SOME COMMENTS ON URBAN TRAVEL DEMAND ANALYSIS, MODEL CALIBRATION AND THE ECONOMIC EVALUATION OF TRANSPORT PLANS

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1. INTRODUCTION

In the analysis of urban transport plans, it is often necessary to examine the effect of constraints on travel demand. In modal demand forecasting, for example, it may be necessary to test a policy which limits central area parking; and traditional work trip distribution models include constraints which ensure that the predicted number of journeys into a given zone is consistent with independent estimates of trip attractions available in a generation sub-model. In the former case the constraint reflects a direct policy instrument of control, while the latter is traditionally viewed as an artifice of the model building procedure and has not, ostensibly, an economic interpretation.

The purpose of this paper is to examine the role played by constraints in transport analysis, and in particular some of the problems and possibilities arising in the economic evaluation of plans by the generalised surplus method proposed by Neuberger [1] and extended by Williams [2]. Attention will be confined throughout to the distribution and modal split models of the transport planning process. In section 2 the commonly adopted forecasting models and evaluation methodology will be summarised, both to indicate their method of generation and for subsequent comparative purposes. The calculation of suitably defined measures of consumer surplus in the presence of constraints is considered in section 3, and the results applied to the doubly-constrained gravity model in order to motivate the study of amendments to the demand model formulation procedure. An alternative approach to model generation is explored in section 4, based on the maximisation of aggregate consumer surplus, and the resultant models are compared with their entropy maximising model counterparts. The problems which arise in deriving user benefits corresponding to the latter models are shown to disappear when an economic based perspective is adopted in demand model formation. Finally the notion of duality is exploited in section 5 to show how the processes of model calibration, demand analysis and user benefit evaluation may be performed within a single framework.

*Institute for Transport Studies, University of Leeds. It is a pleasure to acknowledge helpful advice from several of our colleagues at Leeds, and in particular Dr Q. M. Dalvi, Dr C. A. Nash, Mr M. L. Senior and Professor A. G. Wilson, and useful discussions with Professors Britton Harris and Tony Smith. As ever, what errors remain are our responsibility alone.
2. DEMAND ANALYSIS AND EVALUATION IN THE PRESENCE OF CONSTRAINTS

**Entropy Maximising models of travel demand**

In a path-breaking article, Wilson [3] presented a family of distribution and modal split models which have formed the basis for those implemented in the majority of subsequent major British transport studies. The entropy maximising method not only gave a probabilistic interpretation to spatial interaction models which had hitherto been viewed primarily in deterministic terms, but also provided a natural framework within which to consider constraints on demand. The now widely used trip distribution model for the work purpose

\[ T^n_{ij} = A^n_i O^n_i B^n_j D_j e^{-\beta^n c^n_{ij}} \]  

in which

\[ A^n_i = e^{a^n_i} = (\sum_j B^n_j D_j e^{-\beta^n c^n_{ij}})^{-1} \]  

and

\[ B^n_j = e^{-\gamma^n_j} = (\sum_i A^n_i O^n_i e^{-\beta^n c^n_{ij}})^{-1} \]  

was shown to be generated by a mathematical program of the following structure:

\[ \max_{\{T^n_{ij}\}} S = -\sum_n \sum_{ij} T^n_{ij} \left( \log \frac{T^n_{ij}}{O^n_i D_j} - 1 \right) \]  

subject to the trip-end constraints

\[ \sum_j T^n_{ij} = O^n_i \]  

\[ \sum_i T^n_{ij} = D_j \]  

and the cost constraint

\[ \sum_{ij} T^n_{ij} c^n_{ij} = C^n \]  

Here, \( T^n_{ij} \) denotes the number of trips between zones \( i \) and \( j \) by persons of type \( n \) (usually taken as a car-ownership index), while \( O^n_i \) and \( D_j \) correspond respectively to zonal generations and attractions. The perceived costs \( c^n_{ij} \) may be expressed in terms of modal generalised costs \( c^n_{ij} \) through a composite function

\[ c^n_{ij} = h (\ldots c^n_{ij} \ldots) \]  

including only those modes \( k \) available to \( n \)-type persons. \( C^n \) is the total group expenditure on travel.

The Lagrange multipliers or dual variables \( a^n_i \) and \( \gamma_j \) associated with the constraints (5) and (6) will figure prominently in the discussion which follows. The optimal values have the well-known mathematical property of being the partial derivative of the objective function with respect to the corresponding “resources”, which, in this case, are \( O^n_i \) and \( D_j \) respectively.

The beta parameters \( \beta^n \), which are the dual variables corresponding to constraint (7), have an elasticity connotation, essentially recording the sensitivity of demand to changes in cost difference between spatial interaction opportunities. In distribution modelling they are usually referred to, in mechanical terms, as “impedance”, “deterrence”, or simply “trip length” parameters. In general, different classes of persons will be characterised by different \( \beta \)-parameters. \( n = 1 \) and
\( n = 2 \) will be used below to denote persons from car-owning and non car-owning households respectively, and \( \beta \) will denote a vector containing the quantities \( \beta^1 \) and \( \beta^2 \).

A modal-split model of logit form
\[
T^{nk}_{ij} = e^{-\chi_{ij}^* c^{nk}_{ij}}
\]
\[
= T^n_{ij} \frac{e^{-\chi_{ij}^* c^{nk}_{ij}}}{\sum_{k \in H(n)} e^{-\chi_{ij}^* c^{nk}_{ij}}}
\]
is derivable from a program of similar mathematical structure to that embodied in equations (4) to (7). Now, the multiplier \( v^*_i \) appears in order to satisfy a trip constraint
\[
\sum_{k \in H(n)} T^{nk}_{ij} = T^n_{ij}
\]
where \( \sum \) denotes summation over all modes available to persons of type \( n \).

Together with an assignment algorithm which allocates demand to routes through a network, the models (1) and (10) are used in the determination of the comparative value of plans characterised by different cost matrices \( \{c^{nk}_{ij}\} \) — it is usually assumed in transport studies that trip ends are inelastic.

