FREQUENCY DISTRIBUTIONS OF PEDESTRIANS IN A RECTANGULAR GRID

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This paper deals with distributions of pedestrians over rectangular street networks, and the implications of assumptions about how pedestrians select a particular path when walking from an origin, say a house or a subway station, to a destination, e.g., a store or city hall. If a person plans a walk with a purpose in view, points in space take the role of origin and destination. Some destinations are assumed to be independent of the whereabouts of the persons seeking them (e.g., schools and offices); others (e.g., cafes and open spaces) are not. For the second class, it is desirable to maximise their use by locating them on street sections where many people pass by.

Urban growth models predict or forecast the location of land uses by specifying how many square feet will, or should, be developed, over a certain period of time, within a tract or a square mile. Yet in general no statement is made about where within that area the predicted land use will, or should, be located; this is the sort of question I am concerned with here. One aspect of the question is how far the distribution of land uses and the distribution of pedestrians may influence each other.¹

Models such as the ones suggested in this paper may help to answer questions such as these:²

(a) An office building is planned. Given the location of bus or subway stops, how will employees who reach the building by public transport distribute themselves over the streets between the stops and the building?

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¹By focusing on path selection this paper attempts to supplement existing literature which deals with various aspects of pedestrian movement, such as: results of counting surveys [16]; origin-destination surveys and distribution over street networks [6,8,13]; physiological factors and design criteria for pedestrian facilities [20,21]; characteristics of pedestrian flows, such as velocity and density [7,12,18,19]; variations of pedestrian distributions in different street types and over different time periods [9,10]; the image as the basis for movement through an environment [15]; relationships between pedestrian distributions and land values [17]; observation and tracking of pedestrians [1]; path selection and distribution over networks [5]; pedestrian safety [3,11,14,22]; maze studies with infrahumans [2,4].

²Consequences of the binomial model are explored in [5].
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(b) A town is being planned. How will pedestrians distribute themselves between apartment blocks and bus or subway stops? How will children and teenagers distribute themselves between homes and schools?

(c) Assuming several alternatives for the location of the lecture halls, dormitories and libraries of a proposed university, what will be the different distributions of pedestrians on the campus?

In these cases it will not always be necessary to know absolute numbers. i.e. how many persons walk from a particular apartment block to a particular bus or subway station, but only the percentages (probabilities) for the different links and intersections.

I am concerned with the aggregate behaviour of pedestrians averaged over time periods of one or more days. I do not discuss such issues as perception and cognition, variation in speed, events during the trip (e.g. contacts with other persons) and side activities during the trip (e.g. window shopping). I do not distinguish between those who hurry to work and those who stroll. I do not try to answer questions about learning and adaptation (e.g. How do pedestrians’ paths change over time with changes in location of origins and/or destinations? How does the distribution of pedestrians evolve in new communities?).

Since this paper emphasises path selection and distribution over networks, one may ask whether cars should not be included in the investigation. For three reasons I feel that we should restrict our exploration to pedestrians: (1) Even in a grid with no traffic lights, no one-way streets, and no variation in the number of lanes, the assumption of equiprobable choice for cars seems to be unrealistic, since car drivers tend to minimise direction changes. (2) The assumption of a street environment that does not restrict pedestrian path selection is considerably closer to reality than the corresponding assumption for cars, where movement is constrained by traffic lights, one-way streets, and prohibited left (or right) turns and U-turns. One might look into car distribution by starting with the assumptions which are used in this paper for pedestrians, and then introduce constraints, such as traffic lights, to explore the implications; this, however, would imply an analysis of its own. (3) Street networks available to cars seem to differ from those available to pedestrians in the sense that they are more hierarchical and that "gate" situations are more frequent. (By a "gate" I mean a link or intersection in the network which one has to pass almost inevitably if one wants to get to some other link or intersection, e.g. a bridge across a river).

