MAXIMISATION OF PASSENGER MILES
IN THEORY AND PRACTICE

By S. Glaister and J. J. Collings*

I. INTRODUCTION

The idea has recently gained some currency that public transport operators might be directed to run their services so as to maximise the total number of passenger miles they sell, subject to an overall subsidy constraint. The attention given to this strategy has mainly been due to its adoption in January 1975 as a corporate objective for London Transport [6], [10]. It has also been suggested as a possible objective for the railways [9], and has been discussed recently in this Journal by Nash [8] in the context of management objectives for bus transport. The aim of this paper is (a) to compare and contrast (in purely theoretical terms) the implications of using this objective with those of "classical" alternatives, using a somewhat different formulation from that of Nash; (b) to demonstrate that under certain assumptions it is possible to define a simple procedure for weighting passenger miles so that maximisation of weighted passenger miles would be equivalent to maximisation of net social benefit; (c) to investigate the extent to which alternative weighting systems might be devised to encourage the prosecution of policies aimed at congestion relief through changes in modal split, at income redistribution, or at economic efficiency; (d) to evaluate the importance of the differences which are likely to arise in practice between the results of a policy of passenger-miles maximisation and the alternatives, with special reference to London Transport (LT).

Our general conclusion is that the manifest advantages of using the simple, relatively easily understood and market-orientated decision rules which emerge from passenger miles maximisation must be set against a risk that they would cause quite significant welfare losses, in view of what is known to us about market and cost characteristics in the LT and British Rail (BR) systems. There appears to be a danger of under-estimating this difficulty (see Quarmby [10], para. 217). These losses might be less serious in simpler transport systems, such as relatively homogeneous provincial bus services, but are likely to present real difficulties in a system as heterogeneous and complicated as that of British Rail. The "classical" decision rules derived from consideration of the welfare optimum require no more and no less information. In general our conclusions support those of Nash [8] concerning the criterion of maximisation of unweighted passenger miles, though he

*Dr. Glaister is at the London School of Economics. Dr. Collings is at H.M. Treasury (formerly at the Department of Transport). The views expressed in this paper are solely the private views of the authors. They have benefited from comments from Robert Anderson, Alexander Grey, William Tyson, Gilbert Ponsonby and an anonymous reference.
does not analyse the possibilities and implications of any weighting systems. The weighting schemes which we discuss can be of great help in overcoming these difficulties. Thus, if the objective is accepted together with the disciplines on decision-taking which it entails, and if it is modified to bring it more into line with economic efficiency objectives, this might be a development to be encouraged.

II. PASSENGER MILES, PROFIT AND NET SOCIAL SURPLUS

It is convenient to develop a uniform notation to allow a mathematical formulation of the alternative approaches. Suppose that there are three independent services which might be supplied and that the number of passenger-miles per annum demanded at each of the fare levels \( g_1 \), \( g_2 \) and \( g_3 \) pence per passenger-mile are given by

\[
f_i(g_1), f_i(g_2) \text{ and } f_i(g_3).
\]

These might be alternative modes or alternative services provided by the same mode, such as peak, off-peak and weekend bus services. The following analysis can be generalised in an obvious way to handle any number of services, so this restriction of the analysis is unimportant. The assumption of independence is more restrictive, in view of the strong evidence found by Fairhurst [2] and others of significant cross-price elasticities between L.T. modes. It is made as a simplification; its relaxation leads to more complicated but qualitatively similar results.

The variables \( g_i \) may be defined either as money fares in the normal sense, or as "generalised costs":

\[
g_i = p_i + \tau_i t_i,
\]

where \( p_i \) is money fare, \( t_i \) is average time taken per passenger mile on service \( i \), and \( \tau_i \) is the value of time to users of service \( i \). The text of sections II to IV is worded on the first interpretation. However, it should be noted that, if \( t_i \) are constant, changes in generalised cost are equal to changes in money fares. Because of this, it is easy to show that all the results obtained in these sections are valid on the generalised cost interpretation of the \( g_i \) with the \( t_i \) being held constant. Section V discusses the problem of choice of service quality, as measured by the \( t_i \), by allowing these to become choice variables in addition to the money costs.

It is useful to note that on the generalised cost interpretation

\[
\frac{df_i(g_i)}{dg_i} = \frac{df_i(g_i)}{dp_i} = \frac{1}{\tau_i} \frac{df_i(g_i)}{dt_i}.
\]

Further,

\[
\eta_i = \frac{df_i(g_i)}{dg_i} \cdot \frac{g_i}{f_i(g_i)} = \frac{df_i(g_i)}{dp_i} \cdot \frac{p_i}{f_i(g_i)} = \frac{\hat{\eta}_i}{\hat{p}_i} \frac{g_i}{p_i},
\]

where \( \eta_i \) and \( \hat{\eta}_i \) are respectively generalised cost and money cost elasticities. It follows that
This is pertinent to expressions such as (8) and (17) below.

Suppose that \( \alpha_i \) are values such that any fare higher than this will eliminate demand:

\[
g_i \geq \alpha_i \Rightarrow f_i(g_i) = 0, \quad i = 1, 2, 3.
\]

Assume that the demand relationships have the usual property:

\[
\frac{df_i(g_i)}{dg_i} = f_i' < 0, \quad i = 1, 2, 3.
\]

Costs, related to service levels, are given by

\[
C_i\{f_i(g_i)\} \text{ with } \frac{dC_i}{df_i} = C_i' > 0 \quad i = 1, 2, 3.
\]

and \( C_i'' > 0 \).