**The Generalised Surplus method of determining user benefit**

Reverting to a general notation, in which \( \mathcal{D} \equiv (\mathcal{D}_1 \ldots \mathcal{D}_k \ldots \mathcal{D}_N) \) denotes a set of demand functions for \( N \) movements, with associated perceived costs \( \mathbf{c} \equiv (c_1 \ldots c_k \ldots c_N) \), the benefit accruing from a change in the cost vector from \( \mathbf{c} \) to \( \mathbf{\bar{c}} \) may be determined by evaluating the generalised surplus measure (Hotelling [4])
\[
\Delta B = - \sum_{\xi} \int_{\mathcal{P}} d\mathcal{P}_\xi \mathcal{D}_\xi(c_1 \ldots c_N)
\]
along a path \( \mathcal{P} \) between the initial and final cost states of the system. If \( \mathbf{c}(\sigma = 0) = \mathbf{c} \) corresponds to a “do-nothing” policy and \( \mathbf{c}(\sigma = 1) = \mathbf{\bar{c}} \) the test system, the benefit is given by
\[
\Delta B = - \sum_{\xi} \int_{\sigma}^{1} d\sigma \frac{d\mathcal{P}_\xi(\sigma)}{d\sigma} \mathcal{D}_\xi(\sigma)
\]
where \( \mathbf{c} = \mathbf{c}(\sigma) \) denotes the path \( \mathcal{P} \) in cost space. It may be shown (Williams [2]) that the widely used rule-of-a-half measure
\[
\Delta B = \frac{1}{2} \sum_{\xi} (\mathcal{D}_\xi + \mathcal{D}_{\xi}^-)(c_\xi - \bar{c}_\xi)
\]
is a marginal approximation to equation (12) when integration is performed on the linear path between the end-states \( \mathbf{c} \) and \( \mathbf{\bar{c}} \), to which the demand vectors \( \mathbf{D} \) and \( \mathbf{\bar{D}} \) correspond.

The possibility has recently been explored of obtaining analytic measures of consumer surplus by direct integration of (12). Wilson and Kirwan [5] and Neuberger [1] have derived non-marginal benefit measures for the singly-constrained gravity model, a result extended to a general class of spatial interaction and modal split models by Williams [2], who showed that, to a very good approximation, the benefit is given by

269
\[ \Delta B = \sum_n \frac{1}{\bar{\beta}^n} \sum_i O_i^n (\bar{z}_i^n - \bar{z}_i^n) + \frac{1}{\bar{\beta}} \sum_j D_j (\bar{\gamma}_j - \gamma_j) \]  

(15)

when demand is forecast with the model (1). The bar and double bar over the dual variables \( \alpha \) and \( \gamma \) again denote evaluation in the "do-nothing" and "test" systems respectively. \( \bar{\beta} \) may be taken as the weighted mean

\[ \bar{\beta} = \frac{\sum_n \beta^n \sum_i O_i^n}{\sum_n \sum_i O_i^n} \]  

(16)

Equation (15) is an exact result when the parameters \( \beta^n, n = 1, 2, \ldots \) are equal. It is a significant feature that the model (1) is not integrable in the Hotelling sense (Hotelling, [4]) when the parameters are different, because the conditions

\[ \frac{\partial D_{\xi}}{\partial c_{\xi}} = \frac{\partial D_{\xi'}}{\partial c_{\xi'}} \text{ for } \xi, \xi' = 1 \ldots N \]  

(17)

necessary to render the path integral in equation (12) unique, are not satisfied. Now the model generated within the entropy maximising framework, when unencumbered by the constraint (6), does satisfy the integrability conditions (17), and gives rise to an unambiguous measure of surplus change. We will discuss in section 4 the nature of the approximation incurred in deriving equation (15)—inextricably connected with the presence of the destination constraint which allows the segments \( n = 1, 2, \ldots \) of the demand market to interact, the significance of a violation of the conditions (17), and their implication for travel demand modelling and the measurement of user benefit. Before doing so we examine the general problem of evaluation, when demand is subject to constraints, and interpret the special case of the doubly-constrained gravity model.

3. EVALUATION IN THE PRESENCE OF CONSTRAINTS

The Path Integral approach

The demand functions used in travel forecasting are conceptually, if not explicitly, the outcome of an optimisation process within either a utility maximisation or an entropy maximising framework. The purpose of this section is to exploit the mathematical programming formulation in order to illustrate how, in the presence of constraints, alternative equivalent benefit measures may be produced. In the first instance the path integrals over constrained and unconstrained demand functions will be related.

Consider the problem

\[ \max_{\{D_{\xi}\}} G(\ldots D_{\xi} \ldots) \]  

(18)

subject to a cost constraint

\[ \sum_{\xi} D_{\xi} c_{\xi} = C \]  

(19)
and a set of $M$ linear constraints
\[ \sum_{\xi} D_{\xi} a_{\xi \mu} = f_{\mu}, \quad \mu = 1 \ldots M. \]  
(20)

The solution to this programme may be found in the usual way by embedding the problem in a new unconstrained problem through the Lagrangian formulation. Defining the Lagrangian $\mathcal{L}$ as follows
\[ \mathcal{L}(\mathcal{D}, \eta, \beta) = G(\mathcal{D}) + \beta \left( C - \sum_{\xi} D_{\xi} c_{\xi} \right) + \sum_{\mu} \eta_{\mu} (f_{\mu} - \sum_{\xi} D_{\xi} a_{\xi \mu}) , \]  
(21)

the conditions
\[ \frac{\partial \mathcal{L}}{\partial D_{\xi}} = 0, \quad \xi = 1 \ldots N \]  
(22)
\[ \frac{\partial \mathcal{L}}{\partial \beta} = 0 \]  
(23)
\[ \frac{\partial \mathcal{L}}{\partial \eta_{\mu}} = 0, \quad \mu = 1 \ldots M \]  
(24)

are necessary for the maximisation of $\mathcal{L}$ and thereby for the solution of the programme (18)–(20). At the optimal solution the standard results
\[ \beta = \frac{\partial G}{\partial C} , \quad \eta_{\mu} = \frac{\partial G}{\partial f_{\mu}}, \quad \mu = 1 \ldots M \]  
(25)

display the significance of the dual variables.

It is important at this stage to distinguish between two processes: the solution of the (calibration) problem (18) to (20) for a fixed value of $c = \{c_1 \ldots c_N\}$; and the solution of the problem
\[ \max_{\{\mathcal{D}, \eta\}} \mathcal{L}(\mathcal{D}, \eta, \beta) \]  
(26)
for a fixed value of $\beta = \beta^*$, along a path $P : c = c(\sigma)$ in cost space. This distinction is particularly pertinent to current practice in demand forecasting, in which a model is calibrated in a base year, a process involving the calculation of $\beta$ and $\eta$, where $c$ and $C$ are respectively the base movement costs and total travel cost; and changes in cost $c$ are evaluated in a design year for the fixed value $\beta$.

Now it is clear that the problem (26) generates a demand function equivalent to that obtained through the solution of the programme
\[ \max_{\{\mathcal{D}\}} G(\mathcal{D}) - \beta \sum_{\xi} D_{\xi} c_{\xi} \]  
(27)
subject to
\[ \sum_{\xi} D_{\xi} a_{\xi \mu} = f_{\mu}, \quad \mu = 1 \ldots M \]  
(28)
for which the Lagrangian is
\[ (\mathcal{L}, \eta) = G(\mathcal{D}) - \beta \sum_{\xi} D_{\xi} c_{\xi} + \sum_{\mu} \eta_{\mu} (f_{\mu} - \sum_{\xi} D_{\xi} a_{\xi \mu}) . \]  
(29)

This will be exploited in the evaluation problem which will now be considered.

