Optimal Bus Subsidy and Cross Subsidy
with a Logit Choice Model

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1. Introduction
This paper investigates the impact on optimal bus subsidy of using a probabilistic choice model (namely, a logit model) to predict user choice between random and planned behaviour, with respect to how users access buses.

The bus optimisation problem considered here focuses on the user economies of scale effect (Mohring, 1972; Jansson, 1979; Glaister, 1987; Gwilliam et al., 1985; Bly and Oldfield, 1987; Kerin, 1992; Jansson, 1993). User economies of scale exist because increases in scale (through increased frequency) to accommodate additional users will result in user cost declining across all users. Marginal cost is therefore smaller than average cost, resulting in a case for subsidy.

The driving force behind the user economies of scale effect is frequency-related user costs (such as waiting time). A key factor in the modelling of frequency-related user costs is the form of behaviour followed by users. Conventional studies have tended to assume random user behaviour (see, for example, Mohring, 1972; Turvey and Mohring, 1975). More recently, this assumption has been relaxed. Evans (1987) assessed subsidy assuming planned behaviour. Tisato (1991, 1992) and Jansson (1993) have introduced the possibility of users facing a choice between these two behavioural modes (random and planned behaviour). Tisato (1992) demonstrates that this development has a major impact on the size of subsidy, while Jansson (1993) has demonstrated a number of important results under this new modelling framework.

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The aim of this paper is to extend these recent contributions, focusing on one of the key results of Jansson (1993). Jansson shows that there is a sudden and substantial increase in optimal unit subsidy when switching occurs between random and planned behaviour, which creates difficulties for the policy maker. The sudden increase in unit subsidy distorts the conventional negative relationship between optimal unit subsidy and route patronage (that unit subsidy is stronger the thinner, or less patronised, the route) reported in the literature (Jansson, 1980; Waters, 1982; Gwilliam et al., 1985; Nash, 1988). The conventional negative unit subsidy/patronage relationship has provided a simple and important rule of thumb for describing how subsidy per trip varies between bus routes of different demand density, and has been a useful mechanism for explaining a key policy outcome of user economies of scale. Thus its potential demise is significant.

The *a priori* concern here is that this result is driven by the simplistic manner in which choice between random and planned behaviour is characterised in the Jansson model. In that model, random vs planned choice outcomes are predicted using a purely deterministic framework that can predict very sudden, or knife-edge, behavioural changes for small changes in service levels. This paper addresses the behaviour of unit subsidy more closely, by modelling the random vs planned user choice with a more realistic logit probabilistic choice model. The behaviour of total subsidy is then shown to point to an important implication for subsidy policy implementation in practice. Finally, the issue of optimal cross-subsidy between routes (under a break-even constraint (Gwilliam et al., 1985)) is introduced. The case for optimal cross-subsidy is related to the optimal unit subsidy/patronage relationship. Thus, with the behaviour of unit subsidy being complicated by the nature of user cost, this may also have implications for cross-subsidy. Very little work has been previously done on optimal cross-subsidy between routes.

2. The Optimisation Problem

2.1 Assumptions

2.1.1 Producer Cost, $C_p$

$C_p$ is a function of service vehicle-kms, $VK$. Urban buses display constant returns to $C_p$ with respect to $VK$ (Windle, 1988; Evans, 1990; Hensher, 1993); thus $C_p/VK = \partial C_p/\partial VK$ and is constant. Noting that $VK = 2L_rF$, where $L_r$ is route length and $F$ is service frequency, then

$$C_p = (C_p/VK)VK = C_p = (C_p/VK)2L_rF.$$  

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1 Jansson's analysis is complex, mainly because it is generalised by making the value-of-time savings a function of the length of delay experienced by users. The analysis here was restricted to the case where the value-of-time saving is independent of the length of delay, a case also considered by Jansson.
2.1.2 User Cost, $C_u$

$C_u$ is the total time user cost incurred across all users. User time costs are experienced by all users. For example, if catching a bus results in users having to wait for the arrival of a bus, this wait applies to all users who want to use the service. The convention is, therefore, to define the user cost experienced by each user as the average cost, $AC_u$. Thus:

$$C_u = AC_u q.$$  \hspace{1cm} (2)

where $q$ is route patronage (boardings per hour).

User economies of scale (UES) are driven by frequency-related user costs (such as waiting time). It is useful, therefore, to distinguish between these and other user costs (for example, in-vehicle time, walk time, and so on). Accordingly, the subscripts $F$ and $O$ will refer to frequency-related user costs, and other user costs, respectively. Thus:

$$AC_u = AC_F + AC_O.$$ \hspace{1cm} (3)

$AC_F$ is the key to UES analysis. The model for $AC_F$ used here was developed in Tisato (1995) from a detailed study building on and extending previous models. In this model, $AC_F$ consists of two components:

$$AC_F = u(F) + v[LF(q, F, N), F],$$ \hspace{1cm} (4)

where $LF$ is the average load factor on buses, and $N$ is bus size. The direct impact of $F$ (through both components of $AC_F$) is the "Mohring" effect, or UES; that is, an increase in $F$ leads to lower user cost for all users, written as $\partial AC_F/\partial F < 0$. In addition, $v$, a passenger congestion cost, increases with $LF$. From equations given in the appendices (A7), (A8), and (A10), $\partial AC_F/\partial q > 0$ and $\partial AC_F/\partial N < 0$.

Following Tisato (1991) and Jansson (1993), a key feature of the model is that it models the mode of user behaviour. Users can choose between:

- planned behaviour, through use of timetables (obtained at an information cost $I$) to coordinate their arrival time at the bus stop with scheduled bus departure times; and
- random behaviour, where users choose not to consult a timetable, resulting in users arriving at bus stops at times that are randomly related to scheduled bus departure times.