The Transport (London) Act 1969 specified LT's objective as "to provide or secure the provision of such public passenger transport services as best meet the needs for the time being of Greater London". LT's interpretation of this, to provide a corporate "aim" (see [6]), may be formalised as the following simple non-linear programming problem:

\[
\begin{align*}
\text{maximise} & \quad f_1(g_1) + f_2(g_2) + f_3(g_3) \\
\text{subject to} & \quad \sum [p_i f_i(g_i) - C_i(f_i(g_i))] \geq \pi \\
\text{and} & \quad 0 \leq g_1 \leq \alpha_1, \ 0 \leq g_2 \leq \alpha_2, \ 0 \leq g_3 \leq \alpha_3
\end{align*}
\]

where \( \pi \) is the contribution towards fixed costs required from net operating revenues. Because of the lump sum subsidy provided, this may be either positive or negative.

The necessary conditions for this maximisation with respect to the \( p_i \) are

\[
f_i + \lambda_i (p_i - C_i') f_i + \delta_i = 0 \quad i = 1, 2, 3.
\]

where \( \lambda > 0 \) and \( \delta > 0 \) are the Kuhn-Tucker multipliers (or shadow prices) corresponding to the constraints on profit and on fares respectively. We can safely assume that the profit constraint will be binding; hence \( \lambda > 0 \).

For the sake of the argument let us assume that at the optimum services 1 and 2 are provided, but service 3 is not; so

\[
\delta_1 = \delta_2 = 0 \text{ and } \delta_3 > 0 \text{ with } g_3 = \alpha_3.
\]
We then have from (5)

\[
\begin{align*}
    f_1 + \lambda((p_1 - C_1)f_1 + f_1) &= 0 \\
    f_2 + \lambda((p_2 - C_2)f_2 + f_2) &= 0 \\
    f_3 + \lambda((p_3 - C_3)f_3 + f_3) &> 0
\end{align*}
\]

(6)

In each case the term in braces is the marginal net revenue with respect to price (call it \(MR_i\)), and so the first two equations of (6) imply that

\[
\frac{MR_1}{MR_2} = \frac{f_1}{f_2};
\]

(7)

that the ratio of the net marginal revenues should equal the ratio of the slopes of the demand curves. Rewriting (6) gives

\[
\begin{align*}
    p_i - C_i &= -\frac{1}{\lambda} - \frac{f_i}{f_i} \\
    &= -\frac{1}{\lambda} - \frac{g_i}{\eta_i}, \quad i = 1, 2
\end{align*}
\]

(8)

Relations (8) together with the profit constraint determine the optimum fares and service levels, on the assumption that it is optimum not to offer service 3 at all. The parameter \(\lambda\) is the shadow price of the profit constraint, and its value measures the extra passenger miles which could be provided at the optimum if £1 extra subsidy were provided. It has the dimensions of passenger miles per pound and corresponds to the “pass-mark” used in making decisions about the introduction or expansion of services. Rewriting the third relation of (8), remembering that \(f_3(\alpha_3) = 0\), gives

\[
C_i' - \alpha_3 > \frac{1}{\lambda},
\]

(9)

which implies that a service will not be provided if the net cost to the system of the first unit (and hence on our assumptions all other units) exceeds the inverse of the system pass-mark, in pounds per passenger mile. Services which are provided will be provided to the point where

\[
C_i' - p_i \left(1 + \frac{1}{\eta_i}\right) = \frac{1}{\lambda}, \quad i = 1, 2
\]

(10)

that is, where marginal cost net of marginal revenue generated is just equal to the inverse pass-mark. Thus, once the pass-mark \(\lambda\) has been established for the whole system, a decision to adjust a fare on any individual service can be made at a relatively junior managerial level by comparing the marginal costs and revenues and making changes until (10) is satisfied. Similarly decisions to invest or disinvest
so as to provide new services or close old ones can be made on the basis of (9), provided the change is small relative to the whole system. Larger changes would involve a recalculation of the pass-mark.

**Profit maximisation**

Before discussing these results further we develop a related profit maximisation problem:

\[
\begin{aligned}
\text{maximise } & \sum [p_i f_i(g_i) - C_i(f_i(g_i))] \\
\text{subject to } & \sum f_i(g_i) \geq M \\
& 0 \leq g_i \leq \alpha_i \quad i = 1, 2, 3.
\end{aligned}
\]  

(11)

This is the problem of maximising profit subject to a minimum passenger-miles constraint. The necessary conditions, on the assumption that services 1 and 2 are provided and 3 is not, are easily shown to be

\[
\begin{cases}
(p_i - C_i) f_i' + \mu f_i = 0 & i = 1, 2, \\
(p_3 - C_3) f_3' + \mu f_3 > 0
\end{cases}
\]  

(12)

where \( \mu \) is the shadow price attaching to the passenger-miles constraint.

Now compare (12) with (6) above. Assuming that the passenger-miles constraint is binding (so that a profit maximiser would like to reduce services if allowed to), we have that \( \mu > 0 \). Then (7) still applies and (12) is the same as (6) with \( \mu = 1 \).

Mathematically the problems are duals, and if the constraining value of \( M \) in the profit maximising problem is set at the maximum value of the passenger-miles maximising problem, and similarly the constraining value of \( \pi \) in the passenger-miles problem is set at the maximum value of the profit problem, the two problems will yield identical solutions. Hence the problem of maximising passenger miles subject to a minimum profit (or maximum loss) constraint is very closely related to that of maximising profit subject to a minimum passenger-miles constraint.