1. DEFINITIONS

An origin O is a point at which a pedestrian starts a walk to a destination D. The same point may be an origin and/or a destination for the same person at different times. All O’s and D’s are assumed to be located at street intersections. Examples are: a single family house, a subway station, a parking facility, a friend’s apartment, a church, a store. If we are concerned with an area of, say, one square mile, points on the boundary of this area by which pedestrians enter the area are O’s, and boundary points by which pedestrians leave the area are D’s. (For a person entering the area by subway the O is the station where he leaves the subway and starts walking; for a car driver who drives into the area, the O is the place where he leaves his car).

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A decision point is a street intersection where a pedestrian decides whether to continue in the same direction or to change it. (An O is defined not to be a decision point). On a rectangular grid of city streets, a link is a street section one block in length, from one intersection to the next. A step is a path one link long. Grids and boundaries are illustrated in figure 1. A grid consists exactly of all paths from O to D which satisfy two conditions: the paths consist of links of the grid; and they are chosen from those that minimise time-distance, i.e. the paths are equally long. Behaviour is restricted to the selection of paths between O and D.

2. ASSUMPTIONS

Unless specified otherwise (see sections 3.4, 3.5 and Appendix (4)), I will restrict myself, for convenience of exposition, to trips from O to D above and/or to the right of O (north and/or east). The grid is two-dimensional (flat ground), and time-distance is proportional to length of path. There is always at least one path minimising time-distance between O and D. All trips are completely pedestrian. Each trip comprises
one $O$ and one $D$. Different persons may have the same $O$'s and/or $D$'s. Every person minimises time-distance (no detours); i.e., at any intersection leading from $O$ to $D$ the pedestrian goes either north or east. When more than one path leads from an intersection to $D$, $q$ is the probability that the vertical link is chosen next, and $p$ is the probability that the horizontal link is chosen ($q + p = 1$). I define equiprobable choice as $q = p = \frac{1}{2}$. For points on the eastern boundary, $q = 1$, $p = 0$; for the northern boundary, $q = 0$, $p = 1$.

3. RANDOM PATHS VERSUS RANDOM WALK

Different rules might be chosen to describe the way pedestrians choose paths when walking from $O$ to $D$. Two alternatives are considered here:

(1) All paths from $O$ to $D$ are equally likely: this behaviour we call “random paths”.

(2) Choice at street intersections is equiprobable (except on the northern and the eastern boundary); we call this rule “random walk”.

![Figure 2: Number of Paths Leading to Intersections in a Square Grid](image-url)
3.1. **Number of Paths leading to an Intersection**

Let us first ask how many different paths lead to any particular intersection in a rectangular grid. In figure 2 this number is shown against each street crossing. For example, let us take the intersection two links east and one link north from $O$. This can be reached by three different paths: two links east and one link north; one link east, one link north, and one link east; one link north and two links east. Notice that there is only one path leading from $O$ to any intersection on either of the boundaries meeting at $O$, while several paths lead to the intersections on the boundaries meeting at $D$.

Is the number of paths available to us the same, no matter whether we start our trip to the east or to the north? It is so in the square grid, but not in an oblong grid. A more general way of saying this is that there is a choice of more routes if you start along the long boundary than if you start along the short one. In particular, figure 3 shows that, if we start our trip by walking north, the number of available paths equals the number of paths available in a grid determined by $D$ and by a “pseudo” origin $O_N$ which is one link north from $O$. If we start out to the east, the number of available paths is the number of paths available in the grid determined by $D$ and the pseudo-origin $O_E$, which is one link east from $O$. For example, in case c of figure 3 there are four paths to the north and six to the east, as is shown in the pseudogrids c(iii) and c(ii). See also figure 5.

