$

where primes refer to differentiation with respect to x, p is a real parameter and the function $F \in C^\omega(\mathbb{R})$ satisfies

(**F**) $F(0) = F_u(0) = 0 \quad \text{and} \quad F_{uu}(0) = 1.$

Here, $C^\omega(\mathbb{R})$ denotes the space of real-analytic functions on $\mathbb{R}$ and a subscript u denotes differentiation with respect to u. We shall assume *throughout* that F satisfies assumption (**F**) and is therefore positive in some neighbourhood of $u = 0$. Note that the final condition in (**F**) is not restrictive as it can always be obtained from a suitable scaling of u and of time (denoted x), provided that $F_{uu}(0) > 0$.

Now (1.1) is reversible (see [8] for a discussion of reversible systems) and Hamiltonian, with Hamiltonian

$$H \equiv u'u''' - \frac{1}{2}u''^2 + \frac{1}{2}pu'^2 + F(u), \tag{1.2}$$

and when suitably scaled (see [36]), (1.1) provides the *extended Fisher-Kolmogorov* and *Swift-Hohenberg equations*, with $F_u(u) = \pm u(1 - u^2)$.

Fourth-order equations like (1.1) have a burgeoning literature, as can be seen from the recent studies in [3, 7, 24, 26, 27, 28, 29, 35, 36, 8, 12, 19, 18]; see in particular the recent monograph [25] and also [36, 20]. In these references it is shown using a variety of variational, geometric, functional analytic and elementary techniques that (1.1) may possess periodic, homoclinic and heteroclinic solutions, with applications ranging from geology to buckling theory; in particular, zero-Hamiltonian periodic solutions play an important role in study of cellular buckling (see [12, 19, 8] and the references therein, see also [9]).

We note at this stage that in order for a bifurcation from the trivial solution to occur as p varies, no further restrictions will be required on the nonlinearity F than those already given in assumption (**F**).

As an application of the results of the first part of the paper, we analyse the behaviour of the simple bifurcating branch which connects to $(u,p) = (0,2)$ for the case

$$F(u;\epsilon) = \frac{1}{2}u^2 - \epsilon\left(\frac{1}{4}u^4 - \frac{1}{6}u^6\right), \tag{1.3}$$

where ϵ is a parameter which unfolds the degenerate problem from $\epsilon = 0$. In particular, we prove the existence of a fold bifurcation on this bifurcating branch which was conjectured to exist in [19] and [12]. Finally, the results of some numerical calculations performed in AUTO will be presented which indicated that similar behaviour is observed for the multiple bifurcating branches which connect to the trivial solution at $p > 2$. We also compute the symmetry-breaking bifurcations on these branches and illustrate the subsequent connecting branches of solutions.

2. Bifurcation Theorems. Let X, Y and Z be Banach spaces and $BL(X,Y)$ denote the space of continuous (bounded), linear maps from X to Y. We write X^* for the dual space of continuous linear functionals $BL(X,\mathbb{R})$. If $L \in BL(X,Y)$ and $U \subset X$ is a closed subspace of X then $L|_U \in BL(U,Y)$ will denote the restriction of L to U. We shall use $\|\cdot\|_X$ to denote the norm on X and $\mathrm{Iso}(X,Y)$ denotes the set of continuous, linear, isomorphisms from X to Y. Let $C^r_{2\pi}$ be the Banach space of 2π-periodic, C^r functions from $[0,2\pi]$ to $\mathbb{R}^n$, endowed with a C^r norm.

If $f : X \to Z$ is a given smooth mapping then $df(x)[h]$ will denote the Fréchet derivative of f. For higher derivatives, the k-form $d^k f(x)[h,...,h]$ will also be written as $d^k f(x)[h]^{(k)}$ for brevity. Partial derivatives of a function $f \in C^1(X \times Y, Z)$ will be written as $d_x f(x,y)[h] \in Z$ and $d_y f(x,y)[k] \in Z$ where $(h,k) \in X \times Y$, higher derivatives will be written as in $d^2_{xy} f(x,y)[h,k]$. If $X = \mathbb{R}$ we will identify $d_x f(x,y)[h]$ with $h d_x f(x,y)[1]$, and we shall also write

$$d^k f(x)[h_1,...,h_k] = (\Pi_{j=1}^k h_j) d^k f(x)[1,...,1],$$

although we shall often omit the k-vector $[1,...,1]$ in this expression where no confusion results. Given $u, u_1, u_2 \in X$, we will write $\langle u \rangle \equiv \mathbb{R} \cdot u$ and $\langle u_1, u_2 \rangle = \{\alpha_1 u_1 + \alpha_2 u_2 : \alpha_1, \alpha_2 \in \mathbb{R}\}$. For any continuous function v, we denote the delta functional by $\delta(v) \equiv v(0)$.

For completeness, let us recall the following. A linear mapping $L \in BL(X,Y)$ is said to be *Fredholm* if its range $\mathrm{ran}(L)$ is a closed subspace of Y with finite codimension, and its null space $\ker(L)$ is a finite-dimensional subspace of X. Then

$$\mathrm{ind}(L) = \dim\ker(L) - \mathrm{codim}\ \mathrm{ran}(L)$$

is said to be the *Fredholm index* of L. We recall the following theorem which gives a useful collections of facts which can be found in [32].

THEOREM 2.1. *If $L \in BL(X,Y)$ is Fredholm and $K \in BL(X,Y)$ is a compact linear operator, then $L + K \in BL(X,Y)$ is also Fredholm and* $\mathrm{ind}(L+K) = \mathrm{ind}(L)$. *As a consequence, if L is Fredholm of index zero and it is injective then it is an isomorphism. The set $\mathcal{F}$ of Fredholm operators is open in $BL(X,Y)$ and* ind *is constant on connected components of $\mathcal{F}$.*

2.1. Statement of the Abstract Problem. Let $\psi \in Z^*, M \in C^\omega(X \times \mathbb{R}^2, Y)$ and $g \in C^\omega(X \times \mathbb{R}^2, Z)$, now consider the bifurcation problem

$$\mathcal{H}(u,p,\mu) \equiv \begin{pmatrix} M(u,p,\mu) \\ \psi(g(u,p,\mu)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.1}$$

where $u \in X$ and $\mu, p \in \mathbb{R}$ are parameters. Throughout we shall write $\mathcal{H} = M \times \psi(g)$ for brevity. We intend that (2.1) represents an abstract formulation of finding periodic solutions of (1.1) with the property of having zero Hamiltonian; accordingly we shall call the functional $\psi \circ g$ the *energy* of (2.1). Since the functional H in (1.2) is of *quadratic order* at the origin, we impose this degree of degeneracy into the operator g. Hence we assume that

$$M(0, p, \mu) \equiv 0, \tag{2.2}$$

and

$$g(0, p, \mu) \equiv 0, \qquad d_u g(0, p, \mu) \equiv 0. \tag{2.3}$$

By the term *bifurcation* from the trivial solution $u = 0$ of (2.1) at $(u, p, \mu) = (0, p_0, \mu_0)$, we mean that there is a sequence $(u_n, p_n, \mu_n) \subset X \backslash \{0\} \times \mathbb{R}^2$ which satisfies

$$\mathcal{H}(u_n, p_n, \mu_n) \equiv 0, \quad u_n \to 0 \quad \text{and} \quad (p_n, \mu_n) \to (p_0, \mu_0) \text{ as } n \to \infty.$$

2.2. Local Bifurcations. Let us now seek conditions under which there is a bifurcation of (2.1) from the trivial solution. The implicit function theorem applied to (2.1) shows that $(0, p_0, \mu_0)$ can be a bifurcation point for (2.1) only if

$$d_u M(0, p_0, \mu_0) \notin \mathrm{Iso}(X, Y). \tag{2.4}$$

Furthermore, if M is a assumed to be a Fredholm mapping then a bifurcation can only occur when $d_u M(0, p_0, \mu_0)$ is *not* injective. Motivated by this, we shall now consider two such cases:

(i) $\dim \ker(d_u M(0, p_0, \mu_0)) = 1$,
(ii) $\dim \ker(d_u M(0, p_0, \mu_0)) = 2$.

Case *(i)* is reminiscent of the theorem on bifurcation from a simple eigenvalue and will give rise to a *unique* bifurcating continuum. In case *(ii)*, however, we will be able to locate exactly *two* distinct bifurcating continua. Let us now proceed with the promised results.

THEOREM 2.2 (Simple Abstract Hopf Bifurcation). *Suppose that (2.2-2.3) hold and* $d_u M(0, p_0, \mu_0) \in BL(X, Y)$ *is Fredholm of index zero where* $\ker(d_u M(0, p_0, \mu_0)) = \langle k \rangle$. *Suppose also that* $X = \langle k \rangle \oplus U$, $V = \mathrm{ran}(d_u M(0, \mu_0, p_0), Y = \langle K \rangle \oplus V, P : Y \to V$ *is the projection operator along* $\langle K \rangle$ *and* Q *is the projection onto* $\langle K \rangle$ *which is identified with* $\mathbb{R}$.

Suppose further that $\psi\left(d^2_{uu} g(0, p_0, \mu_0)[k, k]\right) = 0$ *and that the operator* $D \in BL(U \times \mathbb{R}^2, V \times \mathbb{R}^2)$ *given by*

$$D = \begin{pmatrix} P d_u M(0, p_0, \mu_0) & P d^2_{up} M(0, p_0, \mu_0)[k, 1] & P d^2_{u\mu} M(0, p_0, \mu_0)[k, 1] \\ 0 & Q d^2_{up} M(0, p_0, \mu_0)[k, 1] & Q d^2_{u\mu} M(0, p_0, \mu_0)[k, 1] \\ \psi(d^2_{uu} g[k, \cdot]) & \psi(d^3_{uup} g[k, k, 1]) & \psi(d^3_{uu\mu} g[k, k, 1]) \end{pmatrix}$$

is an isomorphism.

Then, $(0, p_0, \mu_0)$ *is a bifurcation point for (2.1). Moreover, there is an interval* I *containing* 0 *and a unique, analytic branch* $\mathcal{B}$ *of solutions of (2.1) on which* $(u, p, \mu) = (u(\beta), p(\beta), \mu(\beta))$ *for* $\beta \in I$ *and which satisfies* $u(\beta) \neq 0$ *for* $\beta \neq 0, (u(0), p(0), \mu(0)) = (0, p_0, \mu_0)$. *Moreover, there results* $\|u(\beta) - \beta k\|_X = O(\beta^2)$ *as* $\beta \to 0$.

Proof. Let us express u in terms of the decomposition of X as $u = \beta k + r = \beta(k+\rho) \in \langle k \rangle \oplus U$, then (2.1) is equivalent to

$$(P+Q)M(\beta(k+\rho), p, \mu) = 0, \tag{2.5}$$

$$\psi(g(\beta(k+\rho), p, \mu)) = 0. \tag{2.6}$$

Using analyticity, it follows that there are analytic mappings $\tilde{M}$ and $\tilde{g}$ such that

$$M(\beta(k+\rho), p, \mu) = \beta d_u M(0, p, \mu)[k+\rho] + \beta^2 \tilde{M}(\beta, \rho, p, \mu),$$

and

$$\psi(g(\beta(k+\rho), p, \mu)) = \psi\left(\frac{\beta^2}{2} d^2_{uu} g(0, p, \mu)[k+\rho, k+\rho] + \frac{\beta^3}{2}\tilde{g}(\beta, \rho, p, \mu)\right).$$

As we are seeking non-zero solutions to (2.1), we can divide by appropriate powers of β in (2.5-2.6) and solve the equivalent problems

$$(P+Q)d_u M(0, p, \mu)[k+\rho] + \beta\tilde{M}(\beta, \rho, p, \mu) = 0, \tag{2.7}$$

$$\psi\left(d^2_{uu} g(0, p, \mu)[k+\rho, k+\rho] + \beta\tilde{g}(\beta, \rho, p, \mu)\right) = 0. \tag{2.8}$$

In turn, (2.7-2.8) is equivalent to

$$P\left(d_u M(0, p, \mu)[k+\rho] + \beta\tilde{M}(\beta, \rho, p, \mu)\right) = 0 \in V, \tag{2.9}$$

$$Q\left(d_u M(0, p, \mu)[k+\rho] + \beta\tilde{M}(\beta, \rho, p, \mu)\right) = 0 \in \mathbb{R}, \tag{2.10}$$

$$\psi\left(d^2_{uu} g(0, p, \mu)[k+\rho, k+\rho] + \beta\tilde{g}(\beta, \rho, p, \mu)\right) = 0 \in \mathbb{R}, \tag{2.11}$$

where the one-dimensional space $\langle K \rangle$ is identified with $\mathbb{R}$.