This fact immediately establishes two points. First it confirms that the objective of maximisation of passenger miles has some validity as a means of relating the "needs" for services as expressed through the market to the costs of providing them, much as a quantity-controlled profit-maximising monopolist would. Second, it warns us that the objective may contain an element of monopolistic pricing which is not wholly consistent with the interpretation of "the public interest" in terms of economic efficiency. Specifically, price discrimination and cross-subsidisation may be introduced because of a tendency to push the fare on an inelastic service above marginal operating costs, in order to earn surplus revenue which can be used to finance a service which exhibits a relatively high elasticity (and hence easily provides an increase in passenger miles). In fact, a service may be operated for which the level of demand (willingness to pay) is never high enough to allow price to cover marginal cost. From the point of view of the dual, the temptation is to earn extra profit on the inelastic service in order to subsidise the elastic but unprofitable service, so that it will make a good contribution to satisfying the minimum
passenger miles constraint. The practical importance of this observation clearly
develops upon the homogeneity of demand elasticities. This is something we
evaluate in section IV below.

The foregoing discussion also establishes the validity of an alternative solution to
the passenger miles maximisation problem which has been proposed by BR. The
Lagrangean for (11) is
\[
\sum_i [p_i f_i(g_i) - C_i(f_i(g_i))] + \mu \left[ \sum_i f_i(g_i) - M \right]
\]
which may be rewritten as
\[
\sum_i \left( p_i + \frac{1}{\lambda} \right) f_i(g_i) - C_i(f_i(g_i)) \right) - \mu M
\].

This shows that solution of (11), and therefore of the constrained passenger miles
maximisation problem (4), is equivalent to simple profit maximisation, so long as
the prices are increased by the appropriate inverse pass-mark. The latter has been
termed the "social fare" (see Parker [9] Chapter 3, and also cf. Nash [8], p. 77).

Economic efficiency

As an alternative we now consider the possibility of maximising some measure of net social benefit. We shall take total net consumer surplus as our objective. The
"modified marginal cost" results which emerge are necessary conditions for economic efficiency. This can be demonstrated by arguments completely independent of the notion of consumer surplus. It only becomes necessary to actually calculate consumer surplus—as opposed to just knowing it has been maximised—if some measure of the benefits of alternative policies is needed. This is the way it is
employed in section IV. Its use as an objective function throughout this paper is ex-
pository.

Thus consider the problem:
\[
\begin{align*}
\text{maximise} & \quad \sum_i \left[ \int_{g_i}^{\alpha_i} f_i(z) dz + p_i f_i(g_i) - C_i(f_i(g_i)) \right] \\
\text{subject to} & \quad \sum_i [p_i f_i(g_i) - C_i(f_i(g_i))] = \pi \\
& \quad 0 \leq g_i \leq \alpha_i \quad i = 1, 2, 3.
\end{align*}
\]

The first two terms in square brackets in the objective represent total "willingness to pay", from which is subtracted the cost of providing the service. Under the
generalised cost interpretation of the $g_i$, it is implicit in this formulation that the
time component of generalised cost is provided by users. Consumer surplus is to be
maximised subject to the same net revenue constraint as in (4). This problem is a
standard one in the literature and is analysed in great detail by Baumol and Brad-
ford [1].

The necessary conditions for the problem are
\[
-f_i + (1 + \sigma)(p_i - C_i)((f_i + f_i) - \gamma_i) = 0, \quad i = 1, 2, 3.
\]

where the $\gamma_i \geq 0$ are the shadow prices of the price constraints and $\sigma$ is the shadow price of the profit constraint—representing the extra net social welfare which could
be generated by the granting of an extra £1 of subsidy. The sign of $\sigma$ will depend upon the circumstances. Given the way it is used in (14), if the required operating profit is greater than would be earned by marginal cost pricing $\sigma$ will be positive (and vice versa). In that case, reducing $\pi$ by giving £1 of subsidy will allow a movement towards the social optimum. Assuming again that routes 1 and 2 are provided but 3 is not, we have

$$\gamma_1 = \gamma_2 = 0; \gamma_3 > 0, p_3 = \sigma_3.$$  

Hence

$$-f_i + (1 + \sigma)(p_i - C_i) f_i' + f_i = 0 \quad i = 1, 2.$$  \hspace{1cm} (15)$$

Thus

$$\frac{MR_1}{MR_2} = \frac{f_i}{f_z}.$$  \hspace{1cm} (16)$$

Comparing this with (7), we see that here the ratio on the right is the ratio of demand levels rather than the ratio of demand slopes. This is consistent with the fact that with passenger miles maximisation one is asking: "Can I transfer marginal funds from one market to another in such a way that the loss of miles from one market is more than compensated by the gain in miles in the other?", but with net-social-benefit maximisation one is comparing gains and losses of net social benefits brought about by the marginal transfer of funds—which depend upon "willingness to pay" net of costs on the margin, and hence on demand levels.

Rewriting (15)

$$p_i - C_i' = -\left(\frac{\sigma}{1 + \sigma}\right) \frac{f_i}{f_i'}$$

$$= -\left(\frac{\sigma}{1 + \sigma}\right) \frac{g_i}{\eta_i} \quad i = 1, 2.$$  \hspace{1cm} (17)$$

$$\alpha_3 - C_3' < 0.$$  

We see from this that for services which are provided the proportionate deviation from marginal cost should be inversely proportional to price elasticity. For instance, if an operating profit has to be raised relative to marginal cost pricing (that is, if the profit constraint is binding and $\sigma > 0$), the margin of fare over marginal cost should be higher in inelastic markets, because then least distortion is caused relative to marginal cost pricing. Note that no prices should fall below marginal costs, and that if the profit constraint is not binding (so $\sigma = 0$) then all prices should equal marginal costs. This is in contrast to the passenger miles case, where some fares may be above and some below marginal costs. For instance, in markets with very high elasticities ($\eta_i \to -\infty$), (8) implies a limiting price below marginal cost by an amount $1/\lambda$, whereas (17) implies marginal cost pricing.
It is also interesting to note that as the required profit becomes very large, \( \sigma \to \infty \) and \( \lambda \to \infty \), and passenger miles and net consumer surplus maximisation yield the same result, as one would expect. On the other hand, as the subsidy becomes more generous, allowing smaller net revenues to be earned, maximisation of net consumer surplus will yield marginal cost pricing while maximisation of passenger miles will allow prices to continue to fall, some below marginal costs; and, as before, services which would never be offered under the former will be made available under the latter.