Differentiating $\mathcal{L}(\sigma)$, along the path of integration $c = c(\sigma)$, we obtain
\[ \frac{d \mathcal{L}}{d\sigma} = \sum_{\xi} \frac{\partial \mathcal{L}}{\partial D_{\xi}} \frac{d D_{\xi}}{d\sigma} - \beta \sum_{\xi} D_{\xi} \frac{d c_{\xi}(\sigma)}{d\sigma} + \sum_{\mu} \frac{\partial \mathcal{L}}{\partial \eta_{\mu}} \frac{d \eta_{\mu}}{d\sigma} . \]  
(30)
If there is a local maximum for all cost states along the path \( P \), the conditions (22) and (24) may be invoked to reduce equation (30) to the form

\[
\frac{d\tilde{L}}{d\sigma} = -\beta \sum_{\xi} \mathcal{D}_{\xi} \frac{dc_{\xi}(\sigma)}{d\sigma},
\]

a result which may be used to prove that conditions (17) are satisfied. Employing the relations (31) and (29), the quantity \( \Delta B \) can be written

\[
\Delta B = -\sum_{\xi} \int_{0}^{1} d\sigma \cdot \frac{dc_{\xi}}{d\sigma} \mathcal{D}_{\xi}(\sigma)
\]

\[
= \frac{1}{\beta} \int_{0}^{1} d\sigma \frac{d\tilde{L}}{d\sigma}
\]

\[
= \frac{1}{\beta} (\tilde{G} - \bar{G}) + (\tilde{C} - \bar{C}).
\]

The change in total surplus is thus equal to the total cost change plus the difference, scaled to money units by \( \beta \), between the objective function \( G \) evaluated in the initial and final cost states. Note that the demand function which enters equation (32) is one in which the constraints operate along the path of integration. We shall now examine the corresponding surplus integral over a model not subject to constraints, by invoking a linear transformation which defines, for all points along the path, a cost vector \( \mathbf{K}(\sigma) \) defined by

\[
K_{\xi}(\sigma) = c_{\xi}(\sigma) + \sum_{\mu} \eta_{\mu}(\sigma) \frac{d_{\xi\mu}}{\beta} \quad 0 \leq \sigma \leq 1.
\]

The path integral \( \Delta B^* \) of the unconstrained model \( \mathbf{D}^*(\mathbf{K}(\sigma)) \)

\[
\Delta B^* = -\sum_{\xi} \int_{0}^{1} d\sigma \frac{dK_{\xi}(\sigma)}{d\sigma} \mathcal{D}_{\xi}^*(\sigma)
\]

can be related to \( \Delta B \) as follows.

Consider the Lagrangian \( \tilde{\mathcal{L}}^* \) defined by

\[
\tilde{\mathcal{L}}^*(\mathcal{D}^*) = G(\mathcal{D}^*) - \beta \sum_{\xi} \mathcal{D}_{\xi}^* K_{\xi} + \sum_{\mu} \eta_{\mu} f_{\mu}
\]

where \( K_{\xi}(\sigma) \), defined in equation (34) is regarded as an input with known parameters \( \eta \).

\( \mathcal{D}^* \), which is the solution of the problem

\[
\max_{\{\mathcal{D}^*\}} \tilde{\mathcal{L}}^*(\mathcal{D}^*)
\]

may now be regarded as an unconstrained model with respect to the costs \( K_{\xi} \). If the quantities \( \eta \) in \( \mathbf{K} \) are the optimal values associated with (28), then at all points along the path \( P \)

\[
\mathcal{D}^*(\mathbf{K}(\sigma)) = \mathcal{D}(\mathbf{c}(\sigma))
\]

and

\[
\tilde{\mathcal{L}}^*(\sigma) = \tilde{\mathcal{L}}(\sigma) \quad 0 \leq \sigma \leq 1.
\]

The demand model \( \mathcal{D}^* \) is a function of the path parameter \( \sigma \) through its dependence on \( \mathbf{K} \).

\( \Delta B^* \) may now be determined as before by computing the derivative of \( \tilde{\mathcal{L}}^* \) with respect to the path parameter \( \sigma \).
\[
\frac{d \hat{P}^*}{d \sigma} = -\hat{\beta} \sum \xi \mathcal{D}_\xi \frac{dK_\xi(\sigma)}{d \sigma} + \sum \mu \hat{f}_\mu \frac{d \check{q}_\mu(\sigma)}{d \sigma} \tag{39}
\]

giving immediately
\[
\Delta B^* = \frac{1}{\hat{\beta}} (\bar{G} - \bar{G}) + \bar{c} \mu - \frac{1}{\hat{\beta}} \sum \mu \hat{f}_\mu (\bar{\eta}_\mu - \bar{\eta}_\mu) \tag{40}
\]
\[= \Delta B - \frac{1}{\hat{\beta}} \Sigma \mu \hat{f}_\mu (\bar{\eta}_\mu - \bar{\eta}_\mu) \tag{41}\]
or
\[
\Delta B = \Delta B^* + \frac{1}{\hat{\beta}} \Sigma \mu \hat{f}_\mu (\bar{\eta}_\mu - \bar{\eta}_\mu) \tag{42}\]

The relations (40) and (42) will be used below both to interpret and to evaluate the effect of cost changes on travel demand.

Equation (42) may be written symbolically as
\[
- \sum \xi \int \bar{c}_\xi \mathcal{D}_\xi (\ldots e_\xi \ldots) = - \sum \xi \int \bar{c}_\xi + \frac{1}{\hat{\beta}} \Sigma \mu \check{q}_\mu (\bar{\eta}_\mu - \bar{\eta}_\mu) \tag{43}\]

because the integrability conditions are satisfied, although it should be remembered that the integrals are in general unspecified until a particular path is chosen.

Equation (43) has been shown to hold for demand functions generated by specific optimisation problems, and these considerations turn out to be sufficient for the general discussion below. It should be noted, however, that the relation (43) holds along any path of integration in cost space even if the integrability conditions (17) are not satisfied, and holds moreover for general non-linear inequality constraints for a suitably modified transformation of variables.

In the applications to follow the right-hand side of equation (43) will be decomposed into a perceived user benefit contribution, and what will normally be interpreted as a transfer payment. The left-hand side or path integral over the constrained demand function between initial and final cost states thus represents the total benefit (apart from resource corrections). As the rule-of-a-half expression (14) is a marginal approximation to this quantity, it is clear that any transfer payments associated with constraints are implicitly accounted for. The doubly-constrained gravity model will be interpreted in the light of this result.