A detailed listing of the sub-models comprising $u$ and $v$ is provided in Appendix A.

Only user cost component $u$ is influenced by the choice of behavioural mode (Tisato, 1995). The optimisation analysis requires the expected value of $u$:

$$E(u) = u_R R + u_p (1 - R),$$ \hspace{1cm} (5)

where $R$ is the proportion of users acting in a random fashion, and the proportion of planned users $P = 1 - R$. $R (0 \leq R \leq 1)$ is determined by a logit choice model (Ben-Akiva and Lerman, 1985):

$$R = 1/(1 + e^{u-R_p}),$$ \hspace{1cm} (6)

where

$$d u_{rp} = u_r - u_p.$$ \hspace{1cm} (7)

As Appendix A indicates, the key variable that influences behavioural mode choice is headway, $H$, the time between consecutive scheduled services. When $H$ is low (that is, frequency, $F$, is high$^2$), $u_r < u_p$, making random behaviour more likely. At high $H$ (that

$^2$Where $F = 60/H$. 

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is, low \( F \), \( u_r > u_p \), making planned behaviour more likely. When \( u_r = u_p \), random and planned behaviour are equally likely; that is, \( R = P = 0.5 \). The critical \( H \) and \( F \) values at which this occurs are denoted \( H_c \) and \( F_c \).

The parameter \( \mu \) in (6) is a logit scale parameter, whose value (see discussion below) determines the rate at which \( R \) increases as \( F \) increases. \( \mu \) is therefore effectively a model calibration parameter.

2.1.3 Demand, \( q \)

Route patronage, \( q \) (boardings/hour), is given by

\[
q = q(g),
\]

where \( g \), the generalised cost of travel (money plus non-monetary costs, in money units) can be written as:

\[
P + AC_u = P + AC_F + AC_O = P + u(F) + v LF(q, F, N) + AC_O.
\]

The exponential demand model (Glaister, 1987; Evans, 1987; Tisato, 1992) is used:

\[
q = \alpha \exp(-\beta q),
\]

where \( \alpha \) is the "potential" demand level, that is, \( q \) when \( g = 0 \); and \( \beta \) is a constant.

2.2 The Bus Optimisation Problem

Consider a representative bus route. Route spacing and bus size are given. The policy variables to be optimised are price \( P \) and frequency \( F \). Many previous studies have considered this bus optimisation problem (Mohring, 1972; Else, 1985; Gwilliam et al., 1985; Evans, 1987; Tisato, 1992; Jansson, 1993). This first-best optimisation problem is then:

\[
\max_{\mu} ES, \text{ where } ES = CS - S, \ S = C_p - P q, \text{ and } s = S/q.
\]

where \( ES \) is economic surplus, \( CS \) is consumer surplus, \( s \) is unit subsidy, and \( S \) is total subsidy. Solving the two first-order conditions \( \partial ES / \partial P = 0 \) and \( \partial ES / \partial F = 0 \) (Else, 1985; Gwilliam et al., 1985) yields:

\[
P = q \frac{\partial v}{\partial LF} \frac{\partial LF}{\partial q},
\]

and

\[
-q \left( \frac{\partial u}{\partial F} + \frac{\partial v}{\partial LF} \frac{\partial LF}{\partial q} + \frac{\partial v}{\partial LF} \right) = \frac{\partial C_p}{\partial F}.
\]

Equation (12) indicates that optimal price equals the marginal passenger congestion negative externality associated with an additional user (the increase in the user cost component \( v \) across all \( q \) users). Equation (13) suggests that marginal benefit and marginal cost of frequency enhancement must be equated.

Finally, optimal unit subsidy (Else, 1985; Gwilliam et al., 1985) is:

\[
s^* = -\left[ \frac{\partial u}{\partial F} (F^*) + \frac{\partial v}{\partial LF} \right] \left( F^* \right) F^*,
\]

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which reduces to (Mohring, 1972; Findlay, 1983):

$$s^* = ATC(q^*) - MSC(q^*),$$

that is, optimal unit subsidy is the gap between average total cost (ATC) and marginal social cost (MSC) at the optimal patronage level ($q^*$).

3. Optimal Unit Subsidy

The first thing to consider is the behaviour of optimal unit subsidy, $s^*$. The starting point with respect to $s^*$ is the conventional negative relationship between it and patronage ($q$), that is, optimal unit subsidy is greater the thinner or less patronised the route (see, for example, Waters, 1982; Gwilliam et al., 1985). This relationship is evident in Figure 1. The two curves $s_r^*$ and $s_p^*$ are the optimal unit subsidy curves for random and planned behaviour respectively. Both curves have a continuous negative slope throughout. While the relationship holds under both behavioural modes, $s^*$ is greater under random behaviour at all $q$ values (Jansson, 1993).

If user behaviour is, however, the outcome of a bi-modal choice process, optimal unit subsidy will lie between, and be a combination of, these polar schedules. The shape of the unit subsidy schedule will be determined by how this choice is modelled.

3.1 Deterministic user choice

Consider first deterministic user choice (Tisato, 1991; Jansson, 1993). This case arises when $\mu = \infty$ in equation (6), thus $R = 0$ if $F < F_c$, 1 if $F > F_c$. Therefore, there is no stochastic element in user choice (Ben-Akiva and Lerman, 1985). Choice is instead driven purely by the deterministic user cost expressions given in Appendix A. When patronage, $q$, is low, optimal frequency, $F^*$, will also be low, yielding planned behaviour and optimal unit subsidy in accordance with the $s_p^*$ schedule. As $q$ increases, so too does $F^*$. Eventually, as $q$ increases, $F$ increases sufficiently to reach and exceed $F_c$, at which point random behaviour will become the lower cost option. Accordingly, behaviour will switch to random behaviour, and optimal unit subsidy will suddenly increase to coincide with the $s_r^*$ schedule (Jansson, 1993). If the patronage level coinciding with $F_c$ is denoted as the critical patronage, $q_c$, the "effective" $s^*$ schedule will then consist of the composite curve $abcd$ illustrated in Figure 1, following the $s_p^*$ schedule when $q < q_c$, and the $s_r^*$ schedule when $q > q_c$. Throughout this study, $q_c \approx 220$ (an outcome of the parameter values used).