We described earlier how the objective of passenger miles maximisation could be made operational by instructing decision-makers to compare the marginal costs (savings) of each project, net of the revenue it would generate (lose), with a standard "pass mark". For net social surplus maximisation, (17) shows that a similar decision rule can be developed. It may be rewritten

\[
p_i\{(1 + \frac{\alpha}{1 + \sigma}) \frac{1}{\sigma_i} \} - C'_i = 0 \quad i = 1, 2
\]

for a provided service. Implementation requires that the modified marginal revenue be equal to marginal cost, the modification being to increase each elasticity by the constant, \((1 + \sigma)/\sigma\). The constant \( \sigma \) is the extra net social benefit attainable by a £1 increase in subsidy, and it takes the place of the pass-mark. It will be noted that decision rule (18) requires no more and no less information than (10).

It is also apparent that the two decision rules will yield different solutions if either elasticities or marginal costs (or both) vary across markets. Equality of elasticities is not sufficient (cf. Quarmby [10], para. 217).

The differences between the policies may become clearer if the simplifying assumptions of linear demand curves and constant marginal costs are imposed (note that marginal cost pricing will then yield exactly zero net revenue). Suppose that

\[
f_i(p_i) = \varepsilon_i + \beta_i p_i \quad \varepsilon_i > 0, \beta_i < 0, \quad i = 1, 2, 3.
\]

Then

\[
\sigma_i = \frac{-\varepsilon_i}{\beta_i}
\]

The pricing rules then become

\[
p_i = \frac{1}{2}(C'_i + \sigma_i - 1/\lambda) \quad i = 1, 2.
\]

\[
\sigma_i < C'_i - 1/\lambda
\]

\[
p_i = \frac{(C'_i + \lambda)\sigma + C'_i}{(1 + 2\sigma)} \quad i = 1, 2.
\]

\[
\sigma_i < C'_i
\]
Finally, we note the connection between Nash’s [8] results and our own. The main difference is that he attributes all costs to the provision of vehicles and (with the exception of a parenthetic remark on p. 74) assumes that demand characteristics will be such that an excess of capacity will always be justified on grounds of service quality. It follows that numbers of passengers can be varied without changing costs, so that optimum fares are zero in the case of unconstrained welfare maximisation. We find this implausible in general, and have assumed that “on the average” carrying more passengers will require more capacity. The distinction between the two approaches becomes much less important when financial constraints are introduced, as his results in his sections 3.1 and 3.2 and his numerical example show when compared with our own results. In any case it is a simple matter to show that if a binding capacity constraint of the form \( Q = kB \) is introduced (where \( k \) is the capacity of a vehicle), and if the resulting Lagrange multiplier is eliminated by substitution from the resulting pairs of conditions, then in every case Nash’s results become identical to our own.

III. WEIGHTED PASSENGER MILES

It has been suggested that it may be possible to devise systems of weighting passenger miles so as to make concessions to various policy objectives. We now show that this is indeed so. First we demonstrate that, if the demand relationships can be satisfactorily approximated by constant elasticity functions (with respect to generalised cost), then simple weights exist so that weighted passenger miles maximisation yields the same result as net social surplus maximisation. The weights are:

\[
W_i(g_i) = \frac{-g_i}{1 + \eta_i}, \quad \eta_i \neq -1, \quad i = 1, 2, 3
\]

where \( \eta_i \) is a constant by assumption.

Weights to achieve economic efficiency

Consider the problem:

\[
\begin{align*}
\text{maximise} & \quad \sum_i W_i(g_i)f_i(g_i) = \sum_i \frac{-g_if_i(g_i)}{(1 + \eta_i)} \\
\text{subject to} & \quad \sum [p_i f_i(p_i) - C_i(f_i(p_i))] = \pi \\
& \quad 0 \leq g_i \leq \alpha_i \quad i = 1, 2, 3.
\end{align*}
\]

The problem has now become, in effect, to minimise the sum of expenditures weighted by \( 1/(1 + \eta_i) \). There are several other ways of looking at the objective. Denoting \( R_i(g_i) \) as the total expenditure in market \( i \)

\[
R_i(g_i) = g_i f_i(g_i)
\]

Then

\[
MR_i(g_i) = \frac{dR_i(g_i)}{dg_i} = f_i(g_i)(1 + \eta_i).
\]

\[1\text{If } \eta_i = -1 \text{ the weight is } W_i(g_i) = -g_i \log (g_i)\]

312
Hence the objective becomes

\[ -\sum_i \left[ \frac{R_i(g_i)}{MR_i(g_i)} \right] f_i(g_i) \]

so that the weights are (minus) the ratios of expenditures to marginal expenditures.