**An economic interpretation of the doubly-constrained gravity model**

For the model
\[
T_{ij} = A_i q_i D_j e^{-BCij} \tag{44}
\]
in which the dual variables \( \alpha \) and \( \gamma \) are determined to satisfy trip end constraints, the relations (42) and (33) may be invoked to express total benefit arising from modifications in the cost vector as follows:

273
\[ - \sum_{y} \int \frac{\overline{c}_{iy}}{\overline{c}_{ij}} d\overline{c}_{ij} T_{ij}(c) = - \sum_{y} \int \frac{\overline{c}_{ij}}{\overline{c}_{ij}} \frac{O_{i}D_{j} e^{-\beta C_{ij}}}{\sum_{j} D_{j} e^{-\beta C_{ij}}} + \frac{1}{\beta} \sum_{j} D_{j}(\overline{\gamma}_{j} - \overline{\gamma}_{i}) \]  

(45)

\[ = - \sum_{y} \int \frac{\overline{c}_{ij}}{\overline{c}_{ij}} \frac{O_{i}D_{j} e^{-\beta C_{ij}}}{\sum_{j} O_{i} e^{-\beta C_{ij}}} + \frac{1}{\beta} \sum_{i} O_{i}(\overline{\alpha}_{i} - \overline{\alpha}_{i}) \]  

(46)

\[ = - \sum_{y} \int \frac{\overline{c}_{ij}}{\overline{c}_{ij}} \frac{O_{i}D_{j} e^{-\beta C_{ij}}}{\sum_{j} O_{i} e^{-\beta C_{ij}}} + \frac{1}{\beta} \sum_{i} O_{i}(\overline{\alpha}_{i} - \overline{\alpha}_{i}) + \frac{1}{\beta} \sum_{j} D_{j}(\overline{\gamma}_{j} - \overline{\gamma}_{i}) \]  

(47)

\[ = \frac{1}{\beta} \{ \sum_{i} O_{i}(\overline{\alpha}_{i} - \overline{\alpha}_{i}) + \sum_{j} D_{j}(\overline{\gamma}_{j} - \overline{\gamma}_{i}) \} \]  

(48)

The result (48) follows from (47) because the total number of trips in the system remains constant. Again it should be remembered that the integral expressions are defined along paths, although uniqueness is assured through the condition (17) being satisfied.

The alternative ways of expressing the total perceived benefit correspond to different behavioural interpretations of the doubly constrained model, which may, as Senior [6] has noted, be viewed as: a residential location model with housing capacity constraints; a workplace location model with zonal employment capacity constraints; or a joint residential-workplace location model. In the three cases, changes in transport costs are considered to bring about: residence switching with fixed workplaces; workplace location switching with fixed residences; and changes of house and job, respectively. In reality, of course, in the transient state before the (assumed) design year equilibrium, some fraction of the population will not respond to cost changes, while the others will fall into one of the three categories described above.

If the model is interpreted as a job-choice model for which

\[ T_{ij} = O_{i} p_{ij} \]

with \( p_{ij} \), the probability of a person selecting a job in \( j \) conditional on his living in zone \( i \), the quantity \( \frac{\gamma_{j}}{\beta} \) is the money cost (shadow price) introduced to satisfy the constraints associated with job totals. To paraphrase Cochrane [7], "the opportunities will be taken up by the trips for which the surplus is greatest and will be brought about either by the trip end utilities being bid down or the costs bid up". In the classical theory of land rent, increasing the competition for employment in a zone will lower wage rates and will, other things being equal, cause rents on the commercial/industrial sites to be bid up. The effects of transport network changes which induce changes in the shadow prices \( \gamma_{j} \) will thus, in principle, percolate through
the labour and land markets, and result in a transfer of money between those seeking workplaces and the land owners. Similar considerations hold for the residential location model in terms of the mechanism of a competitive housing market. Beckmann and Wallance [8] have discussed a linear programming utility model in which rents emerge as the dual variables associated with the constraint on housing capacity in a zone. As the authors note, “in predicting the residential and travel choices they (the rents) are the essential link between the supply of housing on the one hand and demand on the other hand as given by the preference and attribute characteristics of the urban areas”.

Because the values of \(a\) and \(\gamma\) are not unique, but may be calculated up to an arbitrary constant, only the relative changes of benefit between different zone pairs are significant, until a baseline is selected from which benefits are to be measured. This is true whichever interpretation is considered. It is not of course the practice, nor indeed is it necessary, to view the doubly constrained model in terms of one or other of these three interpretations—the same total benefit, which may be determined uniquely, is involved for each.

4. ALTERNATIVE DEMAND MODEL FORMS

**Group surplus models of travel demand**

We have so far accepted without question both the generalised surplus (12) as representing an acceptable measure of total welfare change, and the demand models (1) and (10) as forming a satisfactory basis for forecasting. Yet the entropy maximising approach is characterised by its lack of assumptions regarding micro-behaviour, only requiring that it be consistent with constraints or information available at the macro-level. In the language of statistical mechanics, the maximum entropy state of the system is that which has associated with it the largest number of micro-arrangements. The process is one in which an aggregation over micro-relations is performed with very limited information used in generating macro-demand.

If, however we are to infer preferences from behaviour, the forecasting methodology should be based on a behavioural model specified at an appropriate level of aggregation. It is not sufficient for the purpose of economic evaluation to adopt a descriptive relation for the functions which are used in spatial interaction and modal split modelling. An explanatory theory is needed which invokes in some sense the choice processes of individual decision-makers. While it is relatively straightforward to generate micro-models based on classical utility theory as Golob *et al.* [9], Neidercorn and Bechdolt [10] and others have done, or through probabilistic choice theory (see, for example, the papers by Cochrane [7], Smith [11] and Williams [12]), the severe problems of aggregating over micro-relations within a spatial context remain. Some of the difficulties have been discussed by Wilson [13] and by Cesario and Smith [14].

It is clear that, if the aggregation problems form a stumbling block in the formation of macro-demand models from behavioural micro-theories, a concomitant problem is faced when macro-evaluation forms are sought. The general approach adopted here, essentially the same as that adopted by Neuberger [1], is to accept the path
integrate over the macro-demand model as the change in welfare for a group of
travellers, and generate an optimisation problem which is consistent with the
notion of surplus maximisation for the group. In this sense we shall view the state
of the system as arising from the optimal decisions of the individuals comprising it,
although it is, of course, not entirely satisfactory that the aggregation process by
which the macro-relations are derived from utility maximising behaviour of indi-
vidual decision-makers within a spatial context remains unspecified. It is usually
assumed that, apart from additive constants, individuals in the same zone have
identical utility functions.