Let us now denote (2.9-2.11) as $\Phi_1(\beta, \rho, p, \mu) = 0$ where Φ_1 is an analytic mapping of Banach spaces

$$\Phi_1 : \mathbb{R} \times U \times \mathbb{R}^2 \to V \times \mathbb{R}^2. \tag{2.12}$$

Under the stated assumptions it is clear that $\Phi_1(0, 0, p_0, \mu_0) = 0$ and one can show that $D = d_{\rho,p,\mu}\Phi_1(0, 0, p_0, \mu_0)$, noting $Q[d_u M(0, p_0, \mu_0)] = 0$ by definition. It now follows by the implicit function theorem that we may locally solve (2.9-2.10) for ρ,p and μ as a function of β. The fact that $\rho(0) = 0$ completes the proof. □

The following result tells us that Theorem 2.2 is a special case of a more general result which says that the branch $\mathcal{B}$ of zero energy solutions from this theorem is formed from the intersection of a manifold of solutions of $M(u, p, \mu) = 0$ with the zero-energy surface $\{(u, p, \mu) : \psi(g(u, p, \mu)) = 0\}$.

THEOREM 2.3. *The curve of zero-energy solutions $\mathcal{B}$ from Theorem 2.2 is contained within a (locally) two-dimensional, analytic manifold $\mathcal{M}$ of solutions of*

$$M(u, p, \mu) = 0 \tag{2.13}$$

of both positive and negative energy.

Proof. (sketch) Repeat the same argument as Theorem 2.2 but for the system

$$\mathcal{H}_\epsilon(u, p, \mu) \equiv \begin{pmatrix} M(u, p, \mu) \\ \psi(g(u, p, \mu)) - \epsilon \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.14}$$

One obtains a mapping $\Phi_1(\beta, \rho, p, \mu, \epsilon)$, entirely analogous to (2.12), such that

$$\Phi_1(0, 0, p_0, \mu_0, 0) = 0 \text{ and } D = d_{\rho,p,\mu}\Phi_1(0, 0, p_0, \mu_0, 0).$$

One can then solve ρ, p and μ locally as analytic functions of (β, ϵ). $\square$

The following technical lemma shows that one can parameterise the bifurcating branch from Theorem 2.2 using one of p or μ as parameters, and this result will be used at a later stage in the paper.

LEMMA 2.1. *If $(u, p, \mu) = (u(\beta), p(\beta), \mu(\beta))$ is an element of the bifurcating branch $\mathcal{B}$ obtained in Theorem 2.2, then at least one of*

$$d_{u,p}(M \times \psi(g)) \text{ or } d_{u,\mu}(M \times \psi(g)) \in BL(X \times \mathbb{R}, Y \times \mathbb{R})$$

(evaluated at (u, p, μ)) is an isomorphism for sufficiently small, non-zero $|\beta|$.

Proof. Note from Theorem 2.2 that

$$\left|Qd^2_{up}M(0, p_0, \mu_0)[k, 1]\right| + \left|Qd^2_{u\mu}M(0, p_0, \mu_0)[k, 1]\right| \neq 0,$$

and let us therefore assume for definiteness that

$$Qd^2_{up}M(0, p_0, \mu_0)[k, 1] \neq 0. \tag{2.15}$$

Now define the one-parameter family of linear mappings

$$L(\beta) \equiv \begin{pmatrix} d_uM(u(\beta), p(\beta), \mu(\beta)) & d_pM(u(\beta), p(\beta), \mu(\beta)) \\ \psi(d_ug(u(\beta), p(\beta), \mu(\beta)) & \psi(d_pg(u(\beta), p(\beta), \mu(\beta)) \end{pmatrix},$$

and note that this is an (at most) rank two perturbation of a Fredholm mapping with index zero. It follows that we only need to prove that $L(\beta)$ is injective for $\beta \neq 0$.

Using analyticity, a straightforward but lengthy calculation shows that we may write $L(\beta) = L_0(\beta) + \beta L_1(\beta) + \frac{\beta^2}{2}L_2(\beta) + O(\beta^3)$, where

$$L_0(\beta) = \begin{pmatrix} d_uM & 0 \\ 0 & 0 \end{pmatrix},$$

$$L_1(\beta) = \begin{pmatrix} d^2_{uu}M[k + O(\beta), \cdot] & d^2_{up}M[k + O(\beta), \cdot] \\ \psi(d^2_{uu}g[k + O(\beta), \cdot]) & 0 \end{pmatrix},$$

and

$$L_2(\beta) = \begin{pmatrix} d^3_{uuu}M[k + O(\beta), k + O(\beta), \cdot] & d^3_{uup}M[k + O(\beta), k + O(\beta), \cdot] \\ \psi(d^3_{uuu}g[k + O(\beta), k + O(\beta), \cdot]) & \psi(d^3_{uup}g[k + O(\beta), k + O(\beta), \cdot]) \end{pmatrix},$$

where each of the given derivatives is evaluated at $(u, p, \mu) = (0, p(\beta), \mu(\beta))$. For $\beta \neq 0$, one can see that $L(\beta)$ is injective if and only if $T(\beta)$ is, where

$$\begin{aligned} T(\beta) = &\begin{pmatrix} d_uM & 0 \\ \psi(d^2_{uu}g[k + O(\beta), \cdot]) & 0 \end{pmatrix} + \\ &\frac{\beta}{2}\begin{pmatrix} 2d^2_{uu}M[k + O(\beta), \cdot] & 2d^2_{up}M[k + O(\beta), \cdot] \\ \psi(d^3_{uuu}g[k + O(\beta), k + O(\beta), \cdot]) & \psi(d^3_{uup}g[k + O(\beta), k + O(\beta), \cdot]) \end{pmatrix} \\ &+O(\beta^2). \end{aligned}$$

Again, each of the derivatives is evaluated at $(u,p,\mu) = (0, p(\beta), \mu(\beta))$.

Clearly $T(0)$ is not injective, but the fact that $d^2_{uu}g(0,p_0,\mu_0)[k,k] \neq 0$ implies

$$\ker(T(0)) = \langle \kappa \rangle \subset X \times \mathbb{R},$$

where $\kappa = (0_X, 1) \in X \times \mathbb{R}$. To prove that $T(\beta)$ is injective for small β it suffices to prove that

$$T'(0)\kappa \notin \operatorname{ran}(T(0)). \tag{2.16}$$

But

$$T'(0) = \frac{1}{2}\begin{pmatrix} 2d^2_{uu}M[k,\cdot] & 2d^2_{up}M[k,\cdot] \\ \psi(d^3_{uuu}g[k,k,\cdot]) & \psi(d^3_{uup}g[k,k,\cdot]) \end{pmatrix},$$

evaluating derivatives at $(u,p,\mu) = (0,p_0,\mu_0)$, and $\operatorname{ran}(T(0)) = \operatorname{ran}(d_uM(0,p_0,\mu_0)) \times \operatorname{ran}(\psi(d^2_{uu}g(0,p_0,\mu_0)))$. It follows that $T'(0)\kappa \in \operatorname{ran}(T(0))$ can only be satisfied if

$$d^2_{up}M(0,p_0,\mu_0)[k,1] \in \operatorname{ran}(d_uM(0,p_0,\mu_0)),$$

but this contradicts (2.15). Finally, one can use an analogous argument to cover the case whereby $Qd^2_{u\mu}M(0,p_0,\mu_0)[k,1] \neq 0$. □

Next we consider case *(ii)* where $d_uM(0,p_0,\mu_0)$ has a two-dimensional null-space.

THEOREM 2.4 (Double Abstract Hopf Bifurcation). *Suppose that (2.2-2.3) hold and* $\ker(d_uM(0,p_0,\mu_0)) = W$ *where* $\dim(W) = 2$ *with* $W = \langle k_1, k_2 \rangle$. *Suppose further that* $X = W \oplus U$ *and*

$$Y = Z \oplus V, \qquad V = \operatorname{ran}(d_uH(0,p_0,\mu_0)),$$

where V *is closed,* $\dim(Z) = 2$ *and* $Z = \langle u_1, u_2 \rangle$.

Now, let $P : Y \to V$ *be the projection along* Z *and* $Q = I - P$. *For* $i = 1,2$, *let* Q_i *be the projection of* Y *onto* $\langle u_i \rangle$ *(which we identify with* $\mathbb{R}$*) such that* $Q[y] = Q_1[y]u_1 + Q_2[y]u_2$. *Put*

$$A = \psi(d^2_{uu}g(0,p_0,\mu_0)[k_2,k_2]), B = \psi(d^2_{uu}g(0,p_0,\mu_0)[k_1,k_2])$$

and

$$C = \psi(d^2_{uu}g(0,p_0,\mu_0)[k_1,k_1]).$$

Suppose that $C \neq 0, B^2 > AC$ *and let* $\alpha_\pm$ *be the two (real, non-zero, distinct) roots of the quadratic equation* $A\alpha^2 + 2B\alpha + C = 0$. *Suppose also that*

$$\det\begin{pmatrix} Q_1d^2_{up}M[k_1+\alpha_\pm k_2,1] & Q_1d^2_{u\mu}M[k_1+\alpha_\pm k_2,1] \\ Q_2d^2_{up}M[k_1+\alpha_\pm k_2,1] & Q_2d^2_{u\mu}M[k_1+\alpha_\pm k_2,1] \end{pmatrix} \neq 0$$

when $(u,p,\mu) = (0,p_0,\mu_0)$.

Then $(0,p_0,\mu_0)$ *is a bifurcation point for (2.1). Moreover, there is an interval* I *containing* 0 *and exactly two analytic branches* $\mathcal{B}_\pm$ *of solutions of (2.1) on which* $(u,p,\mu) = (u_\pm(\beta), p_\pm(\beta), \mu_\pm(\beta))$ *for* $\beta \in I$, *with* $u_\pm(0) \neq 0$ *for* $\beta \neq 0$ *and* $(u_\pm(0), p_\pm(0), \mu_\pm(0)) = (0,p_0,\mu_0)$. *Moreover, there are analytic functions* $\alpha_\pm : I \to \mathbb{R}$ *and* $\rho : I \to V$ *such that* $\alpha_\pm(0) = \alpha_\pm$, $\|\rho(\beta)\|_Y = O(\beta)$ *as* $\beta \to 0$ *and* $u_\pm(\beta) = \beta k_1 + \beta\alpha_\pm(\beta)k_2 + \beta\rho(\beta)$.

Proof. Using the analyticity of M let us write

$$M(u,p,\mu) = d_u M(0,p,\,\mu)[u] + \mathcal{O}(2)$$

and

$$g(u,p,\mu) = g(0,p,\mu) + d_u g(0,p,\mu)u + \frac{1}{2}d^2_{uu}g(0,p,\mu)[u,u] + \mathcal{O}(3),$$

where $\mathcal{O}(n)$ represents any function, $\Theta(u,p,\mu)$ say, where there is a $\gamma > 0$ such that $\|\Theta(u,p,\mu)\| \leq \gamma\|u\|^n$ for all (u,p,μ) in a neighbourhood of $(0,p_0,\mu_0)$.

Now let $u = \beta(k_1+\alpha k_2+\alpha\rho) \in W \oplus U$ and note that there is an analytic function ϕ_1 such that the equation $M(u,p,\mu) = 0$ is locally equivalent to

$$\beta d_u M(0,p,\mu)[k_1 + \alpha k_2 + \alpha\rho] + \beta^2\phi_1(\alpha,\beta,\rho,p,\mu) = 0. \tag{2.17}$$

We may also use the analyticity of g to write

$$\begin{aligned} g(\beta(k_1 + \alpha k_2 + \alpha\rho),p,\mu) = \frac{\beta^2}{2}d^2_{uu}g(0,p,\mu)[k_1 + \alpha k_2 + \alpha\rho]^{(2)} \\ + \frac{\beta^3}{2}\phi_2(\alpha,\beta,\rho,p,\mu), \end{aligned} \tag{2.18}$$

where ϕ_2 is another suitably defined analytic function. Now the equation $M(u,p,\mu) = 0$ is equivalent to

$$(P + u_1Q_1 + u_2Q_2)M(u,p,\mu) = 0,$$

and, after dividing (2.17) and (2.18) by β and β^2 respectively, we obtain the locally equivalent problem

$$P\left[d_u M(0,p,\mu)[k_1 + \alpha k_2 + \alpha\rho] + \beta\phi_1(\alpha,\beta,\rho,p,\mu)\right] = 0 \in V, \tag{2.19}$$

$$Q_1\left[d_u M(0,p,\mu)[k_1 + \alpha k_2 + \alpha\rho] + \beta\phi_1(\alpha,\beta,\rho,p,\mu)\right] = 0 \in \mathbb{R}, \tag{2.20}$$

$$Q_2\left[d_u M(0,p,\mu)[k_1 + \alpha k_2 + \alpha\rho] + \beta\phi_1(\alpha,\beta,\rho,p,\mu)\right] = 0 \in \mathbb{R}, \tag{2.21}$$

$$\psi\left(d^2_{uu}g(0,p,\mu)[k_1 + \alpha k_2 + \alpha\rho]^{(2)} + \beta\phi_2(\alpha,\beta,\rho,p,\mu)\right) = 0 \in \mathbb{R}. \tag{2.22}$$

Setting $\beta = 0, \rho = 0, p = p_0$ and $\mu = \mu_0$ in (2.19-2.22) we find an equation for α:

$$\psi(d^2_{uu}g(0,p,\mu)[k_1 + \alpha k_2, k_1 + \alpha k_2]) = 0. \tag{2.23}$$

From the definitions of A, B and C given in the statement of the theorem, (2.23) is simply the equation $C + 2B\alpha + A\alpha^2 = 0$ with solutions $\alpha = \alpha_\pm$.