Using \( \omega \) as a shadow price, the first order conditions corresponding to (6) are

\[
\begin{align*}
    -\frac{g_if_i}{(1 + \eta_i)} - \frac{f_i}{(1 + \eta_i)} + \omega(p_i - C_i)f_i' + f_i &= 0 & i = 1, 2 \\
    -\frac{g_3f_3}{(1 + \eta_3)} - \frac{f_3}{(1 + \eta_3)} + \omega(p_3 - C_3)f_3' + f_3 &> 0
\end{align*}
\]

These conditions may be rearranged as

\[
\begin{align*}
    p_i - C'_i &= -\left( \frac{\omega - 1}{\omega} \right) \frac{g_i}{\eta_i} & i = 1, 2 \\
    \alpha_3 - C'_3 &> -\left( \frac{\omega - 1}{\omega} \right) \frac{\alpha_3}{\eta_3}
\end{align*}
\]

Comparing (26) with (17), it will be noted that these conditions are identical.\(^2\) Hence we can be sure that the solution of (26) with the profit constraint will yield the same set of fares and services, with

\[
\left( \frac{\omega - 1}{\omega} \right) = \left( \frac{-\sigma}{I + \sigma} \right)
\]

or equivalently

\[
\sigma = \omega - 1
\]

Thus, by subtracting unity from the shadow price on the weighted passenger miles objective, one obtains the marginal net social benefit of an extra £1 of subsidy. The decision rule is, of course, identical with that embodied in (18). But (18) may be rewritten as follows, using (27):

\[
C'_i - p_i(1 + \left( \frac{\omega - 1}{\omega} \right) \frac{1}{\eta_i}) = 0
\]

or

\[
C'_i - p_i(1 + \frac{1}{\eta_i}) = -\frac{p_i}{\eta_i} \frac{1}{\omega}
\]

Comparing this with the decision rule (10) for maximisation of unweighted passenger miles, we see that the weighted criterion is equivalent to the unweighted one, but with the inverse pass mark \( \frac{1}{\omega} \) replaced by \(-\frac{p_i}{\eta_i} (1/\omega) = (-g_i/\eta_i)(1/\omega)\).

\(^2\) With constant elasticity functions it does not make sense to talk about an \( \sigma \), such that \( f_i(\sigma) = 0 \). This explains the difference between the last relationships of (26) and (17).
In other words, the new "inverse pass mark" $\omega$ should itself be weighted in a manner analogous to the weighting used in the objective. It should be noted that these "pass mark" weights are all positive, while for markets with $-1 < \eta < 0$ the weights in the objective function are negative, but if $\eta > -1$ they are positive. This is because, as we have observed, the objective is equivalent to minimisation of weighted expenditures. This is in order to maximise total net consumer surplus, and that will be best done by raising fares least in elastic markets. Hence elastic markets are given negative weights in the minimisation, or positive weights in the maximisation. The problem and its solution are directly comparable with the dual to the passenger miles maximisation one, in (11).

There is an alternative explanation for the emergence of negative weights. If there is a service with a constant elasticity between 0 and $-1$, and its revenue is not penalised in the objective, it will be possible in that market to raise the relevant price indefinitely and earn an indefinite revenue. This revenue could then be used to generate an indefinite number of passenger miles on other services. Thus it can be shown that first-order conditions such as (8) would define a minimum rather than a maximum.

**Weights to take account of congestion**

It has been suggested that marginal cost pricing is in any case not socially optimum because of "second best" arguments concerning externalities such as traffic congestion, and that this reduces the "errors" introduced by maximisation of passenger miles. To an extent this may be so. If an elastic market is also an off-peak one, both approaches would suggest depressing its fare below marginal cost. However, this would not be true of an inelastic peak market: externality arguments would suggest a fare again below marginal social cost, but passenger miles maximisation would lead to one much above it. A characteristic of maximisation of passenger miles is that it may involve the earning of profits in some markets in order to subsidise others, whereas this is no part of "modified marginal cost pricing". In any case, there is no a priori reason to expect that, in those cases where the two approaches do suggest a movement in the same direction, the discounts will be of similar magnitude. They occur for different reasons.

This leads us to enquire whether a system of weighting passenger miles can be devised which would assist the attainment of "second best" policies and income redistribution policies.

Suppose, then, that $W_i$ are any constant weights, and relax the assumption of constant elasticity demand relations. Replacing the objective in (4) by

\[
\text{maximise} \quad W_i f_i (g_i) + W_j f_j (g_j) + W_k f_k (g_k) \tag{30}
\]

and using the same constraints, it is easily shown that (8) becomes

\[
\begin{align*}
\rho_i - C_i' &= \frac{W_i}{\lambda} \frac{g_i}{\eta_i} , \quad i = 1, 2 \\
\alpha_i - C_i' &= -\frac{W_i}{\lambda} \frac{f_i (\alpha_i)}{f_i'}
\end{align*} \tag{31}
\]
It follows immediately that attaching constant weights to passenger miles is equivalent to attaching the same constant weights to the inverse shadow prices used in the decision values. Thus, if for some reason different types of trips are valued differently by a policy maker who will specify appropriate weights, it is easy to make this operational.

Unfortunately, not many policy considerations are naturally expressed in terms of weightings for passenger miles. For instance, Glaister and Lewis [5] show that in the constant elasticity case the "second best" congestion-reducing argument would imply pricing rules of the form

$$ p_i - C_i = \frac{\theta k_i}{f_i(g_i)} \quad (31a) $$

where $\theta$ is a constant depending on the marginal social costs of private car use and $k_i$ depends upon the cross-elasticity between the cost of service $i$ and private car use. We cannot see how this could be achieved by weighting.