The model to be developed, and indeed any model based on the optimal behaviour
of decision-makers, makes the usual assumptions about perfect information being
available in perfectly operating markets associated with location choices. The
entropy maximising methodology is not, of course, restricted by such conditions, and
the framework may readily embrace “fuzziness” or “sub-optimality” in the system.
Whether equation (1) is underpinned by optimal or sub-optimal behaviour is
completely unspecified. To use the surplus measure (12), however, we should assume
the former.

From the equilibrium demand configuration of trips generated by the programme
(26) or (27) the generalised surplus change is given by (equation 33)

\[ \Delta B = \frac{1}{\beta} \left( \bar{G} - \bar{G} \right) + \bar{C} - \bar{C} \]

The quantity \( G(D)/\beta \) may thus be regarded as a group utility function in money
units. To take a special case, \( G(D) \) for the family of spatial interaction models,
would be

\[ - \sum_{\hat{q}} T_{\hat{q}} \left( \log \frac{T_{\hat{q}}}{O_i D_j} - 1 \right) \]

as Neuberger [1] noted.

Now a fundamental question arises. For a given value of \( \beta \), is the state of maximum
entropy identical to that obtained by maximising generalised surplus subject to a set
of constraints, as for example with the programme

\[ \max_{\{\hat{T}_{\hat{q}}\}} - \frac{1}{\beta} \sum_{\hat{q}} \hat{T}_{\hat{q}} \left( \log \frac{\hat{T}_{\hat{q}}}{O_i D_j} - 1 \right) - \sum_{\hat{q}} \hat{T}_{\hat{q}} C_{\hat{q}} \]

subject to

\[ \sum_{\hat{q}} \hat{T}_{\hat{q}} = O_i \]  

\[ \sum_{\hat{q}} \hat{T}_{\hat{q}} = D_j \]  

The demand variables \( T_{\hat{q}} \) and \( \hat{T}_{\hat{q}} \) will be used to distinguish between the entropy
maximising model and that based on group surplus.

By examining the Kuhn-Tucker optimality conditions for the programme (49)—
(51) and the corresponding entropy maximising problem, Coelho and Wilson [15]
have shown that the demand variables \( T_{\hat{q}} \) and \( \hat{T}_{\hat{q}} \) are indeed equivalent when \( \beta \) is
taken as the optimal Lagrange multiplier associated with the cost constraint on the
demand functions \( T_{\hat{q}} \). This equivalence will be maintained for models involving

276
different person classes \( n \), provided that the demand variables \( T^n_j \) do not interact. When a destination constraint of the form (6) is present, however, \( T^n_j \) and \( \hat{T}^n_j \) are no longer equal, as will now be shown.

The maximum surplus problem corresponding to the entropy maximising programme (4)—(7) is

\[
\text{Max} \quad \left\{ \sum_n \frac{1}{\beta^n} \sum_j \hat{T}^n_j \left( \log \frac{\hat{T}^n_j}{O^n_i D_j} - 1 \right) - \sum_n \sum_j T^n_j c^n_j \right\}
\]

subject to

\[
\sum_j \hat{T}^n_j = O^n_i \quad (53)
\]

\[
\sum_n \sum_i \hat{T}^n_i = D_j \quad (54)
\]

where \( \beta^n, n = 1, 2 \) are the values of the multipliers associated with equation (7). The solution to this problem is given by

\[
\hat{T}^n_j = O^n_i D_j e^{-\beta^n \hat{c}^n_i + \gamma_j + c^\gamma_j}
\]

where \( \hat{c} \) and \( \hat{\gamma} \) are computed to satisfy the constraints (53) and (54). From equations (1)—(3) and (55) it can be seen that the models are equivalent when

\[
\beta^n \hat{\alpha}^n_i = x^n_i \quad n = 1, 2 \quad \text{for all } i
\]

and

\[
\beta^1 \hat{\gamma}_j = \gamma_j \quad \text{for all } j
\]

\[
\beta^2 \hat{\gamma}_j = \gamma_j
\]

and this will be true only if \( \beta^1 = \beta^2 \).

Now if (52) is by construction a measure of total generalised surplus the benefit change \( \Delta \hat{B} \) is given by

\[
\Delta \hat{B} = \sum_n \frac{1}{\beta^n} \sum_j \left( \hat{T}^n_j \log \hat{T}^n_j - \overline{T}^n_j \log \overline{T}^n_j \right) + \sum_n (\overline{C}^n - \overline{C}^n)
\]

\[
= \sum_n \sum_i O^n_i (\hat{c}^n_i - \overline{c}^n_i) + \sum_j D_j (\hat{\gamma}_j - \overline{\gamma}_j)
\]

It is now surprising that the demand function (55) is integrable in the Hotelling sense, and it can readily be checked that the results (59) and (60) may be generated by expression (12).

The Lagrange multiplier \( \hat{\gamma}_j \) is already in money units. In the entropy maximising model, different money prices \( \gamma_j/\beta^1 \) and \( \gamma_j/\beta^2 \) must be imposed on car owners and non-car-owners, and it is precisely because of this that an exact result for expression (12) is unattainable. We note in passing that introducing the \( \beta \) values into the objective function as in equation (52) is analogous to the weighting process in which the utilities associated with different groups are added by conversion into money units through division by the marginal utility of income. Here \( \beta \) may be regarded as the marginal utility of expenditure on transport for the particular purpose concerned.

\( \hat{\alpha}_i \) and \( \hat{\gamma}_j \) have through equation (25) the mathematical property of being the increment of benefit per unit increment of \( O_i \) and \( D_j \), respectively, which translates into an economic interpretation of rent. Because of this the dual variables have a significant role to play in land use planning (see Williams [2]).
Evaluation under changing constraint capacities
In the previous discussion it has been assumed that the constraint capacities are invariant along any path of integration. The effect of varying transport costs, or capacities (such as the upper-bounds on parking in a zone), may be determined either directly from the generalised surplus function if it is known, or, by path integration, in the following way. Writing the surplus \( S \) as a function of \( f \) and \( c \), the differential \( dS \) is given by

\[
dS = \sum_{\xi} \frac{\partial S}{\partial c_{\xi}} dc_{\xi} + \sum_{\mu} \frac{\partial S}{\partial f_{\mu}} df_{\mu}
\]

(61)

and, for a finite change in the variables, the benefit becomes

\[
\Delta S = - \sum_{\xi} \int_{c_{\xi}}^{\tilde{c}_{\xi}} \mathcal{D}_{\xi}(c, f) - \sum_{\mu} \int_{f_{\mu}}^{\tilde{f}_{\mu}} \eta_{\mu}(c, f)
\]