3.2. **Random Paths: Choice Probabilities at Intersections**

In a $2 \times 2$ grid there are six different paths from $O$ to $D$ (as shown in Figure 4(a)); hence, under the assumption of equally likely trips, each path has $\frac{1}{6}$ as the probability of occurrence. Figure 4(c) gives the corresponding “transitional probability” for each link, and the probability that a path starting at $O$ reaches a particular intersection. These probabilities are defined as follows: Take the intersection one link due east from $O$. From figure 4(a) we see that out of the six paths starting at $O$ three lead to this intersection. The other three lead to the intersection one link due north of $O$. Hence $\frac{1}{2}$ of the trips leaving $O$ reach each intersection, or, in other words, if a path starts at $O$ the probability that it will reach the intersection one link east from $O$ is $\frac{1}{2}$; and this is the “transitional probability” between these two intersections. Now take the intersection two links due east from $O$. Figures 4(a) and (4b) show that of the six different trips one leads to this intersection, i.e., the probability that a trip starting at $O$ reaches this intersection is $\frac{1}{6}$. Three of the six trips will reach the intersection one link east, but of these three only one will continue eastward; this means that $\frac{1}{6}$ of the trips reaching the intersection one link east from $O$ will continue eastward, while $\frac{1}{3}$ will change direction and continue northward; i.e., $\frac{1}{6}$ and $\frac{1}{3}$ are the respective transitional probabilities between the intersection one link east from $O$ and the two next intersections. For an oblong grid ($3 \times 2$) Figure 5 gives the probability that a path starting at $O$ reaches a particular intersection, and the transitional probabilities between intersections.

Transitional probabilities measure the choice probabilities at the preceding intersection. As can be seen from Figures 4 and 5, these may or may not be $\frac{1}{6}$. Equi-probable trips do not imply equiprobable choice at intersections.

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3I am grateful to Mr. D. L. Munby, who pointed this out to me.
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FIGURE 3. NUMBER OF PATHS LEADING TO INTERSECTIONS, STARTING ALONG THE LONG OR THE SHORT BOUNDARY
\[ P = \frac{1}{2} \times \frac{1}{3} \times \frac{1}{1} = \frac{1}{6} \]

\[ P = \frac{1}{2} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{6} \]

\[ P = \frac{1}{2} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{6} \]

\[ P = \frac{1}{2} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{6} \]

4(a) The different paths and their probabilities

4(b) Number of paths leading to an intersection

4(c) Probabilities at links and at intersections

4(d) Persons on links and at intersections (60 walking from O to D)

FIGURE 4. EQUIPROBABLE PATHS IN A SQUARE GRID (RANDOM PATHS)
3.3. **Random Walk: Likelihood of Different Paths**

Figure 6(c) shows the probability that a particular trip reaches a certain intersection, and the transitional probabilities, under the assumption of equiprobable choice at intersections. (The likelihood of a trip is measured by multiplying the transitional probabilities of the links through which the trip goes.) Though we still have the same number of different trips through the grid, namely six, we can see that they are no longer equally likely. Notice that trips using only boundary links are now more probable than trips that go also through links in the interior of the grid.

This last relation does not hold for an oblong grid (Figure 7). Here some trips going through the interior are as likely as one of the two trips using only boundary links, the two paths along boundaries being no longer equally probable. In this $3 \times 2$ grid, 25 per cent of the persons walking from $O$ to $D$ select the path along the western and northern boundaries; 12.5 per cent take each of the three routes, including the one along the southern and eastern boundaries; 6.25 per cent take each of the remaining six routes.

We conclude that if transitional probabilities are $\frac{1}{2}$ throughout the grid (except on the northern and eastern boundaries) the different trips are not equally likely; equiprobable choice at street intersections does not imply equiprobable trips.