By identifying this composite optimal unit subsidy schedule, Jansson (1993) has challenged the conventional negative relationship between optimal unit subsidy and patronage, which no longer applies throughout as a general rule. Denoting point $e$ as the planned unit subsidy exactly equal to random unit subsidy at point $c$, and point $f$ as the random unit subsidy exactly equal to the planned unit subsidy at point $b$, a number of observations can be made.

- The negative relationship applies when the patronage levels being compared are either both bigger, or both smaller, than $q_c$. 

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For comparisons of patronage levels either side of $q_c$, that is, one bigger and one smaller than $q_c$, the negative relationship still applies, provided there is a large enough gap between the patronage levels being compared. For example, unit subsidy at all points to the left of point $e$ is greater than unit subsidy at all points to the right of point $c$. The same is true of comparisons of points to the left of point $b$ and to the right of point $f$.

There are many comparisons, however, of points in the range $eb$ with points in the range $cf$ for which the negative relationship breaks down. For example, unit subsidy at a point half way between $b$ and $e$ has lower unit subsidy than a point just to the right of $c$.

3.2 Probabilistic (logit) user choice

Now consider probabilistic user choice as modelled by the logit model (6). As the value of $\mu$ is relaxed from the polar value of $\infty$ discussed above, a stochastic element becomes apparent in user choice (Ben-Akiva and Lerman, 1985). As the value of $\mu$ gradually decreases, choice outcomes are increasingly explained by the stochastic element, and less by the deterministic element. In the polar case of $\mu = 0$, $R = P = 0.5$ throughout, with the stochastic element of choice totally dominating (Ben-Akiva and Lerman, 1985).

In contrast to the deterministic choice situation discussed above where there is a sudden switching between random and planned behaviour, $R$ now increases more gradually
as $F$ increases. With $F^*$ increasing gradually as patronage ($q$) grows, the transition between planned and random behaviour, and the transition of $s^*$ from $s_p^*$ to $s_r^*$ also occurs gradually over a range of patronage levels, rather than at $q_c$ as in the deterministic choice case.$^3$

This is evident in Figure 2, which presents the $s^*$ curves generated by the logit choice model for a range of $\mu$ values (plus the $s_p^*$ and $s_r^*$ curves as reference schedules). Lack of data prevented estimation of actual $\mu$ values. Instead, $\mu$ values were identified that generate certain reference switching patterns. In all cases $R = 0.5$ at $H_c$. The reference patterns were then defined by the $H$ value, $H'$, at which $R = 0.1$; that is, at which only 10 per cent of users act in a random manner, with 90 per cent acting in a planned fashion. Four reference patterns were used, with $H' = H_c + 2.5$, $H_c + 5$, $H_c + 10$ and $H_c + 15$ respectively. The resulting $\mu$ values are 0.03, 0.05, 0.11, 0.22.

In Figure 2, the vertical discontinuity in $s^*$ at $q_c$ has now disappeared, and has been replaced, in each $\mu$ case, by a smooth continuous curve which approximates $s_p^*$ at low demand levels, and gradually approaches $s_r^*$ as patronage level and $F$ increase. All the logit unit subsidy curves in Figure 2 display a gradual transition of $s^*$ from $s_p^*$ to $s_r^*$. The lower is $\mu$, the more gradual is the switching from planned to random behaviour, and thus the more gradual is the transition of $s^*$ from $s_p^*$ to $s_r^*$.

$^3$ Note that similar results would occur if all individuals made deterministic choices but $F_c$ varied between individuals.
Figure 2 illustrates that, even when a logit choice model is used (rather than the simpler deterministic choice model), the potential still exists for the conventional negative relationship between optimal unit subsidy, \( s^* \), and patronage, \( q \), to be broken. In Figure 2, the relationship is broken in each of the \( \mu \) cases considered, evidenced by the existence of upward sloping segments in the \( s^* \) schedules, over which \( s^* \) increases with \( q \). However, as the value of \( \mu \) falls, and thus switching becomes more gradual, a pattern emerges.

The positively sloped portion of the \( s^* \) schedule becomes flatter (and for small enough \( \mu \) values, would revert to having a negative slope throughout). As a result, the smaller \( \mu \) is, the fewer the number of patronage comparison cases where the conventional negative relationship would be broken. A final result to note is that, in some circumstances (for example the \( \mu = 0.03 \) case), optimal unit subsidy can be relatively constant over a wide range of patronage levels.

In summary, Jansson's analysis, and the behaviour of optimal unit subsidy in Figure 2 here, suggest that the conventional negative relationship between optimal unit subsidy and patronage level does not hold as a general rule. The introduction of bi-modal user behavioural choice raises the possibility for the conventional relationship to be broken, making it difficult to speculate, \textit{a priori}, the direction of the relationship for any given pair of patronage levels. However, the more gradually switching between planned and random behaviour occurs, the fewer the cases where the conventional relationship is broken.