(62)

where the relations

\[
\mathcal{D}_{\xi} = - \frac{\partial S}{\partial c_{\xi}} \quad \text{for all } \xi
\]

(63)

and

\[
\eta_{\mu} = - \frac{\partial S}{\partial f_{\mu}} \quad \text{for all } \mu
\]

(64)

hold at the optimal solution of the programme used to generate the demand function. The path integrals in equation (62) are evaluated in the product space \( \{c, f\} \) between the initial and final states, \( (\tilde{c}, \tilde{f}) \) and \( (\check{c}, \check{f}) \) respectively. Uniqueness of \( \Delta S \) is assured if the following integrability conditions are satisfied.

\[
\frac{\partial \mathcal{D}_{\xi}}{\partial c_{\xi'}} = \frac{\partial \mathcal{D}_{\xi'}}{\partial c_{\xi}} \quad \text{for all } \xi, \xi'
\]

(65)

\[
\frac{\partial \mathcal{D}_{\xi}}{\partial f_{\mu}} = \frac{\partial \eta_{\mu}}{\partial c_{\xi}} \quad \text{for all } \xi, \mu
\]

(66)

\[
\frac{\partial \eta_{\mu}}{\partial f_{\mu'}} = \frac{\partial \eta_{\mu'}}{\partial f_{\mu}} \quad \text{for all } \mu, \mu'
\]

(67)

and this in turn is assured if the demand functions and shadow prices are (notionally) determined within the framework from which the results (63) and (64) were derived. The value of \( \eta(\sigma) \) along the path of integration is not normally required, and the second term in equation (62) may usually be “transformed out”, requiring only the values \( \tilde{\eta} \) and \( \check{\eta} \) at the end states to be determined, as for example in equation (75) below.

The modal split dimension
It is now usual to adopt a three-way person classification, involving 0, 1, 2 + car ownership groups, in post-distribution modal split modelling. The results of the previous section may be applied directly in evaluating the competition by the
different car ownership classes \((1, 2+, 3+)\) for parking spaces. An appropriate form of aggregate demand model for modal split may be generated by the programme

\[
\max_{\{T_{nk}^j\}} \sum_{ik} \sum_{H(n)} \lambda^k \frac{\hat{T}_{nk}^j}{\lambda^k} \left( \log \hat{T}_{nk}^j - 1 \right) - \sum_{ik} \sum_{H(n)} \frac{\hat{T}_{nk}^j}{\lambda^k} c_{ij}^k
\]

subject to

\[
\sum_{H(n)} \hat{T}_{nk}^j = T_{nj}^i \quad (69)
\]

\[
\sum_{ik \neq l(k)} \hat{T}_{nk}^j \leq D_{ij}^k \quad (70)
\]

where \(\sum_{ik \neq l(k)}\) denotes summation over all person types which have mode \(k\) available.

Equation (70) represents a constraint on the total number of trips by mode \(k\) (taken here as the private mode) into zone \(j\). Within a policy-testing context, an important issue concerns the effects of varying the level of restraint embodied in the vector \(D^k\) and of changes in the transport system. In analytic terms the problem translates itself into evaluation of the change \(\{c, D^k\} \rightarrow \{\bar{c}, \bar{D}^k\}\).

By invoking the path integral approach, or in this case appealing directly to the objective function (68), the total benefit associated with this change may be shown to be

\[
\Delta B = \sum_{ik} \hat{T}_{nk}^j \log \left( \frac{\sum_{H(n)} e^{-\kappa^k \eta_{ij}^k}}{\sum_{H(n)} e^{-\kappa^k \bar{\eta}_{ij}^k}} \right) + \sum_{j} \frac{\Delta D_{ij}^k}{\Delta \eta_{ij}^k} \left( \bar{D}_{ij}^k \bar{\eta}_{ij}^k - \bar{D}_{ij}^k \bar{\eta}_{ij}^k \right)
\]

where \(\eta_{ij}^k\) is the multiplier corresponding to the constraint (70).

The third term in equation (71) involving the shadow prices \(\bar{\eta}, \bar{\eta}\) is interpreted, as in conventional analyses, as a tax transfer of money from car drivers to the community (assuming of course zero collection cost).

If the fixed matrix \(\left(\hat{T}_{nk}^j\right)\) assumption is now relaxed and the distribution pattern is taken to be dependent on modal costs and other terminal penalties, the following analysis may be used for post-distribution modal split models of the form

\[
T_{nk}^j = T_{nj}^i M_{ij}^{nk} \quad (72)
\]

Williams [12] has shown that the only form of composite cost (8) which is consistent with the modal share form (10) is

\[
e_{ij}^n = -\frac{1}{\lambda^k} \log \left( \sum_{H(n)} e^{-\kappa^k \eta_{ij}^k} \right)
\]

for which

\[
\frac{\partial e_{ij}^n}{\partial \eta_{ij}^k} = M_{ij}^{nk}
\]

It may be shown by a straightforward application of equations (42) and (62) that the total change in generalised surplus \(\Delta B\) is given by

\[
\Delta B = \sum_{ik} \sum_{n} O_{ij}^n \left( \bar{e}_{ij}^n - \bar{e}_{ij}^n \right) + \sum_{j} D_{ij} \left( \bar{\eta}_{ij}^k - \bar{\eta}_{ij}^k \right) + \sum_{ik} \sum_{l} \left( \bar{D}_{ij}^l \bar{\eta}_{ij}^k - \bar{D}_{ij}^l \bar{\eta}_{ij}^k \right)
\]

The \(\alpha\) and \(\gamma\) variables implicitly involve the modal costs through the composite function (73).

In the following section the computation of these dual variables is discussed.
5. THE PROCESS OF CALIBRATION

The use of duality in calibration

The calibration of the model (1) is conventionally understood to involve the calculation of $\beta^n$ such that the predicted mean trip costs ($c^n$) are equal to the corresponding observed quantities. Williams [16] has reviewed and compared several techniques for this process. The determination of the multipliers $\alpha^n$ and $\gamma$, or rather their exponential transforms $A^n$ and $B$, is achieved through the Furness method, which, for a given value of $\beta$, involves the convergent iterative process

\[
B_j(m+1) = \{ \Sigma B_j(m)D_j e^{-\beta c^m_{ij}} \}^{-1} \quad m = 1, 2, \ldots
\]

\[
A_i^n(m+1) = \{ \Sigma A_i^n(m)O_i^n e^{-\beta c^m_{ij}} \}^{-1} \quad m = 1, 2, \ldots
\]