**Weights for welfare redistribution**

Turning now to considerations of welfare distribution, let the average income of individuals using service $i$ be $y_i$ and let $u'(y_i)$ be the social valuation to be attached to the welfare for this group, as a function of $y_i$. Thus

$$ \begin{align*}
\text{maximising} & \quad \sum_i \left[ u'(y_i) \int_{g_i}^{a} f_i(z)dz + p_i f_i(g_i) - C_i f_i(g_i) \right] \\
\text{subject to} & \quad \sum_i \left[ p_i f_i(p_i) - C_i f_i(p_i) \right] \geq \pi \\
& \quad \text{yields} \quad p_i - C_i = -\left(1 - \frac{u'(y_i)}{(1+\alpha)} \right) \frac{g_i}{\eta_i} \quad (31c)
\end{align*} $$

which is a simple generalisation of (17). Inspection of the argument leading from (24) to (26) shows that in the constant elasticity case this may be simply achieved by using weighted passenger miles, with weights

$$ W_i(g_i) = -\frac{u'(y_i)g_i}{(1+\eta_i)} \quad (31d) $$

However, a full analysis of welfare redistribution must take into account the means by which the subsidy $\pi$ is to be raised. The following argument illustrates how this might be done; alternative taxation schemes would require different analyses. Following Feldstein [3], let the number of individuals in group $i$ be $n_i$ and

$$ N = \sum n_i. \quad (32) $$

Suppose that each member of the population must (say through a property tax independent of income) pay the same proportion of operating losses:

$$ T = \frac{1}{N} \left( \sum [C_i f_i] - p_i f_i(g_i) \right) + F \quad (33) $$

where $F$ is the system total fixed cost.
Net benefit to group $i$ is then
\[ \int_{g_i}^{\alpha} f_i(z) \, dz - \frac{n_i}{N} \sum_{i} \left[ C_i(f_i) - p_i f_i(g_i) + F \right] \] (34)

and the weighted sum of net benefits can therefore be written
\[ \sum_{i} u'(y_i) \left( \int_{g_i}^{\alpha} f_i(z) \, dz - \frac{n_i}{N} \sum_{i} \left[ C_i(f_i) - p_i f_i(g_i) + F \right] \right) \]
(35)

where $u'(y_i)$ gives the weight to be attached to group $i$ as before. This is to be maximised without constraint. The first-order conditions are
\[-u'(y_i)f_i(g_i) - \left( \frac{\sum n_i u'(y_i)}{N} \right) ((C_i' - P_i)f_i' - f_i) = 0 \]
(36)

which with arrangements yields
\[ p_i - C_i' = \left( \frac{u_i(y_i)}{\sum n_i u'(y_i)} - 1 \right) \frac{g_i}{n_i} \]
(37)

Note that this simply implies that, for groups with a "social valuation" greater than the average across all individuals, the price will be below marginal cost (since $\eta_i < 0$) and vice versa. If social valuations were independent of income, marginal cost pricing would again result.

Assuming once again that the demand functions have the constant elasticity form, the decision rules embodied in (37) may be obtained by maximising weighted expenditures subject to constraint. Here the weights are $-W_i$ rather than $-1/(1 + \eta_i)$ as they were in the net social surplus maximisation of (24). Consider then

maximise
\[ \sum_{i} \left[ -W_i g_i f_i(g_i) \right] \]
subject to
\[ \sum_{i} \left[ p_i f_i(g_i) - C_i(f_i(g_i)) \right] = \pi \]
(38)

An exactly parallel argument to (25) and (26) shows that the first order conditions will be
\[ p_i - C_i' = -\left( \frac{\omega - (1 + \eta_i)W_i}{\omega} \right) \frac{g_i}{\eta_i} . \]
(39)

Comparing (39) with what is required in (37) shows immediately that the weights $W_i$ must be given by
\[ W_i = \frac{\omega u'(y_i)}{\frac{1}{\sum n_i u'(y_i)} \left( 1 + \eta_i \right)} \]
(40)

or
\[ W_i = \frac{\omega u'(y_i)}{\left( 1 + \eta_i \right)} \]
(41)

\[ ^1 \text{This is the same as Feldstein's (6), p. 178, except that he constrains all prices to be the same.} \]
Equation (39) can then be written as

$$C_i' = p_i \left( \frac{1}{1 + \hat{\eta}_i} \right) = \frac{-p_i}{\hat{\eta}_i} \cdot \frac{1}{\sum N_i u'(y_i)}$$

(42)

for comparison with (29) and (10).

Equations (40) state that the weights on revenues should be proportional both to the inverses of the elasticities increased by unity and to the "marginal social valuation" of income to the users of the service, normalised by the weighted average of these valuations. Equation (42) says that in investigating projects marginal costs net of marginal revenues should be compared with an "inverse pass mark" which is \((p_i/\hat{\eta}_i)\) as before, weighted by the normalised distributional weight. Markets which are given a high social valuation relative to the average will have marginal revenues which are small relative to marginal costs.

IV. NUMERICAL EVALUATIONS

In this section we illustrate the arguments of earlier sections by presenting the results of numerical calculations. They should be regarded as illustrative only; although the data used are intended to be representative of London Transport’s bus and rail services, they are inevitably subject to considerable uncertainty. Throughout this section linear demand relations are used, and it is assumed that there is no cross elasticity between modes. This latter assumption is almost certainly false in the case of L.T.'s services (see Fairhurst[2]). In this section time costs are suppressed, and generalised cost is therefore identified with money fare.

Calculations are carried out with two sets of own-price elasticities. In the first, bus elasticity is –0.64 and rail is –0.4, to correspond with the values determined by London Transport [6]. In the second case bus elasticity is –0.8 and rail is –0.2, to cover the range of 0.2:2 quoted by Quarmby ([10], para. 217).

Marginal costs are assumed constant and equal to 7 and 2 pence per passenger mile for bus and rail respectively. These values are a rough average of most recent results obtained by LT from its bus scheduling model and other sources (see Glaister and Lewis [5]). Demand levels at 1975 fares levels were taken as 27.5\(\times\)10\(^6\) and 32.5 \(\times\) 10\(^6\) passenger miles per five-day week respectively, and shares in total Greater London expenditure were calculated as 0.82% and 0.87% from Family Expenditure Survey and LT data.