We now write (2.19-2.22) as $\Phi_2(\rho,p,\mu,\alpha,\beta) = 0$, say, where Φ_2 is an analytic mapping $\Phi_2 : U \times \mathbb{R}^4 \to V \times \mathbb{R}^3$. The derivative $d_{\rho,p,\mu,\alpha}\Phi_2(0,p_0,\mu_0,\alpha_\pm,0)$ is given by the operator matrix

$$L \equiv \begin{pmatrix} A_0 & B_0 & 0 \\ 0 & D_0 & 0 \\ E_0 & F_0 & G_0 \end{pmatrix} \in BL(U \times \mathbb{R}^3, V \times \mathbb{R}^3)$$

where

$$A_0 = \alpha_\pm P d_u M^0,\ B_0 = \left[P d^2_{up}M^0[k_1 + \alpha_\pm k_2, 1]\middle| P d^2_{u\mu}M^0[k_1 + \alpha_\pm k_2, 1]\right],$$

$$E_0 = 2\alpha_\pm \psi(d^2_{uu} g^0[\cdot, k_1 + \alpha_\pm k_2]), \quad G_0 = 2\psi(d^2_{uu} g^0[k_2, k_1 + \alpha_\pm k_2]),$$

$$F_0 = \left[\psi(d^3_{uup} g^0[1, k_1 + \alpha_\pm k_2, k_1 + \alpha_\pm k_2]) \middle| \psi(d^3_{uu\mu} g^0[1, k_1 + \alpha_\pm k_2, k_1 + \alpha_\pm k_2])\right]$$

and

$$D_0 = \begin{pmatrix} Q_1 d^2_{up} M^0[k_1 + \alpha_\pm k_2, 1] & Q_1 d^2_{u\mu} M^0[k_1 + \alpha_\pm k_2, 1] \\ Q_2 d^2_{up} M^0[k_1 + \alpha_\pm k_2, 1] & Q_2 d^2_{u\mu} M^0[k_1 + \alpha_\pm k_2, 1] \end{pmatrix},$$

and a superscript zero (^{0}) denotes evaluation of a function at $(u, p, \mu) = (0, p_0, \mu_0)$.

Clearly, for L to be an isomorphism we require $G_0 \neq 0$, that is $\psi(d^2_{uu} g^0[k_2, k_1 + \alpha_\pm k_2]) \neq 0$, but this is just $B + \alpha_\pm A \neq 0$ which is true by assumption. Since $A_0 \in \mathrm{Iso}(U, V)$, L is an isomorphism if $\det(D_0) \neq 0$ and this is also an assumption. Using the implicit function theorem we can now determine all of the variables as analytic functions of β locally to the *two points* $(\rho, p, \mu, \alpha, \beta) = (0, p_0, \mu_0, \alpha_\pm, 0)$. □

As was the case for Theorems 2.2 and 2.3, we can prove that the branches of zero-energy solutions $\mathcal{B}_\pm$ from Theorem 2.4 are obtained from the intersection of solutions of $M(u, p, \mu) = 0$ with the zero-energy surface.

THEOREM 2.5. *The two curves of zero-energy solutions $\mathcal{B}_\pm$ from Theorem 2.4 are each contained within (locally) two-dimensional, analytic manifolds $\mathcal{M}_\pm$ of solutions of $M(u, p, \mu) = 0$ of both positive and negative energy. Moreover, $\mathcal{M}_+ \cap \mathcal{M}_- = \{(0, p_0, \mu_0)\}$.*

Proof. This is almost a ver batim repetition of the proof of Theorem 2.4, but modified to deal with an energy constraint of the form $\psi(g(u, p, \mu)) = \epsilon$. □

3. The existence of bifurcations for fourth and sixth-order systems.

3.1. Preliminaries. In this section we shall apply Theorems 2.1 and 2.2 to find bifurcating branches of periodic solutions of (1.1) which have the zero-Hamiltonian property. To do so, we shall presume that a periodic solution of (1.1) has period T, where

$$T = 2\pi/\mu,$$

and μ is a-priori unknown. Upon setting

$$t = \mu x,$$

a simple rescaling of (1.1) and (1.2) leads us to consider the *two-parameter* problem

$$M(u, p, \mu) \equiv \mu^4 u'''' + p\mu^2 u'' + F_u(u) = 0, \tag{3.1}$$

$$\psi(g(u, p, \mu)) \equiv \delta \left(\mu^4 u' u''' - \frac{1}{2}\mu^4 u''^2 + \frac{1}{2} p\mu^2 u'^2 + F(u)\right) = 0, \tag{3.2}$$

where primes now denote differentiation with respect to t. With regard to (2.1), ψ corresponds to δ and g is the Hamiltonian which appears in (3.2); again we shall write $\mathcal{H} = M \times \psi(g)$ so that (3.1-3.2) corresponds to the equation $\mathcal{H} = 0$.

A natural setting for the application of these Theorem 2.1 and 2.2 is in the space of even functions of period 2π. Accordingly, let $X_e = \left\{u \in C^4_{2\pi} : u(t) = u(-t)\right\}$ and $Y_e = \left\{u \in C^0_{2\pi} : u(t) = u(-t)\right\}$, both endowed with their usual norms. We also define the *even-odd* subspaces,

$$X_{eo} = \left\{u \in X_e : u(\frac{\pi}{2} - t) = -u(\frac{\pi}{2} + t)\right\}$$

and

$$Y_{eo} = \left\{ u \in Y_e : u(\frac{\pi}{2} - t) = -u(\frac{\pi}{2} + t) \right\}.$$

For a given subspace $S \subset Z \subset L^2(0,\pi)$ we define its orthogonal complement by $S^\perp = \left\{ u \in Z : \int_0^{2\pi} u(t)s(t)dt = 0 \ \ \forall s \in S \right\}$. In this way we obtain a map $\mathcal{H} \in C^\omega(X_e \times \mathbb{R}^2, Y_e \times \mathbb{R})$ and if F is even then $\mathcal{H}$ also provides a map $\mathcal{H} \in C^\omega(X_{eo} \times \mathbb{R}^2, Y_{eo} \times \mathbb{R})$.

3.2. Simple bifurcation from $p = 2$. The following theorem shows that the zero-Hamiltonian problem associated with (1.1) has a simple bifurcation point to a locally unique and smooth branch of solutions from the point $p = 2$.

THEOREM 3.1. *Suppose that assumption* (**F**) *holds. Then there is an interval $I \subset \mathbb{R}$ and a unique analytic branch $\beta \mapsto (u(\beta), p(\beta)) \in X_e \times \mathbb{R}$ defined on I of non-trivial, even, periodic solutions of (1.1) with* zero Hamiltonian *and period $T(\beta)$. Moreover, $u(\beta) \neq 0$ if $\beta \neq 0$,*

$$T(0) = 2\pi, \ u(0) = 0, \ p(0) = 2 \text{ and } \|u(\beta)(t) - \beta\cos(t)\|_{C^4} = O(\beta^2)$$

as $\beta \to 0$. If F is even then the function $t \mapsto u(\beta)(t)$ is an element of X_{eo} for all $\beta \in I$.

Proof. To prove this result we apply Theorem 2.2 to $\mathcal{H}(u,p,\mu) = 0$ with $X = X_e$ and $Y = Y_e$. Let $L(p,\mu)[a] \equiv d_u M(0,p,\mu)[a] = \mu^4 a'''' + p\mu^2 a'' + a$ and note that the bilinear form $\psi(d^2_{uu} g(0,p,\mu)[a,b])$ from Theorem 2.2 is given by

$$B(p,\mu)[a,b] = \delta\left(\mu^4(a'b''' + a'b''' - a''b'') + p\mu^2 a'b' + ab\right).$$

In order to verify the hypotheses of Theorem 2.2 let us seek a non-zero solution $a \in X_e$ to $L(p,\mu)a = 0$, that is

$$\mu^4 a'''' + p\mu^2 a'' + a = 0, \quad \delta\left(\mu^4(2a'a''' - (a'')^2) + p\mu^2(a')^2 + a^2\right) = 0.$$

Since a is even and of period 2π, we seek solutions of the form $a(t) = \cos(mt)$ where m is an integer. This provides the equations $\alpha^4 - p\alpha^2 + 1 = 0, -\alpha^4 + 1 = 0$ where $\alpha = \mu m$, whence $\alpha^2 = 1$, so that $p = 2$ and $\mu = 1/m$. Seeking the solution of minimal period, we may put $m = 1$ and thus define $k(t) \equiv \cos(t)$ and record the fact that $\ker(L(2,1)) = \langle k \rangle$. Let us also define $K \equiv k$ for the purposes of Theorem 2.2 and note for the moment that k is an *even-odd* function.

We now form the decompositions $X_e = \langle k \rangle \oplus \langle k \rangle^\perp$ and $Y_e = \langle k \rangle \oplus \langle k \rangle^\perp$, so that $U = \langle k \rangle^\perp \subset X_e$ and $V = \langle k \rangle^\perp \subset Y_e$ in accordance with Theorem 2.2 and define the projection $Q : L^2(0,2\pi) \to \mathbb{R}$ by

$$(Qu)(t) = \frac{1}{\pi}\int_0^{2\pi} u(t)k(t)dx, \quad u \in L^2(0,2\pi),$$

and let $P = I - k \cdot Q$. Now, $\psi(d^3_{uup} g(0,p,\mu)[a,b,1]) = B_p(p,\mu)[a,b] = \delta(\mu^2 a'b')$, $d^3_{uu\mu} g(0,p,\mu)[a,b] = \delta\left(4\mu^3(a'b''' + a'b''' - a''b'') + 2p\mu a'b'\right)$ and $d^2_{u\mu} M(0,p,\mu)[a] = L_\mu(p,\mu)[a] = 4\mu^3 a'''' + 2p\mu a''$. Finally, $d^2_{up} M(0,p,\mu)[a,1] = \mu^2 a''$. Thus the remaining hypothesis of Theorem 2.2 is satisfied if the operator matrix D is non-singular. On inspection of the relevant derivatives we find

$$D = \begin{pmatrix} P \circ L(2,1) & 0 & 0 \\ 0 & -1 & 0 \\ * & 0 & -4 \end{pmatrix} \in BL(U \times \mathbb{R}^2, V \times \mathbb{R}^2),$$

where $P \circ L(2,1) : U \to V$ is an isomorphism and $*$ is irrelevant to the calculation at hand. It follows that D is an isomorphism and the result follows.

The second part of the theorem is proven in exactly the same way, simply observing the change of space, using X_{eo} rather than X_e. The uniqueness of the bifurcating branch in both X_e and X_{eo} and the fact that $X_{eo} \subset X_e$ implies that $u(\beta) \in X_{eo}$ if F is even. □

Remark 1. *In order to demonstrate that the application of Theorem 2.2 is not limited to fourth-order problems, we present the following example. In [33] the authors study the problem of finding periodic solutions for sixth-order problems using a variational approach, of which*

$$u^{vi} + 5u^{iv} + pu'' + u - u^3 = 0, \tag{3.3}$$

is an example (see also [10]). Equation (3.3) has Hamiltonian

$$H \equiv \frac{1}{2}(u'''')^2 + u^v u' - u^{iv}u'' + 5(u'u''' - \frac{1}{2}(u'')^2) + \frac{p}{2}(u')^2 + \frac{1}{2}u^2 - \frac{1}{4}u^4. \tag{3.4}$$

Theorem 3.2. *The points $(u,p) = (0, 4\frac{1}{4})$ and $(0, 1+4\sqrt{2})$ are simple bifurcation points to even, periodic solutions of (3.3) with zero-Hamiltonian and with period near $2\pi\sqrt{2}$ and $2\pi/\sqrt{1+\sqrt{2}}$ respectively.*

The proof of Theorem 3.2 is very similar to that of Theorem 3.1 so we omit it; note that the existence of a locally two-dimensional manifold of positive and negative Hamiltonian solutions also follows from Theorem 2.3.

3.3. Double bifurcations from $p > 2$. The following result shows that the interval $[2, \infty)$ contains a dense set of bifurcation points for (3.1-3.2).

Theorem 3.3. *Suppose that assumption* (**F**) *holds. Then to each $n, m \in \mathbb{N}$ such that $n \geq m+1$ and $\gcd(n,m) = 1$ there is an interval $I \subset \mathbb{R}$ and exactly two analytic branches $\beta \mapsto (u_\pm(\beta), p_\pm(\beta)) \in X_e \times \mathbb{R}$ defined on I of even, periodic solutions of (1.1) with zero Hamiltonian and period $T_\pm(\beta)$. Moreover, $u_\pm(\beta) \neq 0$ for $\beta \neq 0$,*

$$T_\pm(0) = 2\pi\sqrt{nm},\ u_\pm(0) = 0,\ p_\pm(0) = \frac{n}{m} + \frac{m}{n},$$

and

$$\|u_\pm(\beta)(t) - \beta\,(m\cos(nt) \pm n\cos(mt))\,\|_{C^4} = O(\beta^2)$$

as $\beta \to 0$.