3.4. **Multiple Origins and Destinations**

Both random path and random walk models can be applied to multiple $O$–$D$ situations. Assume we have $O$'s and $D$'s as given in Figure 8(b), pedestrian flows between these as indicated by the arrows, and given magnitudes for these flows (in this case, 500 persons per flow). We can then disintegrate this network into $O$–$D$ pairs, each determining a grid. Using either the rules of the random path model or those of the random walk model, we distribute the number of persons walking from a particular
6(a) The different paths and their probabilities

6(b) Number of paths leading to an intersection

6(c) Probabilities at links and at intersections

6(d) Persons on links and at intersections (60 walking from O to D)

FIGURE 6. EQUIPROBABLE CHOICE AT INTERSECTIONS IN A SQUARE GRID (RANDOM WALK)
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\[ \text{Figure 7. Equi-probable choice at intersections in an oblong grid (random walk)} \]

Probabilities on links and at intersections

\( O \) to a particular \( D \) (which equals the magnitude of the flow) over the grid as determined by this \( O-D \) pair. Figure 8(a) is based on the random walk model of equi-probable choice at street intersections. Frequencies were computed by a simulation program.\(^4\)

3.5. Implications and Examples

In a planning situation one may be interested to know which intersection and/or links will be the least, and which will be the most, congested. Going back to Figure 7 – random walk in an oblong grid – we can see that the most heavily used intersection is one link due west of \( D \), while the least heavily used is \( K \), two links due south of \( D \). If 1024 persons start at \( O \), 704 will walk through the intersection west from \( D \) and through the link between this intersection and \( D \), while 128 will walk through \( K \) and through the boundary links immediately west and north of it. The central link (one link from the eastern and western boundaries, and one link from the northern and southern boundaries) will be traversed by 256 persons (\( = 1024 \times \frac{1}{4} \times \frac{1}{4} \)).

Let us compare this with the random path model applied to the same grid (Figure 5). The intersection which was the most congested in the random walk case is now used by 614, or \( \frac{3}{8} \) of all persons, compared with 704, or \( \frac{1}{4} \), in the random walk case. There are also three other intersections with the same frequency. The least heavily

\(^4\)The simulation was run on an IBM-360. The program used provides for a decision at each street intersection by comparing the probability \( p \) of taking one of the two links with a random number generated by a subroutine (I am indebted to Wren McMains, who suggested this procedure). The program can compute aggregate distributions of pedestrians for several different overlapping grids. It is possible to assign different values to \( p \) along the diagonals determined by an equal number of steps from the origin.
FIGURE 8(a) NUMBERS ON EACH LINK AND AT INTERSECTIONS, WITH EQUIPROBABLE CHOICE AT INTERSECTIONS: MULTIPLE GRID SITUATION
used intersection is now used by 130, which amounts to 102 persons, compared with 128 under the assumption of random walk. How about the central link? Here we have a frequency of 410 (= 1024 \times 0.4 \times 0.2), which is approximately 60 per cent more than in the random walk case. The random path model results in lower frequencies than the random walk model on the northern boundary and at K, but considerably higher ones in the interior of the grid. The square grids of figures 4 and 6 show a somewhat similar pattern, though less pronounced and different in detail.

Figures 9 and 10 (cases A and B) represent a multiple O-D situation consisting of three overlapping grids (one square and two oblong), and show that, with the random walk model, frequency distributions over the network change with the direction of flows. This is true for identical oblong grids with identical flow magnitudes, as can be seen by reference to Figure 7. In Figure 9 (case A) the flows are from points (3,1), (2,2), and (1,3) to point (5,5). Links near point (5,5) stand out for their high frequency. Variation among the other links is small when compared with Figure 10 (case B), where all three flows are in the opposite direction. In case B, as compared with case A, the frequencies increase in the central diagonal between (2,2) and (5,5), and fall off in the corner areas near (5,1) and (1,5). Case C (Figure 11) gives frequencies with equal flows in both directions. When a situation such as case A is changed so that there are “back” flows of the same magnitude, the frequencies at intersections and in links are not simply doubled. In some links frequencies increase by approximately 30 per cent, in others by almost 200 per cent.