Linear demand relations are used as specified in equations (19) to (22). Approximating 1975 fare levels by 4 pence in each case, this gives

$$\varepsilon_1 = 45.358 \times 10^6, \varepsilon_2 = 45.5 \times 10^6, \beta_1 = -4.479 \times 10^6, \beta_2 = -3.25 \times 10^6$$

for the first pair of elasticities. In the second case the corresponding values are

$$\varepsilon_1 = 49.5 \times 10^6, \varepsilon_2 = 39.0 \times 10^6, \beta_1 = -5.5 \times 10^6, \beta_2 = -1.625 \times 10^6.$$

Table 1 shows the results. For each of a variety of required operating profit levels, \(\pi\), it shows the solutions to the first-order conditions for passenger miles.
### Table 1

<table>
<thead>
<tr>
<th>Profit Lm per week</th>
<th>Profit Lm p.a.</th>
<th>Pass* Mark</th>
<th>Loss %</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0</td>
<td>-52.0</td>
<td>6.48</td>
<td>0.25</td>
</tr>
<tr>
<td>-0.8</td>
<td>-41.6</td>
<td>6.57</td>
<td>0.35</td>
</tr>
<tr>
<td>-0.6</td>
<td>-31.2</td>
<td>6.67</td>
<td>0.73</td>
</tr>
<tr>
<td>-0.3</td>
<td>-15.6</td>
<td>6.82</td>
<td>1.35</td>
</tr>
<tr>
<td>+0.2</td>
<td>10.4</td>
<td>7.13</td>
<td>2.49</td>
</tr>
<tr>
<td>+0.3</td>
<td>15.6</td>
<td>7.19</td>
<td>2.76</td>
</tr>
<tr>
<td>0.4</td>
<td>20.8</td>
<td>7.26</td>
<td>3.05</td>
</tr>
<tr>
<td>0.6</td>
<td>31.2</td>
<td>7.42</td>
<td>3.64</td>
</tr>
<tr>
<td>0.8</td>
<td>41.6</td>
<td>7.61</td>
<td>4.34</td>
</tr>
<tr>
<td>1.0</td>
<td>52.0</td>
<td>7.83</td>
<td>5.21</td>
</tr>
<tr>
<td>1.15</td>
<td>59.8</td>
<td>8.08</td>
<td>6.11</td>
</tr>
<tr>
<td>1.25</td>
<td>65.0</td>
<td>8.34</td>
<td>7.15</td>
</tr>
</tbody>
</table>

### Table 1 (continued)

<table>
<thead>
<tr>
<th>Profit Lm per week</th>
<th>Profit Lm p.a.</th>
<th>Pass* Mark</th>
<th>Loss %</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0</td>
<td>-52.0</td>
<td>6.82</td>
<td>0.01</td>
</tr>
<tr>
<td>-0.8</td>
<td>-41.6</td>
<td>6.85</td>
<td>0.36</td>
</tr>
<tr>
<td>-0.6</td>
<td>-31.2</td>
<td>6.89</td>
<td>0.76</td>
</tr>
<tr>
<td>-0.3</td>
<td>-15.6</td>
<td>6.94</td>
<td>1.37</td>
</tr>
<tr>
<td>+0.2</td>
<td>10.4</td>
<td>7.04</td>
<td>2.44</td>
</tr>
<tr>
<td>+0.3</td>
<td>15.6</td>
<td>7.06</td>
<td>2.66</td>
</tr>
<tr>
<td>0.4</td>
<td>20.8</td>
<td>7.08</td>
<td>2.89</td>
</tr>
<tr>
<td>0.6</td>
<td>31.2</td>
<td>7.13</td>
<td>3.36</td>
</tr>
<tr>
<td>0.8</td>
<td>41.6</td>
<td>7.16</td>
<td>3.85</td>
</tr>
<tr>
<td>1.0</td>
<td>52.0</td>
<td>7.21</td>
<td>4.36</td>
</tr>
<tr>
<td>1.15</td>
<td>59.8</td>
<td>7.25</td>
<td>4.75</td>
</tr>
<tr>
<td>1.25</td>
<td>65.0</td>
<td>7.27</td>
<td>5.03</td>
</tr>
</tbody>
</table>

*In passenger miles per £.

maximisation (8) and the net social surplus maximisation (17). The value of the pass mark (i.e. \(-1/\lambda\)) is also given in passenger miles per £, and so is the percentage welfare loss attributable to the difference between the passenger miles maximising prices and net-social-surplus maximising prices (the latter prices being the economically efficient solutions). These calculations are repeated for the alternative sets of elasticities. It should be noted that the required "profits", \(\pi\), are what must be provided out of operations as a contribution to cover that part of fixed costs which is not covered by external subsidy.

The following points will be noted. The general properties predicted in previous
sections are confirmed. Passenger miles may give one price above and one below marginal cost, while net consumers’ surplus always has both above for positive profits or both below for negative profits. With the second set of elasticities PM gives an interesting “reversal”: \( p_1 < p_2 \) in spite of the much higher bus marginal cost.

There are numerous large differences between the fares calculated under the two policies, especially with the second, more disparate, pair of elasticities. The extreme reluctance to raise the fare in the more elastic market will be noted in this case. The results are qualitatively similar to those obtained by Nash ([8] pp. 80, 81).