Proof. Let us apply Theorem 2.4 to $\mathcal{H}(u,p,\mu) = 0$ and to identify the functions k_1 and k_2 from Theorem 2.4, we consider the linearised problem

$$L(p,\mu)[a] \equiv \mu^4 a'''' + p\mu^2 a'' + a = 0$$

of (3.1) with $a \in X_e$. This linear equation admits an even, 2π-periodic solution of the form $a(t) = \lambda_1\cos(mt) + \lambda_2\cos(nt)$ with integer m and n provided that $\alpha_1 = (\mu m)^2$ and $\alpha_2 = (\mu n)^2$ both satisfy the equation $\alpha^2 - p\alpha + 1 = 0$. From this we obtain $\alpha_1\alpha_2 = 1$, so that $\mu = \mu_{n,m} \equiv \frac{1}{\sqrt{nm}}$ and $p = \alpha_1 + \alpha_2$, whence $p = p_{n,m} \equiv \frac{n}{m} + \frac{m}{n}$.

Hence we define $k_1(t) \equiv \cos(nt), k_2(t) \equiv \cos(mt)$ and $u_1 \equiv k_1, u_2 \equiv k_2$. Moreover, using $\psi(d^2_{uu}g(0, p_{n,m}, \mu_{n,m})[a,b]) = \delta(-\mu^4_{n,m}a''b'' + ab)$, we have

$$A = (n^2 - m^2)/n^2, B = 0 \text{ and } C = -A,$$

so that $\alpha_\pm = \pm\frac{n}{m}$ in the notation of Theorem 2.4. With $Q_i(v) = \frac{1}{\pi}\int_0^{2\pi} v(t)u_i(t)dt$ for $i = 1, 2$, we find $d^2_{up}M(0, p_{n,m}, \mu_{n,m})[a, 1] = \mu^2_{n,m}a''$, and $d^2_{u\mu}M(0, p_{n,m}, \mu_{n,m})[a, 1] = 4\mu^3_{n,m}a'''' + 2p_{n,m}\mu_{n,m}a''$, we evaluate the determinant from Theorem 2.4, which is $-4\mu^5_{n,m}\alpha_\pm n^2m^2(m^2 - n^2)$ and the result now follows since this is non-zero. □

If F is even, then all the bifurcations which occur for $p \geq 2$ are pitchforks because u is then a solution of (3.1-3.2) if and only if $-u$ is. The uniqueness properties of Theorems 2.2 and 2.4 and the symmetry then imply that the parameterisation of the solution branch satisfies $-u(\beta) = u(-\beta)$. From this we infer that the bifurcation diagram of $\|u(\beta)\|$ (with any suitable norm) plotted against $p(\beta)$ has a tongue-like appearance because of the density of the union of $p_{n,m}$ in $[2, \infty)$.

Remark 2. *Theorem 3.3 was essentially known some time ago and can be found in an unpublished letter by Prof. J. F. Toland (1992), as referred to in [9, equation (5.1) and p.2486] for the case $F_u(u) = u - u^2$. This letter was communicated to the present authors by Prof. A. R. Champneys and we express our gratitude for his help in this matter. A singularly perturbed version of (1.1) for this choice of nonlinearity was studied in [15] and more recently in [16], where the authors consider both homoclinic and periodic solutions, although the latter are not of zero-Hamiltonian; see also [23].*

As an aside, consider the equation

$$\frac{1}{12}v^{iv} + v'' + pv + v^3 + \frac{3}{4}v(vv'' + (v')^2), \tag{3.5}$$

taken from [21], with first integral

$$H \equiv \frac{1}{12}(v'''v' - \frac{1}{2}(v'')^2) + \frac{1}{2}(v')^2 + \frac{p}{2}v^2 + \frac{v^4}{4} + \frac{3}{8}(v')^2v^2. \tag{3.6}$$

Note that a parameter λ^2 appears in [21] which has been replaced here by p. This is not in the class of Hamiltonian system given by (1.1), but Theorems 2.2 and 2.4 are still applicable.

Theorem 3.4. *The point $p = 3$ is a simple bifurcation point to even, periodic solutions of (3.5) with zero first integral and with period near $\pi\sqrt{2/3}$. For each $n, m \in \mathbb{N}$ such that $n > m$ and $\gcd(n, m) = 1$, the point $p_{n,m} = 12(\frac{n}{m} + \frac{m}{n})^{-2}$ is a double bifurcation point to such solutions with period near $\pi\sqrt{(n^2 + m^2)/3}$.*

The proof of Theorem 3.4 is an application of Theorem 2.4 which is entirely analogous to Theorem 3.3, so we omit the details.

3.4. Odd solutions for even F. Now let us suppose that F is an even function. If we define the spaces of odd functions, $X_o = \{u \in C^4_{2\pi} : u(t) = -u(-t)\}$ and $Y_o = \{u \in C^0_{2\pi} : u(t) = -u(-t)\}$, then $\mathcal{H}$ provides a map $\mathcal{H} \in C^\omega(X_o \times \mathbb{R}^2, Y_o \times \mathbb{R})$. This means that one can obtain odd, zero-Hamiltonian solutions of (1.1) in an entirely analogous manner to the way we found the even solutions. For this reason we give the following theorem without proof.

Theorem 3.5. *If assumption* (**F**) *holds and F is even, then to each $n, m \in \mathbb{N}$ such that $n \geq m + 1$ and $\gcd(n, m) = 1$ there is an interval $I \subset \mathbb{R}$ and exactly two analytic branches $\beta \mapsto (u_\pm(\beta), p_\pm(\beta)) \in X_e \times \mathbb{R}$ defined on I of odd, periodic solutions of (1.1) with zero Hamiltonian and period $T_\pm(\beta) = 2\pi/\mu_\pm(\beta)$. Moreover, $u_pm(\beta) \neq 0$ for $\beta \neq 0$,*

$$T_\pm(0) = 2\pi\sqrt{nm},\ u_\pm(0) = 0,\ p_\pm(0) = \frac{n}{m} + \frac{m}{n},$$

and

$$\|u_{\pm}(\beta)(t) - \beta\left(m\sin(nt) \pm n\sin(mt)\right)\|_{C^4} = O(\beta^2)$$

as $\beta \to 0$.

Of course, there is little point of formulating a version of Theorem 3.1 in this context, since that theorem already tells us that a branch of odd solutions of (1.1) can be found by shifting time.

It is possible to formulate an extension of the results proven in this section by considering a smooth, one-parameter family of reversible vector fields on $\mathbb{R}^n$ which possesses a trivial branch of equilibrium solutions and a first integral. One could use Theorems 2.2 and 2.4 to formulate sufficient conditions for the bifurcation of zero-energy, symmetric periodic solutions. However, we have not done this for brevity and we restrict our attention to the properties of fourth-order systems.

3.5. Disjointness properties of solution sets. Motivated by Theorems 3.1 and 3.3 we define the following non-empty sets, assuming (**F**) to be true. Let

$$\Sigma \equiv \{(u,p,\mu) \in X_e \times \mathbb{R} \times \mathbb{R} : \mathcal{H}(u,p,\mu) = 0, u \neq 0, \mu > 0\}, \tag{3.7}$$

and let $\overline{\Sigma}$ denote the closure of Σ in $X_e \times \mathbb{R}^2$. For any pair $(n,m) \in \mathbb{N} \times \mathbb{N}$ such that $\gcd(n,m) = 1$, let $C(n,m)$ be the maximal connected subset of $\overline{\Sigma}$ which contains the point $(u,p,\mu) = (0, p_{n,m}, \mu_{n,m})$ and define the functional $\nu : \Sigma \to (2,\infty)$ by

$$\nu(u,p,\mu) = \|u\|_{C^4} + |p| + |\mu| + \frac{1}{|\mu|}. \tag{3.8}$$

Also, let

$$\Sigma_+ = \{(u,p,\mu) \in \Sigma : p > 0\}, \tag{3.9}$$

$\overline{\Sigma}_+$ is the closure of Σ_+ and let $C_+(n,m)$ be the maximal connected subset of $C(n,m) \cap \overline{\Sigma}_+$ which contains the point $(u,p,\mu) = (0, p_{n,m}, \mu_{n,m})$.

We continue with a simple lemma which is used in the subsequent analysis. Throughout this section, # is used to represent the cardinality of a set and we introduce a *potential* $V(u,u'')$ by writing equation (3.2) as

$$-\mu^2 u'(\mu^2 u''' + \frac{p}{2}u') = V(u,u'') \equiv -\frac{1}{2}\mu^4 u''^2 + F(u).$$

LEMMA 3.1. *Suppose that* $n,m \geq 1$ *are distinct integers, then* $\#\{t \in [0,\pi] : n\cos(nt) \pm m\cos(mt) = 0\} = \max(n,m)$ *and* $\#\{t \in [0,\pi] : m\cos(nt) \pm n\cos(mt) = 0\} = \min(n,m)$.

The following two theorems provide bifurcation invariants that are invaluable to the study of the global nature of Σ.

THEOREM 3.6. *If assumption* (**F**) *holds and* $F(u) > 0$ *for* $u \neq 0$ *then the mapping*

$$\iota_1 : \Sigma \to \mathbb{N}; (u,p,\mu) \mapsto \#\{t \in [0,\pi] : u(t) = 0\}$$

is continuous and satisfies $\iota_1(C(n,m) \backslash \{(0, p_{n,m}, \mu_{n,m})\}) \equiv \min(n,m)$.

We postpone the proof of this theorem until after the following preliminary lemma.

LEMMA 3.2. *Let $(u_k, p_k, \mu_k) \subset \Sigma$ be a sequence with $(u_k, p_k, \mu_k) \to (u, p, \mu) \in \Sigma$ and suppose that there is a pair of sequences $(t_k^1), (t_k^2) \subset [0, 2\pi]$ such that $|t_k^1 - t_k^2| \to 0$ and $u_k'(t_k^{1,2}) = 0$. Then $u_k(t_k^1)u_k(t_k^2) > 0$ for sufficiently large k.*

Proof. For definiteness we assume that $t_k^1 < t_k^2$ and, seeking a contradiction, we also assume that $u_k(t_k^1)u_k(t_k^2) \le 0$; we initially also assume that $u_k(t_k^1) < 0 < u_k(t_k^2)$. By the mean-value theorem there is a $T_k \in (t_k^1, t_k^2)$ such that $u_k''(T_k) = 0$ and taking the limit $k \to \infty$ gives the existence of a T such that $u'(T) = u''(T) = 0$. Using (3.2) yields $F(u(T)) = 0$ and therefore $u(T) = 0$, and since u is not identically zero it follows that $u'''(T) \neq 0$.

We now assume without the loss of any generality that $\|u_k\|_{C^4} \le \|u\|_{C^4} + 1 \equiv B$ and that $\frac{1}{2}\mu \le \mu_k \le \frac{3}{2}\mu$. By the smoothness of F and since $F(0) = F_u(0) = 0$ we may assume that there is a C such that

$$|F(w)| \le \frac{C^2}{2}|w|^2 \text{ if } |w| \le B.$$

Since $u_k'(t_k^{1,2}) = 0$ we have $V(u_k, u_k'') = -\frac{1}{2}\mu_k^4(u_k'')^2 + F(u_k) = 0$ at $t_k^{1,2}$ and therefore

$$|u_k''| \le \frac{C}{\mu_k^2}\,|u_k| \le \frac{4C}{\mu^2}\,|u_k| \text{ at } t_k^{1,2}. \tag{3.10}$$

If we define the function $v(t) = V(u_k(t), u_k''(t))$, then the mean-value theorem gives the existence of a $\tau_k \in (t_k^1, t_k^2)$ with $v'(\tau_k) = 0$. From (3.10) the orbit of u_k is a smooth curve in the u, u''-plane that connects $(u_k, u_k'')(t_k^1)$ (in the left half-plane) to $(u_k, u_k'')(t^2)$ (in the right half-plane) with end-points in the region $\{(u, u'') : |u''| \le \frac{4C}{\mu^2}|u|\}$. By considering the tangent vector (u', u''') of this planar curve, it follows that there is a $\hat{\tau}_k \in (t_k^1, t_k^2)$ such that

$$|u_k'''(\hat{\tau}_k)| \le \frac{4C}{\mu^2}\,|u_k'(\hat{\tau}_k)|\,. \tag{3.11}$$

When we take the limit $k \to \infty$, it follows from the estimate

$$\sup_{(t_k^1, t_k^2)} |u_k'| \le |t_k^2 - t_k^1| \sup_{(t_k^1, t_k^2)} |u_k''|\,,$$

that $u_k'''(\hat{\tau}_k) \to 0$. We therefore find that the limiting solution u satisfies $u'''(T) = 0$ which is a contradiction.