Now consider the model

\[
\tilde{T}_ij^n = O_i^n D_j e^{-\beta c^n_{ij} + \tilde{\gamma} + e^n_{ij}}
\]

which will be written in the form

\[
\tilde{T}_ij^n = \tilde{A}_i^n O_i^n \tilde{B}_j^n D_j e^{-\beta c^n_{ij}}
\]

with

\[
\tilde{A}_i^n = e^{-\beta \tilde{\gamma}}
\]

and

\[
\tilde{B}_j^n = e^{-\beta \tilde{\gamma}}
\]

The following relations must hold:

\[
\tilde{A}_i^n = \{ \Sigma \tilde{B}_j^n D_j e^{-\beta c^n_{ij}} \}^{-1} \quad \text{for all } i \text{ and } n
\]

\[
\Sigma \Sigma \tilde{A}_i^n O_i^n \tilde{B}_j^n e^{-\beta c^n_{ij}} = 1 \quad \text{for all } j
\]

if the trip end relations (5) and (6) are to be satisfied. It can be seen that the balancing factor $\tilde{B}_j^n$, which owes its $n$-dependence to the parameter $\beta^n$ in (79), is no longer separable in equation (81), which is in contrast to equation (3). This is a price to be paid for the greater economic realism of the model (77). A way out of this difficulty is to appeal to the notion of duality, which is closely related to the properties of Langrangians and has been discussed elsewhere by Williams [2] in the context of economic evaluation.

Corresponding to the programmes (4) to (7), and (52) to (54), which will be denoted $P$ and $P^*$, are respectively the duals $Z$ and $Z^*$ written

\[
Z: \min_{\alpha^n, \gamma, \beta} \zeta(\alpha^n, \gamma, \beta) = \Sigma \Sigma O_i^n D_j e^{-\alpha_i^n - \gamma_j + \beta c^n_{ij}} + \Sigma \Sigma O_i^n z_i^n + \Sigma D_j \gamma_j + \Sigma C^n \beta^n
\]

\[
Z^*: \min_{\alpha^n, \gamma} \zeta^*(\alpha^n, \gamma) = \Sigma \Sigma \frac{1}{n} O_i^n D_j e^{-\alpha_i^n - \gamma_j + \beta c^n_{ij}} + \Sigma \Sigma O_i^n \tilde{z}_i^n + \Sigma D_j \tilde{\gamma}_j.
\]

The programmes $Z$ and $Z^*$ have a particularly simple structure; they are convex and subject to no constraints.

The values of $\alpha^n$, $\gamma$ and $\beta$ which optimise $\zeta$ correspond to the matrix $\{ T^n_{ij} \}$ which maximises $P$, through the relations (1) to (3). Similarly the values $\hat{\alpha}$, $\hat{\gamma}$ which optimise
\( Z^* \) correspond to the optimal solution \( \{ \hat{T}_{ij}^n \} \) of \( P^* \) through (77) to (79). The first order conditions for minimisation of \( Z \) and \( Z^* \) are:

\[
\frac{\partial Z}{\partial x_i} = O_i^n - \sum_j O_{ij}^n D_j e^{-a_i^n - \gamma_j + \nu x_i^n} = 0 \quad \text{for all } i \text{ and } n \quad (84)
\]

\[
Z:
\frac{\partial Z}{\partial \gamma_j} = D_j - \sum_i O_{ij}^n D_j e^{-a_i^n - \gamma_j + \nu x_i^n} = 0 \quad \text{for all } j \quad (85)
\]

\[
\frac{\partial Z}{\partial \beta^n} = C^n - \sum_{ij} e_i^n O_{ij}^n D_j e^{-a_i^n - \gamma_j + \nu x_i^n} = 0 \quad \text{for all } n \quad (86)
\]

and

\[
\frac{\partial Z^*}{\partial x_i^n} = O_i^n - \sum_j O_{ij}^n D_j e^{-a_i^n (\hat{x}_i^n + \hat{\gamma}_j + \nu x_i^n)} = 0 \quad \text{for all } i \text{ and } n \quad (87)
\]

\[
Z^*:
\frac{\partial Z^*}{\partial \gamma_j} = D_j - \sum_i O_{ij}^n D_j e^{-a_i^n (\hat{x}_i^n + \hat{\gamma}_j + \nu x_i^n)} = 0 \quad \text{for all } j \quad (88)
\]

Calibration of the entropy maximising model may thus be seen as solving the dual programme (82). The conventional method of calibration essentially involves the decomposition into the iterative solution of two processes:

(i) \( \min_{\alpha, \gamma} Z(\alpha^n, \gamma; \beta) \) for fixed \( \beta \) (89)

(ii) \( \min_{\beta} Z(\beta; \alpha^n, \gamma) \) for fixed \( \alpha^n, \gamma \) (90)

### Solving the dual programmes \( Z \) and \( Z^* \)

Every medium-sized computer installation will have at least one algorithm available for the unconstrained multivariate optimisation problem. Several approaches are available based on search and gradient techniques. The general second order Newton-Raphson approach relies on the provision of a gradient vector \( g \) given for \( Z^* \) by

\[
g = \left( \frac{\partial Z^*}{\partial x_i^n}, \ldots, \frac{\partial Z^*}{\partial \gamma_j} \right) \quad (91)
\]

and a Hessian matrix of second order partial derivatives

\[
H = \begin{bmatrix}
\frac{\partial^2 Z^*}{\partial x_i^n \partial x_i^n} & \frac{\partial^2 Z^*}{\partial x_i^n \partial \gamma_j} \\
\frac{\partial^2 Z^*}{\partial \gamma_j \partial x_i^n} & \frac{\partial^2 Z^*}{\partial \gamma_j \partial \gamma_j}
\end{bmatrix}
\quad (92)
\]

Collecting the dual variables \( \hat{x}_i^n, \hat{\gamma}_j \), into a vector \( x \) of dimension \( 1 \times N(n + 1) \), an approximation to \( x_{min} \) the minimum of \( Z^* \) from point \( x \) may be obtained by moving to \( x + \Delta x \) where

\[
\Delta x = -H^{-1}g \quad (93)
\]

Direct use of (93) is limited, however, because the Hessian matrix must be computed and inverted at each step of the iterative procedure.

One possible approach is to use the diagonal elements of \( H \) is successive moves parallel to the co-ordinate axes so that
\[ \Delta x_{\pi} = -\frac{g_{\pi}}{H_{\pi\pi}} \]  

or

\[ \Delta \hat{x}_i^n = \frac{O_i^n - O_i^\pi}{\beta^n \hat{O}_i^n} \]  

\[ \Delta \hat{r}_{ij} = \frac{D_{ij} - \hat{D}_{ij}}{\Sigma \Sigma \beta^n \hat{T}_{ij}^n} \]  

where

\[ \hat{D}_{ij} = \Sigma \Sigma \hat{T}_{ij}^n \]  

\[ \hat{O}_i^n = \sum_j \hat{T}_{ij}^n \]  

with

\[ \hat{T}_{ij}^n = O_i^n \ D_{ij} \ e^{-p(\hat{r}_{ij} + \gamma_j + \epsilon_{ij})} \]  

the dot notation denoting current value.