4. Optimal Total Subsidy

Consider next the behaviour of optimal total subsidy, \( S^* \). Figure 3 plots \( S^* \) for the range of logit \( \mu \) values considered in Figure 2, plus \( S^* \) for the random and planned cases \( (S^*_r, \text{ and } S^*_p) \) as reference schedules. \( S^* \) rises steadily with patronage level in all situations. With a logit model, \( S^* \) displays a gradual transition between \( S^*_p \) and \( S^*_r \) as \( q \) rises, in line with similar behaviour of unit subsidy in Figure 2. The larger the logit scale parameter \( \mu \), the more rapid the switching between modes, and the more rapidly \( S^* \) grows over a progressively shorter \( q \) range while switching occurs. In the polar case of \( \mu = \infty \) (the deterministic choice model) there would be a sudden vertical jump in \( S^* \) from \( S^*_p \) to \( S^*_r \) at the patronage level at which switching occurs, \( q_c \). Such an outcome is best approximated in Figure 3 by the \( \mu = 0.22 \) case.

The behaviour of \( S^* \) is an interesting issue from a policy perspective. First, policy makers concerned with implementing optimal outcomes that lead to economic efficiency must be prepared (based on Figure 3) to continually increase total subsidy as patronage rises. This is not an unusual outcome, however, and need not cause problems.\(^4\) Second, and more importantly, as patronage grows through the range over which behav-

\(^4\) However, there may be \textit{perceived} limits to how high total subsidy may rise when public finance constraints exist, or when scope exists to reduce costs through productivity improvements (Tisato, 1995).
Optimal Bus Subsidy and Cross-Subsidy with a Logit Choice Model

Figure 3
Optimal Total Subsidy for Logit Choice Model

Optimal模式 switching occurs, successive increments in patronage require greater increases in subsidy to maintain optimality. This is suggested by the steepening in Figure 3 of the $S^*$ schedule over this range. Further, the bigger the logit scale parameter, the greater the $S^*$ increments, and the smaller the $q$ range over which they occur. Consequently, in cases of relatively large $\mu$ values (implying rapid switching behaviour), $S^*$ will grow quite rapidly for relatively small increments in demand.

This suggests that, in some circumstances, policy makers interested in maintaining optimal economic outcomes will be faced with a need to implement quite significant jumps in $S^*$ over quite short periods of time as growth in patronage occurs. This would be the case when patronage levels approach the range over which mode switching occurs, and when patronage is growing rapidly. In the real world, there are often political and financial constraints that may limit the rate at which $S^*$ can be increased. If this is the case, policy makers will need at least to be aware of projected $S^*$ values, and if possible attempt to manage the situation, for example by earmarking funds on a more gradual basis in advance. Of course, this in itself would have associated opportunity costs, requiring careful overall consideration of what might constitute optimal policy.
5. Optimal Cross-Subsidy

Next, consider optimal cross-subsidy\(^5\) between routes that arises under a financial break-even constraint applying across a collection of routes (Nash, 1982; Gwilliam et al., 1985; Evans, 1987). It is an important argument, because cross-subsidies are usually associated with undertakings and enterprises in a non-optimal context, whereas here the outcome is part of the optimal solution. With cross-subsidy being closely related to unit subsidy, the focus here is on whether the fluctuating movements in unit subsidy discussed above produce similar movements in cross-subsidy.

5.1 Second-best financially constrained optimisation

With the requirement to meet a subsidy constraint, the bus optimisation problem is now effectively a second-best problem, in contrast to the unconstrained first-best optimisation of subsection 2.2. Reformulating the first-best optimisation problem, the problem becomes:

\[
\max_{P,F} ES \\
\text{subject to } \sum_i S_i = S_T, \text{ where } S_T \text{ is the target subsidy level, and where } S_T = 0 \text{ in the financial break-even case being considered here. This reduces to:}
\]

\[
\max_{P,F} L = ES + \lambda (S_T - S)
\]

where \(\lambda\) is the Lagrange multiplier, the marginal value of relaxing the constraint (\(\lambda \geq 0\)).

This paper considers the case of two routes, denoted 1 and 2. Substituting for \(ES\) and \(S\) from equation (11) then yields:

\[
L = CS_1 + CS_2 + \lambda S_T - (1 + \lambda) (C_{P1} - P_1q_1 + C_{P2} - P_2q_2).
\]  

(16)

There are five first-order conditions (FOCs) for the variables \(P_1, P_2, F_1, F_2,\) and \(\lambda\). The FOC's \(\partial L/\partial P_i = 0, \partial L/\partial F_i = 0,\) and \(\partial L/\partial \lambda = 0\) (evaluated in Appendix C) respectively yield:

\[
P_i = q_i \left( \frac{\partial v}{\partial F_i} \frac{\partial LF_i}{\partial q_i} - \frac{\lambda}{(1 + \lambda)} \frac{\partial q_i}{\partial q_i} \right),
\]

(17)

\[
-\frac{q_i}{\partial F_i} \left( \frac{\partial u}{\partial F_i} + \frac{\partial v}{\partial F_i} \frac{\partial LF_i}{\partial F_i} \bigg|_q + \frac{\partial v}{\partial F_i} \bigg|_{LF_i, q} \right) = \frac{\partial C_{P_i}}{\partial F_i}
\]

(18)

and the budget constraint

\[
C_{P1} - P_1q_1 + C_{P2} - P_2q_2 = S_T (= 0).
\]

(19)

Equation (18) matches (13). Equation (17) is the second-best optimal price, which has two components: the first-best optimal price (12); and a markup required to raise revenue to meet the subsidy constraint. The constrained markup is denoted \(CMU_i\), that is,

\(^5\) Cross-subsidy is defined (Gwilliam et al., 1985) as having economic profits on some routes financing economic losses on other routes.
\[ CMU_i = -\frac{\lambda}{(1 + \lambda)} \frac{q_i}{\partial g_i}. \]  

(20)

As Gwilliam et al. (1985) explain, other things being equal, the markup will be greater: (a) the greater is patronage, \( q_i \); and (b) the less responsive is patronage to generalised cost, that is, the smaller \( \partial q_i/\partial g_i \) is, or alternatively, the greater \( \partial g_i/\partial q_i \) is, or the steeper the generalised cost inverse demand curve.\(^6\)

Zero cross-subsidy can only occur if the markup on each route is exactly equal to the first-best unit subsidy (equation (14)) on each route. If this exact outcome does not eventuate, given the overall breakeven requirement, cross-subsidy must exist, with:

- the profit-making route *cross-subsidising* the loss-making route; and
- the loss-making route *being cross-subsidised* by the profit-making route.