If the signs of u_k at $t_k^{1,2}$ are inverted, so that $u_k(t_k^2) < 0 < u_k(t_k^1)$ then the argument given above holds unchanged. If exactly one of the two values $u_k(t_k^{1,2})$ is zero for all k then the argument holds in a similar way: in this case the curve in the u, u''-plane connects the origin to the other point. The existence of $\hat{\tau}_k$ satisfying (3.11) follows as before. If both of the values of u_k are zero, then the curve is closed, and again a value of $\hat{\tau}_k$ can be found satisfying (3.11). This concludes the proof of the lemma. □

Proof of Theorem 3.6. Suppose that $(u, p, \mu) \in \Sigma$ and $(u_k, p_k, \mu_k) \subset \Sigma$ satisfies $(u_k, p_k, \mu_k) \to (u, p, \mu)$ in $C^4 \times \mathbb{R}^2$.

We first note that if four or more zeros of u_k collide (counted according to algebraic multiplicity), say $0 \le t_k^1 \le t_k^2 \le t_k^3 \le t_k^4 \le 2\pi$ are all zeros of u_k that converge to T then from the mean-value theorem we have $u(T) = u'(T) = u''(T) = u'''(T) = 0$, contradicting the assumption that $u \neq 0$.

On the other hand, if two or more zeros collide then from the mean-value theorem again there is a T such that $u'(T) = 0$ and therefore $u''(T) = 0$ by (3.2). In order to avoid the same contradiction as above, necessarily $u'''(T) \neq 0$. This implies that the zero of u at $t = T$ is topologically transverse which rules out the possibility that two transverse zeros coalesce.

We therefore are left with two cases: either three simple zeros collide, or two zeros of which one is a double zero. In the first case, three simple zeros, there exist $t_k^{1,2}$ such that $u_k'(t_k^{1,2}) = 0$ and $t_k^{1,2} \to T$ as $k \to \infty$, and since the zeros are transverse we can assume that u_k has opposite signs at t_k^1 and t_k^2. The conclusions of Lemma 3.2 show that this situation leads to a contradiction. In the second case we choose t_k^1 to be the non-simple zero, and $\tau_k \in (t_k^1, t_k^2)$ to be an intermediate point such that $u'(\tau_k) = 0$. Again an application of Lemma 3.2 leads to a contradiction and therefore the number of zeros of u_k eventually equals that of u. This shows that ι_1 is continuous on Σ and therefore constant on connected components of Σ.

In order to evaluate $\iota_1(u, p, \mu)$ for $(u, p, \mu) \in C(n, m)$ with $u \neq 0$ we use the representation of $C(n, m)$ at bifurcation given in Theorem 3.3. From Lemma 3.1 there results

$$\|u_\pm(\beta)(t) - \beta\,(m\cos(nt) \pm n\cos(mt))\,\|_{C^4} = O(\beta^2),$$

hence $\#\{t \in [0, \pi] : u_\pm(\beta)(t) = 0\} = \min(n, m)$ follows for sufficiently small and non-zero β and the theorem is proven. □

THEOREM 3.7. *If assumption* (**F**) *holds and* $F(u) > 0$ *for* $u \neq 0$ *then the mapping*

$$\iota_2 : \Sigma_+ \to \mathbb{N}; (u, p, \mu) \mapsto \#\{t \in [0, \pi] : u''(t) = 0\}$$

is continuous.

Proof. Let $(u, p, \mu) \in \Sigma_+$ and suppose that there is a $T \in [0, 2\pi]$ such that

$$u''(T) = u'''(T) = 0.$$

The zero-Hamiltonian condition (3.2) then gives

$$\frac{1}{2}p\mu^2 u'(T)^2 + F(u(T)) = 0,$$

and the hypotheses on F ensure that $u'(T) = 0$ and $u(T) = 0$. It follows that $u = 0$ which contradicts the definition of Σ_+ and this contradiction implies that the zeros of u'' are transverse. Consequently, if $(u_n, p_n, \mu_n) \subset \Sigma_+$ is a sequence such that $(u_n, p_n, \mu_n) \to (u, p, \mu)$ in Σ_+, then $u_n'' \to u''$ in the C^1 topology. Hence u_n'' has the same number of zeros as u'' for all sufficiently large n which shows that ι_2 is continuous as claimed. □

Theorem 3.7 immediately implies that ι_2 is constant on connected components of Σ_+ and from this observation we deduce the following.

COROLLARY 3.1. *Suppose that* (**F**) *holds and* $uF_u(u) \geq 0$ *for all* $u \in \mathbb{R}$, *then* $\Sigma_+ = \Sigma$ *and as a consequence,* $C_+(n, m) = C(n, m)$. *Moreover,*

$$\iota_2(C(n, m)\backslash\{(0, p_{n,m}, \mu_{n,m})\}) = \max(n, m),$$

so that $C(n, m) \cap C(n', m')$ *is empty unless* $(n, m) = (n', m')$.

Proof. Multiplying (3.1) by u and integrating gives

$$p\mu^2 \int_0^{2\pi} (u')^2 dt = \int_0^{2\pi} \mu^4 (u'')^2 + uF_u(u) dt \geq \int_0^{2\pi} \mu^4 (u'')^2 dt \geq 0.$$

Hence if there is a solution of (3.1-3.2) with $p = 0$ and $\mu > 0$ it follows that $u'' \equiv 0$, so $u(t) = At + B$ for constants A and B. As u is periodic, $A = 0$ and as u must have zero Hamiltonian, $F(B) = 0$ is also true. The hypotheses ensure that $F(u) = 0$ only when $u = 0$ so that $B = 0$, hence $u(t) \equiv 0$ and so $\Sigma = \Sigma_+$, from where $C(n, m) = C_+(n, m)$ by definition.

Since $\iota_2 : \Sigma \to \mathbb{N}$ is continuous by Theorem 3.7, the set $\mathcal{C}$ defined by $\mathcal{C} \equiv C(n, m)\backslash\{(0, p_{n,m}, \mu_{n,m})\}$ is a connected subset of Σ, because the intersection of $C(n, m)$ with some small ball, $C(n, m) \cap B_\delta(0, p_{n,m}, \mu_{n,m})$, is path-connected for all sufficiently small $\delta > 0$. Hence ι_2 is constant on $\mathcal{C}$. In order to evaluate $\iota_2(u, p, \mu)$ with $(u, p, \mu) \in \mathcal{C}$ we use the representation of $C_+(n, m)$ at bifurcation from the trivial solution described in Theorem 3.3 and then apply Lemma 3.1. From Theorem 3.3 there results

$$\|u''_\pm(\beta)(t) + \beta mn\,(n\cos(nt) \pm m\cos(mt))\,\|_{C^2} = O(\beta^2),$$

whence $\#\{t \in [0, \pi] : u''_\pm(\beta)(t) = 0\} = \max(n, m)$ follows for sufficiently small and non-zero β.

Finally, as (**F**) ensures that F is positive in a punctured neighbourhood of zero, the condition $uF_u(u) \geq 0$ then ensures that $F(u) > 0$ if $u \neq 0$. Applying Theorem 3.6, we obtain $\iota_j(C(n, m)) = \iota_j(C(n', m'))$ for $j = 1, 2$ and the last part of the corollary follows. □

Finally, we have the following theorem which applies to *path-connected subsets* of Σ, although it provides no information regarding the behaviour of *connected subsets* of Σ which are not path-connected.

THEOREM 3.8. *Suppose that $F(u) > 0$ for $u \neq 0$ and let (u_s, p_s) be a continuous path of solutions of (1.1), where each u_s is defined on a sufficiently large subset of $\mathbb{R}$ and the path does not contain the equilibrium solution. If two zeros of u'_s collide, then multiplicity is preserved in and through the collision.*

Proof. We may assume that all collisions occur at $s = 0$ and $t = 0$. Note that if all u_s are defined on a common interval $I \subset \mathbb{R}$, then by standard elliptic estimates the solution curve is bounded in $C^k(I')$ for any $k \in \mathbb{N}$ and any compact interval $I' \subset I$. Since F is smooth, we can bound F by

$$|F(w)| \leq C^2 w^2/2 \tag{3.12}$$

for, say, $|w| \leq 1$.

Note that $u'(0) = u''(0) = 0$ at $s = 0$, and therefore by equation (1.2), $F(u(0)) = 0$ which implies $u(0) = 0$. The non-constancy hypothesis on each u_s implies that $u'''(0) \neq 0$. This proves that multiplicity is conserved *in* the collision. Note that the zero at $(t, s) = (0, 0)$ is necessarily transverse.

To show that any subsequent perturbation preserves the multiplicity we use some ideas from the analysis of the *configuration space* (u, u'') from [34, 17]; [29] contains a simplified description that is sufficient for our purposes. The structure of the configuration space and the set $\{V = 0\}$ is shown in Figure 3.1. Near the origin in this plane the set $\{V = 0\}$ consists of two curves that intersect in the origin. Near the origin the direction of these curves is bounded from above by $2C$, where C is the constant in (3.12).

At $s = 0$, we have $u(0) = u'(0) = u''(0)$, $u'''(0) \neq 0$, and therefore the orbit near $t = 0$ is represented in the u, u''-plane by a curve that remains inside the set $\{V \leq 0\}$ and intersects $\{V = 0\}$. We can choose appropriate translations of u_s, and small $\bar{t}, \bar{s} > 0$, such that

FIG. 3.1. *The configuration space u, u'' is partitioned according to the sign of V.*

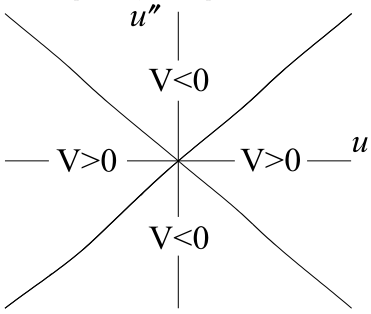
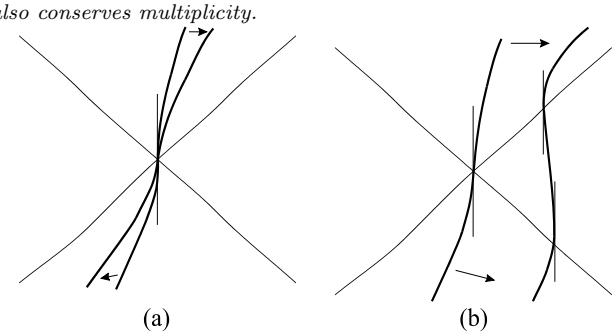


FIG. 3.2. *Two forms of perturbation: (a) if the curve continues to intersect $\{V = 0\}$ in the origin, then the tangent remains vertical; (b) a translation, on the other hand, creates new zeros of u' and therefore also conserves multiplicity.*



1. $u_s(t)$ is defined for $(t, s) \in Q \equiv (-\bar{t}, \bar{t}) \times (-\bar{s}, \bar{s})$;
2. u_s depends smoothly on s in $C^4(-\bar{t}, \bar{t})$;
3. $u_s'''(t) \geq 0$ on Q (if not then reverse time);
4. For each $s \in (-\bar{s}, \bar{s})$ we have $\pm u_s''(\pm\bar{t}) < 0$, and $V\big(u_s(\pm\bar{t})), u_s''(\pm\bar{t}))\big) < 0$. This implies that no intersections of the solutions with $\{V = 0\}$ appear or disappear through the boundary $\pm\bar{t}$;
5. $u_s'''/u_s' \geq 4C$ on Q, where C is the constant in (3.12).

We write $\gamma_s \equiv \{(u_s(t)), u_s''(t)) : -\bar{t} < t < \bar{t}\}$.

We now consider the alternatives for perturbation away from $s = 0$. First assume that $\bar{s}$ can be chosen such that $\gamma_s \cap \{V = 0\}$ has only one intersection for $0 < s < \bar{s}$; let this intersection be at $0 < t_s < \bar{s}$. The lower bound on the angle of the curve γ_s, given by condition 5 above, implies that γ_s intersects $\{V = 0\}$ only at the origin in the u, u''-plane. Since $u_0'(0) = 0$ and $u_0'''(0) \neq 0$, the smooth dependence of u_s on s implies that $u_s'''(t_s) \neq 0$ for s close to zero; therefore the requirement $u'(t_s)(u_s'''(t_s) + pu_s'(t_s)/2) = 0$ forces $u_s'(t_s) = 0$. Combining this with $u_s(t_s) = 0$ and taking the limit yields that the zero of u' at $s = 0$ is a double zero.

To cover the alternative case we assume that $\gamma_s \cap \{V = 0\}$ has at least two intersections for a sequence $0 < s_n < \bar{s}$, $s_n \downarrow 0$, at the points $0 < t_n < \tau_n < \bar{t}$. We have $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \tau_n = 0$. With an argument similar to the one above, it follows that $u_{s_n}'(t_n) = u_{s_n}'(\tau_n) = 0$, and therefore the zero at $s = 0$ is of second order.