Another approach to the problem is to attempt to simulate the Newton-Raphson iteration (93) by making use of the gradient vector, in conjunction with linear search. The approach adopted below, taken from Fletcher and Reeves [17], makes use of conjugate gradient vectors with respect to the Hessian matrix and is very modest in its storage requirements. In place of an approximation to the inverse Hessian matrix required by some second order gradient methods, it requires only two vectors to be stored. The price is a sacrifice on efficiency.

### An example

The group surplus model (55) and the corresponding entropy maximising model (1) have been implemented for an 81-zone system of the West Yorkshire region, using the Fletcher-Reeves (FR) algorithm. The direct minimisation of \( \mathcal{Z}(\alpha^n, \gamma, \beta) \) allows alternative approach for the calibration of the latter model to the method usually adopted, which corresponds to equations (89) and (90). The optimal values of the dual variables are maximum likelihood estimators.

To illustrate the process of calibration and the general problem of model implementation, the minimisation of \( \mathcal{Z} \) has been performed (for 245 variables), and the resultant \( \beta^n \) values injected into the programme (83), which was solved by the same method. Because the values of \( \beta_1 \) and \( \beta_2 \), which were found to be 5.2543 and 6.2090 respectively in the units adopted, are of similar magnitude, it is not surprising that the predicted trip matrices \( T_{ij}^n \) and \( \hat{T}_{ij}^n \) are close. One set of outputs, which is directly comparable for the two models, is the number of destinations into zone \( j \) by person type \( n, \ldots D_j^n \ldots \) which is equal to \( \Sigma_i T_{ij}^n \). The values associated with the centre of Leeds are given in Table 1.

Solution of the programme (82) by the FR algorithm, which successively updates each variable, involved 245 linear searches per iteration. Convergence of the process was attained after three iterations, and required an overall time of 78 seconds and 43 \( K \) words of core on the CDC 7600 computer at Manchester University.

Several comments should be made at this stage regarding the programming
procedure for calibrating and forecasting with entropy maximising models. The speed and efficiency of any calibration procedure depends to a large extent on the starting values for the variables $\alpha^k$, $\gamma$, and $\beta$. Here we have deliberately taken a very poor starting value (all dual variables were set equal to unity) to test the convergence process, and have rejected heuristic rules for selecting initial values of $\beta$. The results in this sense were highly satisfactory. An important advantage of the general optimisation approach is its ease of implementation and its facility for handling inequality constraints (for example, with parking restraint problems). The alternative of writing and testing a balancing factor programme and coupling it to a Newton-Raphson sub-routine is a time-consuming process. It should be remarked however that for large problems, involving say 1,000 zones, any method which updates all dual variables simultaneously is likely to be inefficient because of the extravagantly repetitive calculation of the matrix of exponential functions $e^{-\beta^k}$. In this case there would be sound reasons for treating the dual variables asymmetrically as in the prescription (89) to (90). Because the $\beta$ values are inputs to the programme (83) its solution is very efficient if the exponentiation of the cost matrix is performed a single time. In order to compare numerically the user benefit measures appropriate to the two models, initial and final cost states $\bar{c}$ and $\bar{c}$ which contain the matrix elements $c_{ij}$ must be defined. These are taken to correspond to two alternative network sets $N_1$ and $N_2$ for the West Yorkshire region, and it is the difference between consumer surplus measures computed for $N_1$ and $N_2$ that we shall be concerned with. $N_1$ and $N_2$ both consist of public and private (road) networks, each of which contains approximately 1500 links. The generalised costs $c_{ij}^g$, which are determined by using a shortest path algorithm, are a combination of time and out-of-pocket costs.

The value of the dual objective function $\zeta^*$ may be used directly to determine user benefit for the maximum surplus model. From equations (60) and (83) it can be seen that
\[ \Delta \hat{B} = \vec{\zeta}^*(\bar{\alpha}, \bar{\gamma}) - \vec{\zeta}^*(\bar{\alpha}, \bar{\gamma}) \]  
(100)

where the bar and double-bar refer to the two network sets \( N_1 \) and \( N_2 \) respectively. For the entropy maximising model the approximation (15) is used to evaluate user benefits, and the results are compared with those from equation (100) in Table 1.

**The Parameters \( \beta^n \)**

The parameters in (83), appropriate to the doubly constrained entropy maximising model(1), were adapted for demonstration purposes, but this is not entirely satisfactory. What are the values to be used? One obvious possibility is to generate values of \( \beta^1 \) and \( \beta^2 \) which equate the predicted and observed mean trip cost by nesting the programme (83) within a Newton-Raphson procedure. This could however be an expensive device for a finely zoned region, unless the programme (82) was first solved and its optimal parameters \( \beta \) used as an initial solution.

Another possible approach suggests itself if we assess the significance of the parameters within a utility-maximising context and have due regard to forecasting assumptions. The inverses of \( \beta^1 \) and \( \beta^2 \) are regarded as fundamental behavioural parameters which scale utility to money units, and are values which are traditionally regarded as invariant with respect to changes in the network (and indeed in time). Now the imposition of shadow prices \( \gamma \) may be seen as a change in the cost matrix and therefore in the network. It would therefore seem appropriate to have the same parameters for the model of constrained choice (with shadow price \( \gamma \) included) as for the model which expresses locational preferences in the situation in which there existed an excess supply of jobs (or houses). As the entropy maximising model disaggregated for multi-person types is equivalent to the group surplus model in the absence of destination constraints, the parameters \( \beta^1 \) and \( \beta^2 \), which correspond to the former, would seem to be appropriate for use in (83). Calculation of these parameters by conventional methods poses no difficulties.

6. **CONCLUSION**

The evaluation methodology proposed by Neuberger [1] has been generalised and used for restructuring the distribution and modal split models in current use. The formal similarity between the models proposed and entropy maximising models is a direct result of a particular choice of group utility function. The method used here allows a spectrum of models to be generated by varying this function. For entropy models this flexibility may arise, not through the objective function, but through the form of constraints associated with the moments of the trip cost distribution.

The change in emphasis, which is afforded by an approach of maximising generalised surplus, away from marginal cost differences implicit in the “rule-of-a-half” to changes in commonly interpreted accessibility indices, offers both new insight into the nature of user-benefit measures and computational efficiencies in their calculation. The revised models may be implemented by exploiting a dual programming framework, which offers a unifying context for the processes of calibration, demand analysis and determination of user benefit. A procedure, based on unconstrained optimisation, has been used for the calibration of both group surplus and entropy models.
REFERENCES