The breakeven constraint requires that total subsidy, \( S \), be identical in size, but opposite in sign, for the two routes, that is, \( S_1 = -S_2 \). Thus optimal cross-subsidy, \( CRS \), is the absolute value of \( S \) on either route: \( CRS = |S_i| \). Denoting \( CRS_{12} \) as the cross-subsidy from route 1 to route 2, then:

\[ CRS_{12} = -S_1 = S_2 = -CRS_{21}. \]  

(21)

To illustrate the case for cross-subsidies, and more importantly to investigate the impact on cross-subsidy of using the logit model to predict choice between random and planned behaviour, simulation runs were undertaken using the exponential demand function (equation (10)). From (10), \( (\partial q_i/\partial g_i) = -\beta q_i \); thus equation (20) becomes

\[ CMU_i = \frac{\lambda}{(1 + \lambda)\beta}. \]  

(22)

That is, for any pair of patronage levels \( q_1 \) and \( q_2 \), the markup will be the same on both routes.\(^7\) Then, from equation (17), optimal second-best price becomes:

\[ P_i = q_i \frac{\partial \lambda}{\partial LF_i} \frac{\partial LF_i}{\partial q_i} + \frac{\lambda}{(1 + \lambda)\beta}. \]  

(23)

Thus, when comparing two routes, given the identical markup for both routes, the direction of cross-subsidy will depend completely on the relative size of first-best unit subsidies.

**Result 1:**

*For an overall breakeven outcome, the route with the lower first-best unit subsidy will be cross-subsidising the route with the higher first-best unit subsidy.*

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\(^6\) These two effects can be integrated via generalised cost elasticity of demand \( \varepsilon = (\partial q/\partial g) (g/q) \), yielding (Tisato, 1995) \( (\delta_i - MSC) / \delta_i = -\lambda (1 + \lambda) \varepsilon_i \), the conventional inverse elasticity rule in the literature on optimal taxation (Ramsey, 1927) and optimal pricing (Brown and Sibley, 1986). The rule indicates that the percentage markup of generalised cost above \( MSC \) is inversely proportional to the elasticity of demand.

\(^7\) However, for each different patronage pair \( \lambda^* \) will change, and thus so will the markup.
5.2 Generating a constrained optimal solution

An optimal solution is generated from equations (17) to (19), solving for \( P_1, P_2, F_1, F_2, \) and \( \lambda \), for any given pair of patronage levels \( q_1 \) and \( q_2 \). \( F_1^* \) and \( F_2^* \) are determined by solving (18) for each route, given \( q_1 \) and \( q_2 \). \( P_1^*, P_2^*, \) and \( \lambda^* \) are then determined by solving the remaining three FOCs by substituting (17) for \( P_1 \) and \( P_2 \) in (19), and solving to yield \( \lambda^* \). Thus equation (19) becomes:

\[
S_T = X - Y + \frac{\lambda}{(1 + \lambda)} Z,
\]

(24)

where

\[
X = \sum_i C_{pi},
\]

\[
Y = \sum_i \left( q_i^0 \frac{\partial V}{\partial LF_i} \right),
\]

and

\[
Z = \sum_i \left( \frac{q_i^0}{(\partial q_i)/(\partial g_i)} \right),
\]

and rearranging (24) yields:

\[
\lambda^* = \frac{S_T + Y - X}{S_T + Y - X - Z}.
\]

(25)

\( \lambda^* \) can then be substituted into (17) to yield \( P_1^* \) and \( P_2^* \).

5.3 Simulation results

The impact on cross-subsidy of using a logit choice model was then assessed using a number of simulation runs. All the simulation results were constructed by setting \( q_1 \) at some selected level, then observing how CRS12 changed as \( q_2 \) was varied. The results are summarised in Figures 4 to 6.

Consider first the situation where users act in either a random or planned manner. Figure 4 plots four single behavioural mode schedules of CRS12 (the cross-subsidy from route 1 to route 2). Three schedules are for random user behaviour, one for each of three \( q_1 \) values (100, 200, and 300), and the fourth schedule is for planned user behaviour for the \( q_1 = 100 \) case. In each case, CRS12 > 0 when \( q_1 > q_2 \) and CRS12 < 0 when \( q_1 < q_2 \); that is, the higher patronage route always cross-subsidises the lower patronage route, consistent with findings in the literature (Gwilliam et al., 1985; Evans, 1987). This occurs because, given the conventional negative relationship between \( s^* \) and \( q \) discussed earlier, first-best unit subsidy is always lower on higher patronised routes. It then follows from result 1 that a higher patronage route will cross-subsidise one of lower patronage.

Note also in Figure 4 that CRS12 = 0 when \( q_1 = q_2 \), since both the second-best markup, and the first-best unit subsidy, will be the same on both routes. Further, the big-
Figure 4
Optimal Cross-Subsidy from Route 1 to Route 2:
Random and Planned Behaviour Cases

Figure 5
Optimal Cross-Subsidy from Route 1 to Route 2:
Logit Model, $q_1 = 100$ Case
Figure 6
Optimal Cross-Subsidy from Route 1 to Route 2:
Logit Model, \( q_1 = 300 \) Case

The difference between \( q_1 \) and \( q_2 \), the greater the magnitude of cross-subsidy. Finally, note that the planned behaviour model generates lower cross-subsidies, and thus a flatter cross-subsidy schedule, than does the random behaviour model. This is because the first-best unit subsidy schedule is flatter under planned behaviour (see Figure 1), and so the differences in first-best unit subsidy between a given pair of patronage levels, and correspondingly the level of cross-subsidy, will be lower.\(^8\)

Now consider the more interesting question of how cross-subsidy behaves when planned and random behaviour are considered simultaneously in a discrete choice framework, specifically, the logit choice model. The resulting cross-subsidy outcomes are summarised in Figures 5 and 6 for two cases, where \( q_1 = 100 \) and 300 respectively.\(^9\)

In each case, cross-subsidy results are presented for the range of logit \( \mu \) values, plus the random and planned schedules (\( CRS_{12r} \) and \( CRS_{12p} \)) are also plotted for comparative purposes.