At any point where the orbit intersects $\{V = 0\}$, either $u' = 0$ or $u''' = -pu'/2$. Since at $s = 0$ we have $u' = 0$ and $u''' \neq 0$, under perturbation of s we have $u' = 0$ on the intersection of the orbit with $\{V = 0\}$. If we assume that under perturbation

there is only one zero of u' (locally), then the intersection with $\{V = 0\}$ necessarily occurs at the origin in the u, u''-plane. Therefore the zero remains of multiplicity two.

Note that the only possible scenario is the reduction of multiplicity two to multiplicity zero. Multiplicity zero implies that while $u'(0) = 0$ at $s = 0$, this zero of u' disappears under perturbation. An inspection of the u, u''-plane (Figure 3.2) shows that such a perturbation is only possible if $u''' + pu'/2$, which is non-zero at $s = 0$, jumps to zero for $s \neq 0$. This contradicts the assumption of continuous dependence of the curve of solutions in C^k on the parameter s. $\square$

4. Global Bifurcations. Let us now briefly state some results from global real-analytic bifurcation theory for one-parameter problems as developed in [6]. The utility of this theory with respect to (2.1) is the fact that Lemma 2.1 identifies either p or μ which can be used as the bifurcation parameter. So, suppose that $\mathcal{U} \subset \mathbb{R} \times X$ is a given set and $F : \mathbb{R} \times X \to Y$ is a real analytic map, now define the set

$$S = \{(\lambda, x) \in \mathcal{U} : F(\lambda, x) = 0, d_x F(\lambda, x) \in \mathrm{Iso}(X, Y)\}.$$

Throughout this section the space $\mathbb{R} \times X$ is endowed with the norm $|\lambda| + \|x\|_X$ and a pair $(\lambda, x) \in S$ is said to be a regular solution of $F(\lambda, x) = 0$. While the theory developed in [6] is more powerful than we require, we shall use the following result which is the statement of Theorem 7.4(iii) of this reference.

THEOREM 4.1. *Let $\nu : \mathbb{R} \times X \to [0, \infty)$ be a given function and suppose that*

(i) S is non-empty and $\mathcal{U} \cap \mathcal{S}$ is open in $\mathcal{S}$, where $\mathcal{S} \equiv \{(\lambda, x) \in \mathbb{R} \times X : F(\lambda, x) = 0\}$,

(ii) $d_x F(\lambda, x)$ is Fredholm of index zero for all $(\lambda, x) \in \mathcal{U}$,

(iii) subsets of $\mathcal{S}$ on which ν is bounded have compact closure,

(iv) there are $\delta > 0, \lambda_0 \in \mathbb{R}$ and an analytic function $h : N_\delta(\lambda_0) \backslash \{\lambda_0\} \to S$, where $N_\delta(\lambda_0)$ is a half-neighbourhood of λ_0, such that $\lim_{\lambda \to \lambda_0} h(\lambda) = 0$ but $(\lambda_0, 0) \notin \mathcal{U}$,

(v) if $\mathcal{A}_0$ is the maximal path-connected subset of S which contains the graph of h and if $(\xi_n) \subset \overline{S} \cap \mathcal{U}$ is any convergent sequence with $\xi_n \to \xi \notin \mathcal{U}$ and $\sup_n \nu(\xi_n) < \infty$, then $\xi = (\lambda_0, 0)$ and $\xi_n \in \mathcal{A}_0$ for all n sufficiently large.

Under conditions (i)-(v) the maximal connected component of $\overline{S} \cap \mathcal{U}$ that contains $\mathcal{A}_0$ contains a path-connected subset $\mathcal{P}$ on which ν is unbounded.

If we recall the definition of the sets Σ and Σ_+ in (3.7) and (3.9) respectively, then the reasoning in Corollary 3.1 ensures that $\Sigma_+ = \Sigma$ if $uF_u(u) \geq 0$ for all $u \in \mathbb{R}$. From this observation we can obtain the following lemma.

LEMMA 4.1. *Suppose that* (**F**) *holds and $uF_u(u) \geq 0$ for $u \in \mathbb{R}$, then $\nu(u, p, \mu)$ (defined in (3.8)) is unbounded on a path-connected subset of $C(1, 1)$.*

Proof. Let us verify the hypotheses of Theorem 4.1 in turn, where $\omega = \mu - 1, x = (u, \omega)$ and write λ in place of p. To keep the notation consistent with Theorem 4.1, let $F \equiv M \times \psi(g)$ (the symbol $\mathcal{H}$ was used for this previously) and

$$\mathcal{U} \equiv \{(\lambda, x) = (p, u, \omega) \in \mathbb{R} \times X_e \times \mathbb{R} : \omega > -1, u \neq 0, \iota_1(u) = \iota_2(u) = 1\}.$$

We remark that $S \subset \Sigma \subset \mathcal{S}, \mathcal{S} \cap \mathcal{U} \subset \Sigma$ and the space X referred to in Theorem 4.1 is $X_e \times \mathbb{R}$ and Y is $Y_e \times \mathbb{R}$.

(i) It follows that S is non-empty from Lemma 2.1. (The argument assumes that $d_{u,p}(M \times \psi(g))$ is an isomorphism from Lemma 2.1, for if this is not the case then we can repeat the argument of this proof using μ for λ rather than p.) One can see that the set $\mathcal{U} \cap \mathcal{S}$ is open in $\mathcal{S}$ as follows. Suppose, seeking a contradiction that $\mathcal{U} \cap \mathcal{S}$ is

not open in $\mathcal{S}$, so there is a $(\lambda_0, x_0) = (p_0, u_0, \omega_0) \in \mathcal{S} \cap \mathcal{U}$ and a sequence $(\lambda_n, x_n) = (p_n, u_n, \omega_n) \in \mathcal{S} \backslash \mathcal{U}$ such that $(\lambda_n, x_n) = (p_n, u_n, \omega_n) \to (\lambda_0, x_0) = (p_0, u_0, \omega_0)$.

Since $u_0 \neq 0$ and $u_n \overset{C^4}{\to} u_0$, it follows that $u_n \neq 0$ and as $\omega_0 > -1$ then $\omega_n > -1$, both for all sufficiently large n. Since ι_1 and ι_2 are continuous functions on Σ and Σ_+ respectively, and $\Sigma = \Sigma_+$ by the hypothesis on F, then $\iota_1(u_n) \to \iota_1(u_0) = 1$ as $n \to \infty$, but since ι_1 is integer-valued, this means that $\iota_1(u_n) \equiv 1$ for all n sufficiently large. Similar reasoning applies to ι_2. Consequently, $(\lambda_n, x_n) \in \mathcal{U}$ for all n sufficiently large, which is the required contradiction.
(ii) The operator $d_x F(\lambda, x)$ has the form

$$\begin{pmatrix} d_u M(u,p,\mu) & 0 \\ 0 & 0 \end{pmatrix} + K \ \in BL(X_e \times \mathbb{R}, Y_e \times \mathbb{R}),$$

where K is a continuous operator that has rank at most two and $d_u M(u,p,\mu)[h] = \mu^4 h'''' + p\mu^2 h'' + F_u(u)h \in BL(X_e, Y_e)$. But the latter is a compact perturbation of the operator

$$E : h \mapsto \mu^4 h^{iv} + \theta h, \qquad E \in BL(X_e, Y_e).$$

Since $\mu > 0$, using a Fourier series argument one can easily show that there is a θ such that E is an isomorphism of the given spaces and consequently $d_x F(\lambda, x)$ is a compact perturbation of a Fredholm mapping of index zero, and therefore is itself Fredholm of index zero.
(iii) If $(u_n, p_n, \omega_n) \subset \mathcal{S}$ is a sequence such that $\omega_n = \mu_n - 1 > -1$ and

$$\nu(u_n, p_n, \mu_n) = \|u_n\|_{C^4} + |p_n| + |\omega_n + 1| + 1/|\omega_n + 1|$$

is bounded then there are p_0 and ω_0 such that $\omega_n \to \omega_0 \geq -1$ and $p_n \to p_0 \geq 0$. Now $\omega_0 \neq -1$ by the boundedness of ν and therefore $u_n^v = -(\omega_n + 1)^{-4}(p_n(\omega_n + 1)^{-2} u_n''' + F_u(u_n)u_n')$ is also bounded, whence (u_n) converges to some $u_0 \in C^4$ as the embedding $C^5 \hookrightarrow C^4$ is compact, and therefore (p_n, u_n, ω_n) converges in $\mathbb{R} \times X_e \times \mathbb{R}$.
(iv) This part of the theorem follows from Theorem 2.2 and Lemma 2.1 where the bifurcating branch is represented by an analytic curve of regular solutions.
(v) If a ν-bounded sequence $(\xi_n) = (p_n, u_n, \omega_n) \subset \overline{S} \cap \mathcal{U}$ satisfies $\xi_n \to \xi = (p_0, u_0, \omega_0) \notin \mathcal{U}$ then the only viable possibility is that $u_0 = 0$, so that $(p, \mu) = (p_0, \omega_0 + 1)$ is a bifurcation point from the trivial solution of (3.1-3.2). However, the *only* point at which such a bifurcation occurs into the set S is at the point $(u, p, \mu) = (0, 2, 1)$, so that $(u, \omega) = (0, 0)$. Hence property (v) is satisfied if $\mathcal{A}_0$ is defined to be the maximal path-connected subset of S which contains the graph of the bifurcating branch from Theorem 3.1. In this case let us note that $\lambda_0 = 2$.

This concludes the proof. □

Now define the functional on $X_e \times \mathbb{R}^2$ by

$$\overline{\nu}(u, p, \mu) = \|u\|_{C^4} + |p| + 1/|\mu|.$$

In order to obtain a result analogous to the global Hopf bifurcation theorem of [1], we show that ν can actually become unbounded on $C(1,1)$ if and only if $\overline{\nu}$ is unbounded on $C(1,1)$.

THEOREM 4.2. *If* $(\mathbf{F})$ *is satisfied and* $uF_u(u) \geq 0$, *then* $C(1,1)$ *contains a path-connected,* $\overline{\nu}$*-unbounded subset.*

Proof. Suppose that ν is unbounded on $C(1,1)$ but that $\overline{\nu}$ is bounded on this set, it follows that there is a sequence $(u_n, p_n, \mu_n) \in C(1,1)$ such that $\|u_n\|_{C^4} + |p_n|$ is bounded, $u_n \neq 0$ for each n and $|\mu_n| \to \infty$. However, since

$$\int_0^{2\pi} p\mu^2(u'')^2 dt \geq \frac{1}{2\pi}\int_0^{2\pi} p\mu^2(u')^2 = \frac{1}{2\pi}\left(\int_0^{2\pi} \mu^4(u'')^2 + uF_u(u)dt\right)$$
$$\geq \frac{1}{2\pi}\int_0^{2\pi} \mu^4(u'')^2 dt$$

holds for any non-trivial solution in $C(1,1)$ by the Poincaré inequality, it follows that $\mu_n^2 \leq 2\pi p_n$, which is a contradiction. Therefore, by Lemma 4.1, ν is unbounded on a path-connected subset of $C(1,1)$ and the above contradiction implies that $\overline{\nu}$ must also be unbounded on this set. □

This theorem represents a partial global trichotomy for bifurcations of periodic orbits of (1.1) which says that the solution continuum $C(1,1)$ either has an unbounded sequence of orbits in phase-space or is unbounded with respect to either the parameter (p) or with respect to the period (as occurs in the *blue-sky bifurcation* [13]). Unfortunately, due to assumption *(v)* of Theorem 4.1, it has not proven possible to use the same techniques to study the global existence properties of the branches $C(n,m)$ for $n > 1$.

5. Local Secondary Fold Bifurcations. Another advantage of the approach taken in this paper as opposed to the shooting methods previously used in [35, 36] is that we can investigate the geometry of each bifurcating continuum by introducing an unfolding parameter, ϵ, into (3.1-3.2). We will now show that degeneracies present in (3.1-3.2) at $\epsilon = 0$ can unfold to give secondary fold bifurcations along the bifurcation branch when $\epsilon \neq 0$.