Let us now consider case D (Figure 12), where there is a change of O-D facilities such that 500 persons walking from point (5,5) to (1,3) as in case C now all go first to (1,5) and only then continue to (1,3). These 500 pedestrians are no longer distributed over the whole of their oblong grid, but only over its eastern and southern boundaries. In case C the maximum frequencies are distributed around the central diagonal, with relatively small differences between links as we proceed along the diagonal (except for the four boundary links adjacent to (5,5)). In case D, on the other hand, we have two separate areas with high frequencies: one round the boundary from (5,4) to (5,5) and on to (1,5) and then to (1,3); the other around the link between (3,2) and (3,3).

4. VALIDATION

While I have not described how individual pedestrians select a path through an environment, I have suggested how the outcome of many individual decisions can be described. The preceding notes may be helpful in attempting to relate path selection by pedestrians to the theory of location as applied to stores, public facilities and open spaces. If this approach stimulates others to test other assumptions, this would be a matter of more importance than whether one or other of the two models I have presented is valid.

Data to test these models can be collected in the following ways:

1. **Experiments in an artificial environment**: An environment is built, consisting of a street grid and walls which are 7 feet high. Subjects are asked to walk from specified

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3 Frequencies were computed by simulation (see footnote 4).
FIGURE 9 (a) NUMBERS ON EACH LINK AND AT INTERSECTIONS, WITH EQUIPROBABLE CHOICE AT INTERSECTIONS: MULTIPLE GRID SITUATION, Case A
FIGURE 10(a) NUMBERS ON EACH LINK AND AT INTERSECTIONS, WITH EQUIPROBABLE CHOICE AT INTERSECTIONS: MULTIPLE GRID SITUATION. Case B
FIGURE 11(b) On each route there are 500 walking in each direction.

FIGURE 11(c) NUMBERS ON EACH LINK AND AT INTERSECTIONS, WITH EQUIPROBABLE CHOICE AT INTERSECTIONS: MULTIPLE GRID SITUATION. Case C
FIGURE 12 (b) On routes with double arrows there are 500 walking in each direction.

FIGURE 12 (a) NUMBERS ON EACH LINK AND AT INTERSECTIONS, WITH EQUIPROBABLE CHOICE AT INTERSECTIONS: MULTIPLE GRID SITUATION. Case D
O's to specified D's. The environment is kept uniform or made more or less complex, depending on the hypotheses to be tested.

(2) Experiments in real environments: The experimenter assigns tasks to the subjects: e.g., starting at a subway or bus station, he may say: "walk to D1, then to D2, then to D3, then come back to this station". The chosen path is then recorded.

(3) Handing out numbers: A study area is specified. A person entering that area is given a button with a number, which he is asked to wear till he leaves the area. Observers stationed at selected locations (facilities, street intersections, points between intersections) record the numbers of the persons passing or entering. From these records their paths are reconstructed.6

(4) Entry-exit interviews: Persons entering the study area are asked what their destinations in the area are. When leaving the area they are asked to record on a map the destinations they went to and the paths they took. In a variation of these interviews, persons entering the area would be asked to report their destinations in the area and the streets they intended to use to get there.

(5) Questionnaires: Persons leaving the area are given questionnaires including a map. They are asked to fill them out at home and mail them to the investigator. Questions deal with the destinations visited and the paths taken; each person is also asked to report which destinations he had planned to visit before entering the area.7

(6) Air photography: A camera is mounted on a balloon stationed above an area. The camera takes colour photographs at constant intervals of time, say every 30 seconds. Paths of identifiable pedestrians are then reconstructed from these photographs.

SELECTIVE BIBLIOGRAPHY


7This method is similar to the visitors’ records method described in [8.]

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APPENDIX

Mathematical Generalisations

Generalisations of some of the observations and computations presented above are given in the following appendices.

1. Notation

I use the following notation (see Figure 13):

\( (i,j) \) = intersection of two streets, \( i \) units east from \( O \), \( j \) units north from \( O \).

\( P_{i,j} \) = probability that a path starting at \( O \) reaches the point \( (i,j) \) by paths restricted to the grid; subscripts \( i,j \) are omitted whenever this is possible without loss of clarity.