The calculated welfare losses, though variable, are apparently small, except in extreme circumstances. However, this conclusion requires caution. In the first place, as Glaister [5] has shown, the neglect of cross elasticities can cause a serious under-estimate of true welfare losses—perhaps by a factor of up to 5. Secondly, the smallness must be placed in the context of the assumed small proportions of total Greater London expenditure represented by bus and tube travel. Thus a loss of say 0.5% of all welfare represents a loss of about \( 0.005 \times (0.0082 + 0.0087) \times 100 = 29.6\% \) of expenditure on LT public transport. At 1972 prices a 0.5% welfare loss represents about 20p per household per week, or a total of £28.7m per year. In the case with elasticities in the range \( 1/2:2 \) mentioned by Quarmby [10], losses are of this order of magnitude.

V. SERVICE QUALITY

It was explained in section I that as far as price variations are concerned the results derived in sections II to IV are equally valid whether the “price” variable is interpreted as generalised cost or as money price. This section extends the analysis to consider the effect of variations in generalised cost through changes in service quality, as measured by changes in travel times per mile, \( t_i \).

Generalised cost is given by

\[
g_i = p_i + \tau_i t_i
\]

where \( \tau_i \) is the value of time to users of service \( i \). Suppose that operators can vary both \( p_i \) and \( t_i \). Changes in service quality will, other things being equal, involve changes in costs, so the cost function will become

\[
C_i \{ f_i(g_i), t_i \} \text{ with } \frac{\partial C_i}{\partial t_i} \leqslant 0
\]

It is a simple matter to show that each one of the first-order conditions, and pricing rules following from them, which have been derived so far is unchanged so long as \( C_i \{ f_i(g_i) \} \) is interpreted to mean \( \delta C_i \{ f_i(g_i), t_i \}/\delta f_i(g_i) \).

It is also a straightforward matter to find the relevant first-order conditions for maximisation with respect to the \( t_i \) in each case by differentiation. It happens that in every case these can be reduced to the simple form.
by exploiting the relevant pricing conditions.

Note that to say that all the problems have the same time conditions (45) is not to say that every approach will lead to the same service quality. This is because, in each case, the time conditions have to be solved simultaneously with the respective pricing conditions, and these differ substantially.

Service externalities

These results do depend upon the particular form chosen for the influence of timings on costs. This is fairly general, but it is clearly not the only possible way of formulating the problem. Further, it does not take account of any "externalities" due to the influence of service frequencies on waiting times, of the kind discussed by H. Mohring [7].

A simple means of incorporating such "externalities" into the previous analysis is to assume that, at any given level of demand, there is a minimum service level required to satisfy capacity constraints. Since service levels are inversely related to journey times, we can write

\[ t_i \leq T_i[f_i(g_i)] \]

Should the service level provided be more than the minimum required by capacity constraints, this maximum time constraint will not be binding and the rules for pricing and service provision developed above will be unchanged. On the other hand, if the constraint is binding, a further term will be added to each of the expressions obtained so far for the difference between price and marginal cost. In general this term is negative and of the form:

\[ \frac{\partial_i T_i^i}{\gamma} \text{ with } T_i^i = \frac{\partial T_i}{\partial f_i}, \]

where the \( \theta_i \) are the Kuhn-Tucker multipliers on the maximum time constraints and \( T_i^i \) represents the effect on journey times of patronage changes, operating through service level adjustments. The constant \( \gamma \) is equal to the multiplier on the profit constraint in all cases except for those represented by (31c) and (37). In (31c) the constant is equal to the multiplier on the profit constraint increased by one, and in (37) there is no profit constraint. The term given in (46) measures the amount by which price on a particular service should be reduced relative to marginal cost to allow for the benefits, in terms of higher service levels, that accrue to intramarginal passenger miles as a result of marginal increases in patronage. In determining the size of this price reduction, there are two effects of marginal changes in patronage to be balanced. On the one hand, if patronage on a given service increases, the consequent increase in service levels will tend to increase the maximand. On the other hand, net revenue on the service in question will be reduced, and hence net revenues on other services will have to be increased to meet the profit constraint, with a consequent reduction in the maximand. The term given in (46) takes account of the former effect through \( \theta_i T_i^i \) and of the latter effect through \( \gamma \).
VI. SUMMARY AND CONCLUSIONS

We have compared and contrasted the implications of a public transport organisation operating under a fixed subsidy (or profit) constraint adopting a policy of maximising the total number of passenger miles that it sells as opposed to the "classically correct" prescription of some form of marginal cost pricing. We have shown that "naive" passenger miles maximisation will only yield similar results to the socially optimum policy under stringent conditions which are known to be violated in many transport systems. Illustrative calculations using data representative of London show that the differences between the policies are not trivial, but will lead to significant differences in the net social welfare generated.

However, we have also demonstrated the existence of a weighting scheme which, on the assumption of constant elasticities with respect to generalised cost, would lead to similar solutions for the two policies. Although the weights involved are not constant, it is possible to work out constant approximations which are valid for non-extreme changes in fares or service qualities and to derive rules for the practitioner from them. Similarly, weighting schemes exist to achieve certain objectives in the distribution of welfare. We have been unable to specify a weighting scheme that would approximate to "second best" pricing designed to compensate for traffic congestion externalities.

We have attempted to draw attention to weaknesses in the criterion of passenger mile maximisation, but we would accept some of the claims that have been made in its favour (although we are not convinced that its correct operation requires any less information). As a device for decentralising decisions it is more easily understood and accepted as a relevant criterion by those who have to operate it. It may well be that the welfare losses caused by the price distortions it introduces in practice are totally insignificant when set against the waste due to inefficiencies, mismanagement and a failure to relate what is provided to what the public actually wants. If so, we would agree that its advantages may, in practice, outweigh its drawbacks.

REFERENCES