To illustrate this we shall consider (1.1) for the particular case given in (1.3). This has been studied in [19] (see also [12] for an asymptotic analysis of this problem using multiple scale techniques) as a model for an elastic rock layer on a *restiffening foundation*, with corresponding $\mathbb{Z}_2$-symmetric ODE

$$M_\epsilon(u,p,\mu) \equiv \mu^4 u'''' + p\mu^2 u'' + u - \epsilon(u^3 - u^5), \tag{5.1}$$

and even Hamiltonian

$$H_\epsilon(u,p,\mu) \equiv \mu^4 u' u''' - \frac{1}{2}\mu^4 u''^2 + \frac{1}{2}p\mu^2 u'^2 + \frac{1}{2}u^2 - \epsilon\left(\frac{1}{4}u^4 - \frac{1}{6}u^6\right). \tag{5.2}$$

As in the proof of Theorem 2.2, and therefore also in Theorem 3.1, we can obtain a local representation of the bifurcating branch of the zero-Hamiltonian problem associated with (5.1) from the bifurcation point $p = 2$ in the form $u_\epsilon(\beta) = \beta(k + \rho_\epsilon(\beta))$, where $\rho_\epsilon(\beta) = O(\beta)$ for fixed ϵ and is an analytic function of both β and ϵ near $(\beta, \epsilon) = (0,0)$; here $k(t) = \cos(t)$. We now proceed with a calculation to find the Taylor expansion of $\rho_\epsilon(\beta)$ in order to determine the local geometry of the set of branches $C_\epsilon(1,1)$. Throughout this section we shall write

$$r_\epsilon(\beta) = \beta\rho_\epsilon(\beta).$$

It is important to note that the existence of bifurcating solutions for this problem close to $p = 2$, as determined by Theorem 3.1, does not depend upon the value of ϵ.

Indeed, using the implicit function theorem we simply find that for each ϵ sufficiently small and for suitable m, n, there is a bifurcating branch from $(p, \mu) = (p_{n,m}, \mu_{n,m})$ and this branch (that is, the local parametric representation of this branch) varies analytically with ϵ. This property also holds at $\epsilon = 0$ where the branches are pairs of lines. We also note that since $F_\epsilon(u)$ is an even function of u for each ϵ, $r_\epsilon(\cdot)$ is odd and $p_\epsilon(\cdot)$ and $\mu_\epsilon(\cdot)$ are even functions, forming a pitchfork bifurcation at $p = 2$.

We start our analysis by listing the Fréchet derivatives of the operator M_ϵ:

D1. $d_u M_\epsilon(u, p, \mu)[h] = \mu^4 h'''' + p\mu^2 h'' + h - \epsilon(3u^2 - 5u^4)h$,
D2. $d_u^2 M_\epsilon(u, p, \mu)[h_1, h_2] = -\epsilon h_1 h_2 (6u - 20u^3)$,
D3. $d_u^3 M_\epsilon(u, p, \mu)[h_1, h_2, h_3] = -\epsilon h_1 h_2 h_3 (6 - 60u^2)$,
D4. $d_u^4 M_\epsilon(u, p, \mu)[h_1, h_2, h_3, h_4] = 120\epsilon v h_1 h_2 h_3 h_4$,
D5. $d_u^5 M_\epsilon(u, p, \mu)[h_1, h_2, h_3, h_4, h_5] = 120\epsilon h_1 h_2 h_3 h_4 h_5$,

where $h_i \in X_e$ for each $i = 1, ..., 5$.

We denote the first derivative of $M_\epsilon(u, p, \mu)$ evaluated on the trivial solution branch $u = 0$ by $L \equiv d_u M_\epsilon(0, 2, 1)$; this operator is independent of ϵ. Suppose further that P is the projection of Y_e onto $\mathrm{ran}(L) = \langle k \rangle^\perp$ along $\langle k \rangle$, now define

$$L(p, \mu) \equiv d_u M_\epsilon(0, p, \mu).$$

We can solve the projected differential equation $P \circ N_\epsilon(\beta k + r, p, \mu) = 0$ for some function $r = r_\epsilon(\beta, p, \mu)$ near $(\beta, p, \mu, \epsilon) = (0, 2, 1; 0)$ using the implicit function theorem (we refer to the proof of Theorem 2.2 for details). From the uniqueness properties of the implicit function theorem if follows that $r_\epsilon(0, p, \mu) \equiv 0$, and if we repeatedly differentiate the identity $PM_\epsilon(\beta k + r_\epsilon(\beta, p, \mu), p, \mu) = 0$ with respect to β then we shall obtain the Taylor coefficients of r_ϵ. This is a tedious exercise so we omit the details, but one eventually obtains

R1. $P(d_u M_\epsilon[k + d_\beta r]) \equiv 0$,
R2. $P(d_u^2 M_\epsilon[k + d_\beta r, k + d_\beta r] + d_u M_\epsilon[d_\beta^2 r]) \equiv 0$,
R3. $P(d_u^3 M_\epsilon[k+d_\beta r]^3 + 2d_u^2 M_\epsilon[d_\beta^2 r, k+d_\beta r] + d_u^2 M_\epsilon[d_\beta^2 r, k+d_\beta r] + d_u M_\epsilon[d_\beta^3 r]) \equiv 0$,
R4. $P(d_u^4 M_\epsilon[k + d_\beta r]^4 + 6d_u^3 M_\epsilon[d_\beta^2 r, k + d_\beta r, k + d_\beta r] + 3d_u^2 M_\epsilon[d_\beta^3 r, k + d_\beta r] + 3d_u^2 M_\epsilon[d_\beta^2 r, d_\beta^2 r] + d_u M_\epsilon[d_\beta^4 r] + d_u^2 M_\epsilon[k + d_\beta r, d_\beta^3 r]) \equiv 0$
R5. $P(d_u^5 M_\epsilon[k + d_\beta r]^5 + 10d_u^4 M_\epsilon[d_\beta^2 r, [k + d_\beta^3 r]^3] + 10d_u^3 M_\epsilon[d_\beta^3 r, k + d_\beta r, k + d_\beta r] + 3d_u^2 M_\epsilon[d_\beta^4 r, d_\beta r] + 10d_u^2 M_\epsilon[d_\beta^3 r, d_\beta^2 r] + 2d_u^2 M_\epsilon[d_\beta^4 r, d_\beta r] + d_u M_\epsilon[d_\beta^5 r]) \equiv 0$.

Evaluating these expressions at $\beta = 0$, that is $u = r_\epsilon(0, p, \mu) = 0$, yields the following information. From R1 we have $PL(p, \mu)[k + d_\beta r(0, p, \mu; \epsilon)] = 0$, and because $L(p, \mu)k = (\mu^4 - p\mu^2 + 1)k \in \langle k \rangle$ we have $d_\beta r_\epsilon(0, p, \mu) \equiv 0$. The expression $d_\beta^2 r_\epsilon(0, p, \mu) = 0$ then follows from R2. Also, R3 gives

$$PL(p, \mu) d_\beta^3 r_\epsilon(0, p, \mu) = 6\epsilon P(k^3), \tag{5.3}$$

so that the third derivative of r_ϵ is not zero in general at $(u, p, \mu; \epsilon) = (0, 2, 1; \epsilon)$, but $d_\beta^3 r_\epsilon(0, p, \mu)$ is seen to provide an $O(\epsilon)$ contribution to the Taylor expansion of r_ϵ. Using Taylor's theorem to expand $r_\epsilon(\beta, p, \mu)$ with respect to β and using the symmetry properties of r_ϵ (it is odd with respect to β) we may write $r_\epsilon(\beta, p, \mu) = \frac{\beta^3}{6} R_\epsilon^1(p, \mu) + O(\beta^5)$, for some operator $R_\epsilon^1(p, \mu)$ with range in $\langle k \rangle^\perp$.

We can determine R_ϵ^1 as follows. In (5.3) seek an even Fourier series solution which is also orthogonal to k in X_e of the form $R_\epsilon^1(p, \mu) = \sum_{j=2}^\infty a_j \cos(jt)$, where the

coefficients a_j remain to be determined. Since $k^3(t) = \frac{1}{4}(\cos(3t) + 3\cos(t))$, it follows that the only non-zero coefficient is a_3 and

$$R^1_\epsilon(p,\mu) = \frac{3\epsilon}{2}\frac{\cos(3\cdot)}{81\mu^4 - 9p\mu^2 + 1}.$$

Using R4 and setting $\beta = 0$ we find $d^4_\beta r_\epsilon(0,p,\mu) \equiv 0$, which of course also follows from symmetry. We may evaluate $d^5_\beta r_\epsilon(0,p,\mu)$ from R5, which simplifies to give

$$P(d^5_u M_\epsilon[k]^5 + 10 d^3_u M_\epsilon[k,k,d^3_\beta r_\epsilon] + d_u M_\epsilon[d^5_\beta r_\epsilon]) \equiv 0.$$

To find $d^5_\beta r_\epsilon(0,p,\mu)$ we solve the following linear equation for $w \in X_e \cap \langle k\rangle^\perp$,

$$\mu^4 w'''' + p\mu^2 w'' + w + P\left[60k^2\frac{3\epsilon^2}{2}\frac{\cos(3\cdot)}{81\mu^4 - 9p\mu^2 + 1} + 120\epsilon k^5\right] = 0, \tag{5.4}$$

and then $d^5_\beta r_e(0,p,\mu) = w$. Since $k(t)^5 = \frac{1}{16}(\cos(5t) + 5\cos(3t) + 10\cos(t)), k(t)^2 = \frac{1}{2}(1 + \cos(2t))$ and $\cos(3t)\cos(2t) = \frac{1}{2}(\cos(5t) + \cos(t))$, we also solve (5.4) using a Fourier series expansion. Accordingly, taking $w(t) = \sum_{j=2}^\infty w_j \cos(jt)$, we find that all the coefficients w_j are zero, except when $j = 3$ or $j = 5$. In these cases

$$(81\mu^4 - 9p\mu^2 + 1)w_3 + 150\epsilon - \frac{45\epsilon^2}{81\mu^4 - 9p\mu^2 + 1} = 0,$$

and

$$(625\mu^4 - 25p\mu^2 + 1)w_5 + 30\epsilon - \frac{90\epsilon^2}{2(81\mu^4 - 9p\mu^2 + 1)} = 0.$$

It follows that $r_\epsilon(\beta,p,\mu) = \frac{\beta^3}{6}R^1_\epsilon(p,\mu) + \frac{\beta^5}{120}R^2_\epsilon(p,\mu) + \frac{\beta^7}{720}d^7_\beta r_\epsilon(0,p,\mu) + O(\epsilon^2\beta^3)$, where

$$R^2_\epsilon(p,\mu) = -\epsilon\left(\frac{150}{4(81\mu^4 - 9p\mu^2 + 1)}\cos(3\cdot) + \frac{30}{4(625\mu^4 - 25p\mu^2 + 1)}\cos(5\cdot)\right) + O(\epsilon^2).$$

One can show by further differentiating that $d^6_\beta r_\epsilon(0,p,\mu) \equiv 0$, as we expect from symmetry, and the equation which determines $d^7_\beta r_\epsilon(0,p,\mu)$ shows this term to be of order $O(\epsilon^2)$. Higher derivatives of r_ϵ will also be of order $O(\epsilon^2)$.

Now $p_\epsilon(\beta)$ and $\mu_\epsilon(\beta)$ are even functions of β and applying the zero-Hamiltonian constraint gives

$$\mu^4\delta((k+\rho_\epsilon)''^2) = \delta\left((k+\rho_\epsilon)^2 - \frac{1}{2}\epsilon\beta^2(k+\rho_\epsilon)^4 + \frac{1}{3}\epsilon\beta^4(k+\rho_\epsilon)^6\right), \tag{5.5}$$

using a prime to denote $\frac{d}{dt}$. Seeking an expansion of the bifurcation branch about $(\epsilon,\beta) = (0,0)$ we write

$$p_\epsilon(\beta) = 2 + \epsilon\left(P_1\beta^2 + P_2\beta^4\right) + O(\epsilon^2) \tag{5.6}$$

and

$$\mu_\epsilon(\beta) = 1 + \epsilon\left(\omega_1\beta^2 + \omega_2\beta^4\right) + O(\epsilon^2), \tag{5.7}$$

where P_1, P_2, ω_1 and ω_2 are real numbers to be determined. The highest power of β which exists in these expansions at $O(\epsilon)$ is the quartic because of the Taylor series we have found for $r_\epsilon(\beta)$. This is clear from equation (5.5) which contains terms of the order $\epsilon\beta^2$ and $\epsilon\beta^4$, but not $\epsilon\beta^6$ or higher.

To determine ω_1 and ω_2 we substitute the expressions for $\rho_\epsilon = r_e/\beta$ and μ_ϵ into (5.5). Using $\delta(\rho) = \left(\frac{1}{256}\beta^2 - \frac{23}{4608}\beta^4\right)\epsilon + O(\epsilon^2)$ and $\delta(\rho'') = \left(-\frac{9}{256}\beta^2 + \frac{215}{4608}\beta^4\right)\epsilon + O(\epsilon^2)$, we then find $1/(81\mu^4 - 9p\mu^2 + 1) = 1/64 + O(\epsilon)$ and $1/(625\mu^4 - 25p\mu^2 + 1) = 1/576 + O(\epsilon)$. Putting $v = \beta(k+\rho) \in \langle k\rangle \oplus \langle k\rangle^\perp$ in (5.1) and projecting the result onto the span of $k(t) = \cos(t)$ we obtain

$$\mu^4 - p\mu^2 + 1 - \epsilon\beta^2 \frac{1}{\pi}\int_0^{2\pi} k((k+\rho)^3 - \beta^2(k+\rho)^5)dt = 0. \tag{5.8}$$

We now use this information to equate coefficients at the appropriate orders to find

$$P_1 = -\frac{3}{4}, P_2 = \frac{5}{8}, \omega_1 = \frac{-9}{64} \text{ and } \omega_2 = \frac{5}{48}. \tag{5.9}$$

5.1. Conditions for a Fold bifurcation. The solution branch $C_\epsilon(1,1)$ determined above which branches from $p = 2$ can be continued from bifurcation in p for $p < 2$, or in μ for $\mu < 1$. When considered as a function of p, the branch has a fold bifurcation at some point which we label p_F and the same behaviour is observed when the branch is continued in μ. The numerical calculations presented in the next section also indicate that the solution branch is restricted to the parameter range $p > p_F$ and $\mu > \mu_F$, although we have no proof of this claim.