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\(^8\) Expression (20) can be written as \( CMU_1 = -[\lambda/(1+\lambda)(g_1/g_2)] \). In the exponential demand function case presented here, \( g_1 = -\beta_2 \); that is, elasticity is proportional to \( g \), yielding a constant markup. If a less strong elasticity relationship applied, \( CMU_1 \) would be higher for the route with the higher \( g \), that is, the low demand (low \( F \)) route. As a result, the cross-subsidy from the high demand (high \( F \)) route to the low demand (low \( F \)) route would be smaller in size.

\(^9\) With \( q_2 = 220 \) boardings/hour, these two \( q_1 \) values were chosen to yield a different mode of behaviour on route 1 in each case (planned behaviour when \( q_1 = 100 \), and random when \( q_1 = 300 \)). In contrast, the mode of behaviour on route 2 will vary as \( q_2 \) varies, starting with planned behaviour when \( q_2 \) is low, and gradually switching to random as \( q_2 \) is approached and exceeded.
In Figure 5, consider first the case of $\mu = 2$, in which switching occurs very quickly (and therefore approximates the simple deterministic choice model). For $q_2$ values below $q_\text{cr}$, $CRS_{12}$ is almost identical to that which occurs under pure planned behaviour, since that is in fact the behavioural mode at both patronage levels. Once $q_2$ approaches $q_\text{cr}$, however, there is a rapid switch to random behaviour on route 2, but importantly, with $q_1$ still $< q_\text{cr}$, behaviour on route 1 is still planned. We therefore now have different behavioural modes on the two routes. With a switch to random behaviour on route 2, there is a corresponding sudden increase (see Figure 2) in first-best unit subsidy on that route. With first-best unit subsidy on route 1 unchanged, result 1 suggests that there should be a rapid relative shift in revenue-raising from route 2 to route 1, resulting in the sudden rise in the cross-subsidy schedule evident in Figure 5. The rise in first-best unit subsidy on route 2 is so great that it now exceeds that on route 1, leading to $CRS_{12}$ moving from negative to positive. As $q_2$ increases further, first-best unit subsidy on route 2 declines gradually, and hence so does cross-subsidy.

The other logit curves in Figure 5 reflect less rapid switching from planned to random behaviour on route 2 as $q_2$ approaches $q_\text{cr}$. As a result, the need to increase the relative proportion of revenue raising from route 2 to route 1 changes more gradually, requiring a more gradual change in cross-subsidy compared to the rapid change experienced for $\mu = 2$. Note the close relationship between the cross-subsidy schedules here and the corresponding first-best unit subsidy schedules in Figure 1. The more gradual the change in unit subsidy in Figure 2, the more gradual the change in cross-subsidy in Figure 5.

A similar, but almost symmetrical, pattern of outcomes occurs in Figure 6 for the case of $q_2 = 300$. In this case, $q_1 > q_\text{cr}$, so there is always random behaviour on route 1. When $q_2$ also exceeds $q_\text{cr}$, behaviour on route 2 is also random, and thus the logit cross-subsidy schedules lie close to the pure random schedule. As $q_2$ declines towards $q_\text{cr}$, there is switching of behaviour on route 2 to planned behaviour. This causes route 2 first-best unit subsidy to decline, implying the need to shift relative revenue raising efforts from route 1 to route 2, resulting in a corresponding decline in the cross-subsidy from route 1 to route 2. If $\mu$ is large (for example, $\mu = 2$) switching is very rapid, and the corresponding decline in cross-subsidy is also rapid. As $\mu$ becomes smaller, so switching is more gradual, and so too the rate of change in cross-subsidy is more gradual.

The behaviour of cross-subsidy depicted in Figures 5 and 6 under a logit model differs in several respects from that reported in the literature, and from the single behavioural mode cases in Figure 4. First, it is no longer necessarily the case that high patronage routes always cross-subsidise low patronage routes. In some circumstances the reverse may occur, with high patronage routes being cross-subsidised by low patronage routes. Consider, for example, the case in Figure 6 of $\mu = 0.11$ when $q_2$ lies approximately between 105 and 215. Over this $q_2$ range, $CRS_{12}$ is negative and thus route 1 is being cross subsidised, yet route 1 has the higher patronage (300).

Second, it is no longer necessarily the case that the magnitude of cross-subsidy increases when the difference in patronage between routes becomes larger. For example,
in Figure 6 when $\mu = 0.11$, as $q_2$ falls over the approximate range 165 to 105, the magnitude of cross-subsidy declines, yet the patronage difference has grown.

Third, as the difference in patronage between routes becomes larger, it is no longer necessarily the case that cross-subsidy changes uniformly. It can now change rapidly in some circumstances (such as the $\mu = 2$ example) and slowly in others (for example, when $\mu = 0.03$ there is quite a wide range of $q_2$ values over which cross-subsidy remains relatively stable, especially in Figure 5).