\( P_{i,j}^{i+1,j} \) = probability that a path starting at \( O \) and going to \( D \) leads through links \( (i,j) \) \( i + 1,j \) or \( i,j + 1 \) respectively.

\( P_{i,j}^{i,j+1} \) = probability of step to east from \( (i,j) \).

\( P_{i,j}^{i+1,j} \) = probability of step to north from \( (i,j) \).

\( P_{i,j+2} \)
\( P_{i,j+1} \)
\( P_{i+1,j} \)
\( P_{i+1,j+1} \)
\( P_{i+1,j+1} \)
\( P_{i,j+1} \)
\( P_{i+1,j} \)
\( P_{i+1,j} \)

FIGURE 13
2. Number of paths leading to an intersection

Suppose we have a grid which is determined by $O$ and two lines going through $O$, these lines going east and north. Starting from $O$, we connect all points which can be reached after $n$ steps ($n = 1, 2, \ldots$). Then the number of intersections connected by the $n$th line is $n + 1$, the number of terms in the binomial expansion $(q + p)^n$. The number of different paths leading to an intersection which is on that line, and which is $i$ links east of $O$ and $j$ links north, is the number of permutations of $i + j$ links, $i$ being horizontal and $j$ vertical. For the $n$th line these numbers of permutations equal the binomial coefficients of the terms in the binomial expansion $(q + p)^n$ (Figure 14). These coefficients form Pascal’s triangle:

\[
\begin{array}{ccccccccc}
& & & & & 1 & & & \\
& & & & 1 & & 1 & & \\
& & & 1 & & 2 & & 1 & \\
& & 1 & & 3 & & 3 & & 1 \\
& 1 & & 4 & & 6 & & 4 & \\
& & & & & & & & \\
\end{array}
\]

etc.

For points not on the axes, we get these coefficients by adding those associated with the point one unit south and the point one unit west. This procedure also yields the number of paths going to an intersection of a grid bounded on the north and east (Figure 15).
3. **Probability of getting to an intersection**

The underlying assumption of this appendix is that of equiprobable choice (random walk) as defined above. For the grid unbounded on the north and east the probability of getting to intersection \((i,j)\) is the probability that \(i\) out of \(n\) total steps \((n = i + j)\) are east, and \(j (= n - i)\) are north, which gives

\[
P_{i,j} = \binom{n}{i} p^i q^{n-i} \quad \text{(all } q_{i,j} \text{ equal; } q_{i,j} + p_{i,j} = 1) \quad \text{(see Figure 15)}
\]

and

\[
P_n = (q + p)^n = \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} = 1, \quad n \geq 1
\]
In other words, beginning at $O$, we compute the probability of getting to a point of
the grid by multiplying the probabilities of getting to the point one unit south, and to
the point one unit west, by the respective transitional probabilities:

$$P_{i+1,j+1} = P_{i,j+1} \times p_{i,j+1} + P_{i+1,j} \times q_{i+1,j} \quad \text{where}$$

$$p_{i,j+1} \geq 0, \quad q_{i+1,j} \geq 0, \quad p_{i,j} + q_{i,j} = 1 \quad \text{for all } i,j.$$

By bounding the grid in all four directions and taking $i = j$ we get a square grid
with corners $O$, $D$, $K$, and $L$ (Figure 16). Once a person has arrived at a point on
either boundary opposite $O$, he has no choice but to walk along this boundary; the
transitional probabilities of the northern boundary $LD$ are $p = 1$, while those of the
eastern boundary $KD$ are $q = 1$.