We can now prove the following theorem.

THEOREM 5.1. *There is a neighbourhood $I \subset \mathbb{R}$ of zero such that if $\epsilon \in I$, the zero-Hamiltonian branch $C_\epsilon(1,1)$ associated with equation (5.1) which bifurcates from $p = 2$ at $\mu = 1$ has a fold which occurs with respect to p at*

$$p_F(\epsilon) = 2 - \frac{9}{40}\epsilon + O(\epsilon^2). \tag{5.10}$$

There is also a fold in $C_\epsilon(1,1)$ with respect to μ which occurs at

$$\mu_F(\epsilon) = 1 - \frac{243}{5120}\epsilon + O(\epsilon^2). \tag{5.11}$$

Proof. Using (5.6), (5.7) and (5.9), a fold bifurcation with respect to p occurs on the $C_\epsilon(1,1)$ branch when the conditions $d_\beta p_\epsilon(\beta) = 0$ and $d^2_{\beta\beta}p_\epsilon(\beta) \neq 0$ are met. Applying the implicit function theorem when $\epsilon \neq 0$, these conditions are satisfied when $\beta^2 = \beta_F^2 \equiv 3/5 + \mathcal{O}(\epsilon)$ giving (5.10).

Similarly, a fold bifurcation with respect to μ occurs on the branch $C_\epsilon(1,1)$ when $d_\beta\mu_\epsilon(\beta) = 0$ and $d^2_{\beta\beta}\mu_\epsilon(\beta) \neq 0$ and, provided $\epsilon \neq 0$, these conditions are satisfied when $\beta^2 = 27/40 + \mathcal{O}(\epsilon)$, giving (5.11). □

6. Numerical computations.

6.1. Preliminaries. We now describe a series of numerical calculations to determine solutions of the unscaled differential equation (1.1) with the restiffening foundation whose primary solution branch was studied in the previous section:

$$u'''' + pu'' + u - \epsilon(u^3 - u^5) = 0. \tag{6.1}$$

FIG. 6.1. *Bifurcations of zero-Hamiltonian, periodic solutions from* $p = 2, p = 2\frac{1}{2}$ *and* $p = 3\frac{1}{3}$, *with* p *plotted horizontally against* $\|u'\|_{L^\infty}$ *vertically;* e *are even solutions,* o *are odd solutions and* b *are solution with* broken *symmetry.*

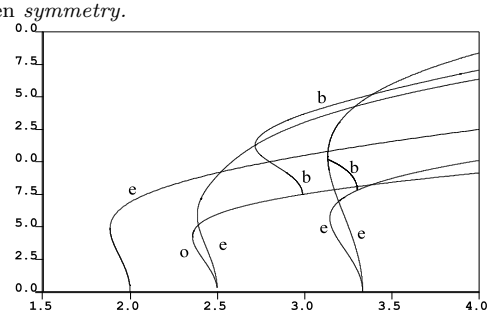
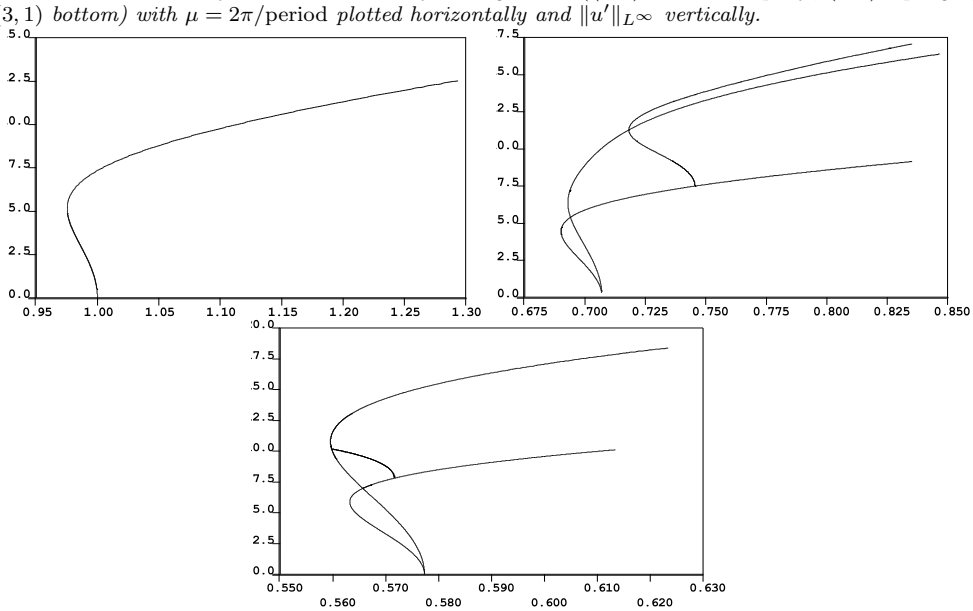


FIG. 6.2. *Period of solutions solutions from Figure 6.1 (*$(1,1)$ *branch is top-left,* $(2,1)$ *top-right,* $(3,1)$ *bottom) with* $\mu = 2\pi/\text{period}$ *plotted horizontally and* $\|u'\|_{L^\infty}$ *vertically.*



We augment this with the periodic boundary conditions $u(0) = u(T), u'(0) = u'(T)$ and $u'''(0) = u'''(T)$, and specify the phase by requiring $u'(0) = 0$. Finally, we impose the constraint that the Hamiltonian is zero, so that

$$u''(0) = \pm\sqrt{2(u(0)^2/2 - \epsilon(u(0)^4/4 - u(0)^6/6)} \tag{6.2}$$

and set $\epsilon = 1/2$ for the purposes of computation.

In (6.2) the positive root corresponds to the solution which is tangential to the rescaled eigensolution $e_-(x) \equiv n\cos(x\sqrt{m/n}) - m\cos(x\sqrt{n/m})$ at the bifurcation point $(u, p, \mu) = (0, p_{n,m}, \mu_{n,m})$, whereas the negative root corresponds to the solution which is tangential to the eigensolution $e_+(x) \equiv n\cos(x\sqrt{m/n}) + m\cos(x\sqrt{n/m})$. In order to follow the solution branches in p and to detect fold bifurcations, the collocation-based code *AUTO* [14] was used.

6.2. Calculation of the solution branches. We now illustrate three cases regarding the bifurcation of solutions of (6.1): the case $(n, m) = (1, 1)$, for which

FIG. 6.3. *Solutions ($u(s)$ for $0 \le s \le 1$) bifurcating from $p = 2$, away from the bifurcation point. In accordance with Theorem 2.2 these are even about zero and odd about one-quarter.*

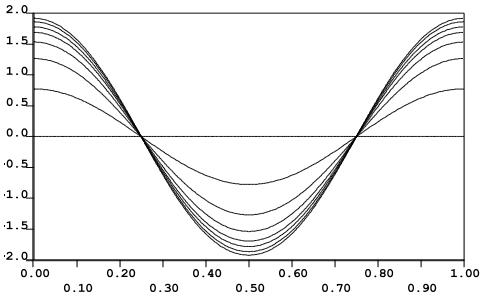
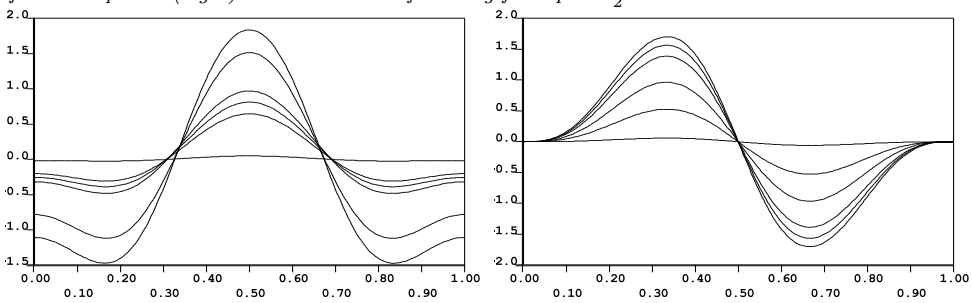


FIG. 6.4. *(left) Even solutions ($u(s)$ for $0 \le s \le 1$) bifurcating from $p = 2\frac{1}{2}$, away from the bifurcation point. (right) Odd solutions bifurcating from $p = 2\frac{1}{2}$.*



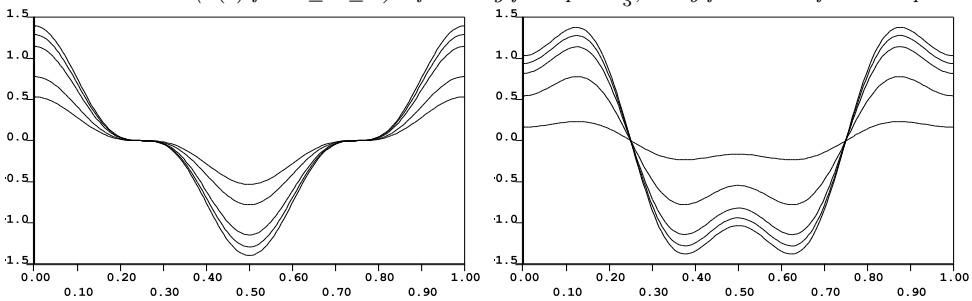
there is a unique bifurcating branch; the case $(n, m) = (2, 1)$, with $p_{2,1} = 2\frac{1}{2}$ and $\mu_{2,1} = 1/\sqrt{2}$ and $(n, m) = (3, 1)$, for which $p_{3,1} = 3\frac{1}{3}$ and $\mu_{1,3} = 1/\sqrt{3}$. Broadly speaking, higher values of n and m lead to similar solution branches.

Figure 6.1 shows the bifurcation branches which are proven to exist in Theorems 3.1 and 3.3, with p plotted against $\|u'\|_{L^\infty}$. The $(1, 1)$ branch has the form described in Theorem 5.1, and for the $p = 2\frac{1}{2}$ and $p = 3\frac{1}{3}$ cases one sees a similar geometry in that the branches initially bifurcate to the left, have fold bifurcations and then persist for all values of p to the right of the fold point.

The following comments are in order regarding Figure 6.1 and the three points $p = 2, 2\frac{1}{2}$ and $3\frac{1}{3}$. Due to the $\mathbb{Z}_2$-symmetry of (6.1) and of the symmetry properties of the eigenfunctions when $(n, m) = (2, 1)$, if $u(t)$ is one even solution on the $(2, 1)$ branch, then so too is $-u(t + T/2)$. Therefore, in order to obtain a second *distinct* periodic solution, we apply Theorem 3.5 to give the existence of two branches of odd solutions. Again, one of these branches of odd solutions can be obtained from the other by symmetry, and we therefore have plotted one of each even and odd branch in Figure 6.1. The solutions on this branch are shown in Figure 6.4.

Figures 6.3, 6.4 and 6.5 each show several solutions chosen from Figure 6.1 on the branches which connect to $p = 2, p = 2\frac{1}{2}$ and $p = 3\frac{1}{3}$ respectively, although the domain of each solution has been normalised to unity. (The information regarding the period of the solutions is given in Figure 6.2.) If we examine Figure 6.5, we notice that each of the solutions is even about zero and odd about one quarter. Consequently, the two branches of solutions shown are in fact identical, up to a shift and a reflection, to the odd solutions which are obtained using Theorem 3.5.

FIG. 6.5. *Solutions ($u(s)$ for $0 \leq s \leq 1$) bifurcating from $p = 3\frac{1}{3}$, away from the bifurcation point.*



Since $F(u)$ and $uF_u(u)$ are both positive for non-zero u when $\epsilon = 1/2$, the global bifurcation theorem (Theorem 4.2) applies to the $(1,1)$ branch and the nodal properties are preserved along the resulting global branch in accordance with Theorems 3.6, 3.7 and 3.8. This is illustrated in each of Figures 6.3, 6.4 and 6.5.

Finally, note that the $(1,1)$ branch in Figure 6.1 appears to have no further bifurcations, whereas the $(2,1)$ and $(3,1)$ branches both have symmetry-breaking, secondary bifurcation points. What is interesting about the resulting branches of unsymmetric solutions is that they form connections between the $(2,1)$ and $(3,1)$ branches. This indicates that it would be futile to seek generalisations of the results of §3.5 to include the space of all periodic, zero-Hamiltonian solutions of (1.1) and that the disjointness properties of the solution branches obtained in this paper are peculiar to spaces of symmetric solutions.

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