Fourth, in the single behavioural mode cases in Figure 4, the only time when $\text{CRS}_{12} = 0$ is when $q_1 = q_2$. However, once a logit model is used to predict choice between modes, $\text{CRS}_{12}$ can also equal zero, even when $q_1 \neq q_2$. In the $\mu = 2$ and 0.11 cases in both Figures 5 and 6, $\text{CRS}_{12} = 0$ at two other $q_2$ values. This occurs because, for these $\mu$ values, switching occurs rapidly enough to cause a change in $\text{CRS}_{12}$ that is large enough for $\text{CRS}_{12}$ to change sign. In contrast, for the $\mu = 0.03$ case, switching does not occur rapidly enough for this to arise, and so $q_1 = q_2$ is the only location where $\text{CRS}_{12} = 0$.

Overall, introduction of the logit model has had a significant impact on optimal cross-subsidy analysis, making it more difficult to predict a priori how cross subsidy will behave.

6. Conclusions

The aim of this paper was to investigate the influence on bus subsidy analysis of using a probabilistic logit model to predict the outcome of discrete user choice between random and planned behaviour.

A recent contribution to the literature (Jansson, 1993) has shown that optimal unit subsidy was found to change suddenly and substantially as users switched between random and planned behaviour. An implication of this result is that the conventional negative relationship between unit subsidy and patronage level breaks down. The analysis of this paper supports Jansson's result, demonstrating that the conventional unit subsidy/patronage relationship may also break down when a logit choice model is used (rather than one of deterministic choice), making it difficult to speculate a priori the direction of the relationship between unit subsidy and patronage. However, when using a logit model, the range of patronage levels over which the relationship falters, and the severity of breakdown in the unit subsidy/patronage relationship, decreases the more gradually mode switching takes place between random and planned behaviour.

The key result in the analysis undertaken here of total subsidy is that, as patronage increases, the growth in optimal total subsidy is greater as user behavioural mode switching occurs than it is before or after switching. Further, the more rapidly switching occurs, the greater is the relative growth of total subsidy during switching. Policy makers concerned with maintaining optimal outcomes in terms of economic efficiency may, therefore, be faced with the prospect of implementing quite significant increases in total
subsidy levels, in potentially short periods of time, as growth in patronage occurs in the
presence of behavioural mode switching.

The literature has previously established that it is optimal to have cross subsidies be-
tween bus routes when an overall breakeven constraint is imposed across a group of
routes. This paper considered how optimal cross-subsidy results were affected by the
introduction of bi-modal user behavioural choice. The paper showed that random user
behaviour generates higher cross-subsidy levels than planned behaviour. In addition,
when either random or planned behaviour applies throughout, the result in the literature
that a high patronage route should cross-subsidise a low patronage route was confirmed.
Further, cross-subsidy varied uniformly as patronage differences between routes varied.
With bi-modal choice between random and planned behaviour (modelled here via a logit
model), however, quite different cross-subsidy outcomes were observed.

With a logit model, it became possible for low patronage routes to cross-subsidise
high patronage routes. In addition, as the patronage difference between routes grows,
the cross subsidy no longer necessarily increases, nor does it necessarily change gradu-
ally. Cross-subsidy can now both increase and decrease, and change both quickly and
slowly, depending on the rate of switching between random and planned behaviour. Fi-
nally, although in a single behavioural mode model cross-subsidy is equal to zero only
when route patronage levels are equal, in a logit model an optimal cross subsidy of zero
is also possible in situations where route patronages differ. On the whole, it becomes
more difficult to predict a priori how cross subsidy will behave.

Appendix A

User Cost Sub-Models

Tisato (1995) reports sub-models for $AC_F$ as follows:

$$u_i = FDC_i(F) + SSDC_i(F) + I_i$$  \hspace{1cm} (A1)

and

$$v = SSDC \left[ LF(q, F, N), F \right].$$  \hspace{1cm} (A2)

where subscript $i$ is either $r$ or $p$ for random and planned behaviour. In addition: (a) $I_r$
$= 0$ and $I_p > 0$; (b) $FDC$ is frequency delay cost, the delay experienced by users because
buses are not scheduled to depart at times which best suit the user; (c) $SSDC$ is stochastic
supply delay, the delay experienced by users because buses fail to depart at their sched-
uled time, and are thus unreliable; and (d) $SDDC$ is stochastic demand delay, the delay
experienced by users because user demand is stochastic, leading to buses being full on
some occasions, resulting in some users missing their intended bus.\footnote{Boarding and alighting time costs have not been modelled here. Although boarding and alighting have played a role in other subsidy studies (for example, Mohring, 1972; Bly and Oldfield, 1987; Jansson, 1993), Bly and Oldfield (1987) find that other user cost components have a much more significant impact on subsidy analysis.}

The specific expressions for these sub-models are as follows, where headway $H = 60/F$:
\[ FDC_r = v_w H/2, \]  
\[ SSDC_r = v_w \sigma^2/H, \]  
\[ FDC_p = fH/4, \]  
\[ SSDC_p = 1.88H^{0.357} \sigma^{0.389} v_w, \]  
\[ SDDC_r = SSDC_p = v_w \times LF_{\text{mult}} H/2, \]  
\[ \text{where } LF_{\text{mult}} = (1/b_1 - b_2 LF) - 1, \]  

where \( v_w \) is unit waiting time cost, \( \sigma \) is the standard deviation of bus departure times from scheduled times (an indicator of service unreliability), \( b_1 \) and \( b_2 \) are constants, and \( f \) is unit planned frequency delay cost.