For probabilities $P_n$ of getting to a point of the grid between $O$ and $LK - LK$
included – after $n$ steps, the relationship stated for the unbounded grid still holds. As
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for probabilities \( P_n \) between \( LK \) and \( D \) \( (i < n < i_K + j_L - 1) \), the binomial "collapses". The probabilities \( P_{\text{inside}} \) between boundaries are the same as in the grid that is unbounded on the north and east. \( P_n \)'s of getting to a point on the northern or eastern boundary are given by the sum of the probability of getting to the same point in the unbounded grid plus the probabilities of getting to the points which are outside the grid if we bound it:

\[
P_{i,j} = P(\text{getting to a point on northern boundary}) = P_{i,j}^n
\]

\[
P_{i,j} = P(\text{getting to same point of unbounded grid}) + \sum P(\text{getting to points north of grid boundary})
\]

Similarly for the eastern boundary. Hence we have

\[
P_n = P(\text{getting to a point on northern boundary}) + \sum P_{\text{inside}}
\]

\[
P(\text{getting to a point on eastern boundary}) = i.
\]

The probability of getting to \( D \) after \( n = i + j \) steps is \( i \) (the binomial collapses completely into one point.). The described way of computing the probabilities of getting to intersections holds for any rectangular grid, \( i = j \) or \( i \neq j \). For \( i \neq j \), notations have to be adjusted. In the case of the degenerate case \((i = j = 0)\) we have \( P(\text{get to any point of the grid}) = 1 \).

The probabilities of getting to the boundary points may be computed in a different way. For the northern boundary \( LD \) we have

\[
P_{i,j} = P(\text{getting to } (i,j)) = q^j \sum_{m=0}^{m=i} (b_m - b_{m-1}) p^m
\]

\((p \text{ and } q \text{ constant for all links of the grid except boundaries through } D, \text{ where probabilities are unity. } b_m \text{ is the binomial coefficient of a boundary point } (m,j). \text{ For summation } I \text{ have changed notation of coordinates from } (i,j) \text{ to } (m,j). \text{ A corresponding rule holds for the points on the eastern boundary } KD.\)

4. Probability of passing through a link

We have

\[
P_{i,j}^{i+1,j} = P(\text{passing through link } (i,j)(i + 1,j))
\]

\[
= P(\text{getting to } (i,j)) \times p(\text{using link } (i,j)(i + 1,j), \text{ given we are at } (i,j))
\]

\[
P_{i,j}^{i+1,j} = P_{i,j} \times p
\]

Now assume that the link \((i,j)(i + 1,j)\) is in two different overlapping grids, and that \( A \) is a person walking north-east while \( B \) is a person walking north-west. What is the probability that \( A \) or \( B \) will pass through this link, which lies in the grid determined by \( O_A \) and \( D_A \) and in the grid determined by \( O_B \) and \( D_B \)? (To simplify notation, I call \((i,i + 1)\) the link determined by \((i,j)\) and \((i + 1,j)\).)

\[
P(A \text{ or } B \text{ passes } (i,i + 1))
\]

\[
= P(A \text{ or } B \text{ both pass } (i,i + 1))
\]

\[
= P(A \text{ takes } (i,i + 1)) + P(B \text{ takes } (i,i + 1))
\]

\[
- P(A \text{ and } B \text{ take } (i,i + 1))
\]

\[
= P(A \text{ gets to } (i,j)) \times P(A \text{ takes } (i,i + 1) \mid A \text{ has got to } (i,j))
\]

\[
\times P(B \text{ gets to } (i+1,j)) + P(B \text{ takes } (i,i + 1) \mid B \text{ has got to } (i+1,j))
\]

\[
- P(A \text{ gets to } (i,j) \text{ and } B \text{ gets to } (i+1,j))
\]

\[
\times P(A \text{ and } B \text{ take } (i,i + 1) \mid A \text{ at } (i,j), B \text{ at } (i+1,j))
\]
\[ P(A \text{ or } B \text{ passes } (i,i+1)) = p_{i,j} \times p_A + p_{i+1,j} \times p_B - p_{i,j} \times p_A \times p_{i+1,j} \times p_B \]

(Assumption: events are independent.)

The probability that \( A \) and \( B \) will pass through \((i,i+1)\) is:

\[ P(A \text{ and } B \text{ pass through } (i,i+1)) = p_{i,j} \times p_A \times p_{i+1,j} \times p_B \]

(Independence of events assumed.)

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