\( LF \) is average load factor. \( LF \) is measured as the ratio of passenger-kms (PK) to seat-kms (SK) (the author is obliged to John Dodgson for assisting him in clarifying this point), and determined as the weighted average across both linehaul (\( LH \)) and backhaul (\( BH \)) directions of flow, weighted by the size of the flows; that is,

\[ LF = \left( LF_{\text{LH}}^{A} + LF_{\text{BH}}^{A} \right) / q, \]  

where \( LF_i = PK_i / SK_i \), \( PK_i = q_i L_i, SK_i = FNL_i, L_i \) is trip length, and \( L_r \) is route length. If \( d \) is the proportion of total boardings in the \( LH \) direction, that is, \( q_{LH} = dq \), then (A9) reduces to

\[ LF = qA / FN, \]  

where

\[ A = \left( 1 - 2d + 2d^2 \right) L_r / L_i. \]

As \( d \) varies, the directional flow multiple in brackets changes as follows: when \( d = 0.5 \) (balanced directional flows), the multiple is smallest and equal to 0.5; as \( d \) grows, the multiple (and thus \( LF \)) grows in size (at an increasing rate); and when \( d = 1.0 \) (flow in one direction only), the multiple is greatest and equal to 1.0. The model is therefore able to predict pronounced differences in \( LF \) between peak and off-peak.

**Appendix B**

**Parameter Values**

The parameter values used in this study, summarised in Table B.1, reflect the Adelaide bus system. All values are in 1993 units and values. A detailed description of their derivation is contained in Tisato (1995). A brief overview follows.

- Estimates of \( L_i, \sigma, AC_o, N, d \), and \( C_p/VK \) were based on data from, and discussions with the regulator and providers of public transport services in Adelaide.

<table>
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<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
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<td>( b_1 )</td>
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<td>unitless</td>
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<td>( v_w )</td>
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<td>( b_2 )</td>
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<td>unitless</td>
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<td>cents/min</td>
<td>( \sigma )</td>
<td>2</td>
<td>mins</td>
</tr>
<tr>
<td>( l )</td>
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<td>cents/trip</td>
<td>( N )</td>
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<td>passengers</td>
</tr>
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<td>( d_{PK} )</td>
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<tr>
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<td>kms</td>
<td>( d_{OP} )</td>
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<td>%::%</td>
</tr>
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<td>cents</td>
<td>( C_p/VK )</td>
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<td>( \beta )</td>
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<td>boardings/hr</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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• \( v_V \) and \( v_w \) were based on reported empirical Australian studies (for example, Hensher, 1989).
• \( f \) and \( l \) were more difficult to determine, given a general lack of data and estimates. \( l = 5 \) cents/trip was used, based on a previous study (Tsato, 1991). The estimate for \( f \) was based on the argument that it should not exceed \( v_V \) (Douglas and Miller, 1974; Mohring, 1976). A value of half \( v_V \) was used.
• With regard to \( \beta \), Evans (1987) has shown that \( \varepsilon = -\beta P \), where \( \varepsilon \) is own price elasticity of demand, and \( P \) is price. \( \beta \) was therefore calibrated using local elasticity and price estimates.
• With regard to \( b_1 \), \( b_2 \), values were chosen which ensured a good fit with the stochastic supply delay model for London reported by Glaister (1982), the only known calibrated model.

Appendix C
Optimal Cross-Subsidy Analysis, Derivation of First-Order Conditions

(a) From equation (1), \( \partial C_p/\partial q_i = 0 \); thus \( \partial C_p/\partial P_i = (\partial C_p/\partial q_i) (\partial q_i/\partial P_i) = 0 \). The FOC \( \partial L/\partial P_i = 0 \) is therefore:

\[
\frac{\partial L}{\partial P_i} = \frac{\partial CS_i \partial g_i}{\partial g_i} P_i -(1+\lambda) (-q_i- P_i \frac{\partial q_i}{\partial P_i}) = 0.
\]

(C.1)

From equation (9),

\[
\frac{\partial g}{\partial P} = \frac{\partial g}{\partial P} - \frac{\partial v}{\partial P} \frac{\partial LF}{\partial P} \frac{\partial q}{\partial P} 1 + \frac{\partial v}{\partial LF} \frac{\partial q}{\partial P}.
\]

(C.2)

Now, \( CS = \int q \cdot dg \); thus \( \partial CS/\partial g = -q \). Substituting this and (C.2) into (C.1) and rearranging yields:

\[
P_i = \frac{1}{(1+\lambda)} q_i \frac{\partial v}{\partial LF} \frac{\partial q_i}{\partial q_i} -(1+\lambda) \frac{\partial q_i}{\partial P_i}.
\]

(C.3)

From equations (9) and (10),

\[
\frac{\partial q_i}{\partial P_i} = \frac{\partial q_i}{\partial g_i} \left( \frac{1 - \frac{\partial v}{\partial LF} \frac{\partial q_i}{\partial q_i}}{1 + \lambda \frac{\partial q_i}{\partial g_i}} \right).
\]

Substituting into (C.3) and rearranging yields:

\[
P_i = q_i \frac{\partial v}{\partial LF} \frac{\partial q_i}{\partial q_i} -(1+\lambda) \frac{\partial q_i}{\partial g_i}.
\]

(C.4)

(b) Tsato (1995) shows that \( \partial L/\partial F = 0 \) reduces to \( \partial TC/\partial F = 0 \). \( TC = C_p + C_u = C_p + AC_u q \). Then, noting equation (4), \( \partial TC/\partial F = 0 \) yields

\[
-q \left( \frac{\partial u}{\partial F} + \frac{\partial v}{\partial LF} \frac{\partial F}{\partial F} + \frac{\partial v}{\partial LF} \frac{\partial F}{\partial q} \right) = \frac{\partial C_p}{\partial F}.
\]

which is identical to equation (13), the result when there is no subsidy constraint.
(c) As usual, the FOC with respect to the Lagrange multiplier \( \lambda \), \( \partial L/\partial \lambda = 0 \), yields the budget constraint

\[
C_{p1} - P_1 q_1 + C_{p2} - P_2 q_2 = S_T = 0.
\]

(C.5)
References


Date of receipt of final manuscript: November 1997