INFERRING ORDINARY ELASTICITIES FROM
CHOICE OR MODE-SPLIT ELASTICITIES

By John H. E. Taplin*

In a report on work done to derive a system of vacation travel elasticities from a limited number of estimates (Taplin, 1980), it was suggested that choice or mode-split elasticities may be valuable for this purpose. If choice elasticities are to be used in this way they must be transformed to ordinary demand elasticities. This paper has two purposes: to set out the general relation between mode-split and ordinary price elasticities, and to consider ways of inferring the latter from the former. It can be done if there is an estimate of at least one ordinary elasticity from which to construct a “second stage” elasticity, and indirectly a set of them, to link mode-split with ordinary elasticities. Even when a separate estimate is not available, the system constraints on the two elasticity matrices narrow the range of feasible values and may enable ordinary elasticities to be inferred.

Many studies of travel demand have dealt only with mode choice, which is one part of the price responsiveness reflected in a system of ordinary demand equations. Only in the extreme case of a fixed total of trips do the ordinary elasticities become identical with the mode-split elasticities. The connection between mode-split and ordinary elasticities is rarely discussed.¹

Intuitively, one would expect mode-split elasticities to be derivable from a set of ordinary elasticities, because the latter embody the effects of substitution between modes as well as income effects and substitutability for other goods. But the derivation of mode-split elasticities from ordinary elasticities means a loss of information, since it uses only the inter-modal substitution component. It would therefore rarely be a useful exercise.

---

* Professor of Transport Economics, University of Tasmania. The paper originated from discussions with colleagues at the University of British Columbia, and has benefited from the comments of W. G. Waters while he was visiting the University of Tasmania. Without his encouragement the paper would not have been written. Thanks are also due to W. Magill of this university for checking the Appendix and to P. W. Blackshaw of the BTE and the World Bank for first drawing the author’s attention to Quandt’s result. The author is grateful to the Editors and a reviewer for suggesting a number of improvements.

¹ A number of writers have mentioned the relation between direct demand and mode choice models (e.g., McGillivray, 1970; Talvitie, 1973; Domeneich and McFadden, 1975; Gaudry, 1979). Various workers have approached demand estimation in two stages, mode-split and total demand (e.g., McLynn et al., 1968; Gaudry and Wills, 1978); and some have commented on a second stage elasticity which related ordinary elasticities to mode-split elasticities (e.g., Wills, 1978; Taplin, 1980; Truong, 1980). The relation between mode-split and ordinary elasticities was shown by Quandt (1968) for the two-mode case.
Conversely, one would not expect to be able to transform directly from mode-split to ordinary elasticities, because additional information is required. However, it would be useful to be able to do so, because ordinary elasticities can be used for direct prediction. The main purpose of the paper is to see what additional information is needed to achieve this transformation.

Mode-split elasticity is defined in the next section. Many estimates of market share or mode choice elasticities, particularly with respect to own-price, are based on models which are consistent with this definition. Nevertheless, anyone drawing inferences from estimates should satisfy himself that they are appropriate for the application in hand. In attempting to infer ordinary elasticities from share or choice elasticities, one should be satisfied not only that the share or choice elasticities are mode-split elasticities as defined, but also that they conform to the constraints which—as will be shown in this paper—mode-split elasticities must satisfy.

The analysis is carried out for \( n \) modes, the first step being to generalise Quandt's (1968) two-mode result. After that a three-mode numerical example is used to illustrate how one can infer ordinary from mode-split elasticities.

**THE GENERAL RELATIONSHIP**

Let \( E \) be an \( n \times n \) matrix of ordinary own-price and cross-price elasticities of demand, in which an element \( \varepsilon_{ij} \) is the ordinary elasticity of demand for trips \( (T_i) \) on the \( i \)th mode with respect to the price or fare \( (p_j) \) of the \( j \)th mode.

\[
\varepsilon_{ij} = \frac{\partial T_i}{\partial p_j} \frac{p_j}{T_i}
\]

\( M \) is the corresponding \( n \times n \) matrix of own-price and cross-price mode-split elasticities, in which an element \( m_{ij} \) is the mode-split elasticity of the \( i \)th mode's share of trips with respect to the price of the \( j \)th mode.

\[
m_{ij} = \frac{\partial s_i}{\partial p_j} \frac{p_j}{s_i} \quad \text{where} \quad s_i = \frac{T_i}{T_1 + T_2 + \ldots + T_n},
\]

\( T_1, T_2, \ldots, T_i, \ldots, T_n \) being trips on the various modes. \( S \) is an \( n \times n \) matrix of mode shares in which the row vector \( s_1, s_2, \ldots, s_n \) is repeated \( n \) times:

\[
\begin{bmatrix}
  s_1 & s_2 & \ldots & s_n \\
  s_1 & s_2 & \ldots & s_n \\
  \vdots & \vdots & \ddots & \vdots \\
  s_1 & s_2 & \ldots & s_n
\end{bmatrix}
\]
The matrix \( I - S \) is formed:

\[
I - S = \begin{pmatrix}
(1 - s_1) & -s_2 & \cdots & -s_n \\
-s_1 & (1 - s_2) & \cdots & -s_n \\
\vdots & \vdots & \ddots & \vdots \\
-s_1 & -s_2 & \cdots & (1 - s_n)
\end{pmatrix}_{n \times n}
\]

It can be shown (see Appendix) that the matrix of mode-split elasticities, \( M \), is found as follows:

\[
M = (I - S)E
\]

The matrix \((I - S)\) is singular because each row sums to zero, so that it is not possible to derive the ordinary elasticities (matrix \(E\)) directly from the mode-split elasticities (matrix \(M\)). Thus, one can readily transform from ordinary to mode-split elasticities, but not the converse.

Nevertheless, each ordinary elasticity is related to the corresponding mode-split elasticity in a straightforward way. From

\[
M = (I - S)E
\]

we obtain

\[
(E - M) = SE
\]

Thus, for element \(ij\),

\[
(E - M)_{ij} = (SE)_{ij} = \sum_{k=1}^{n} S_{ik} E_{kj}
\]

\[
= \sum_{k=1}^{n} S_{kj} E_{kj} \quad \text{(the rows of } S \text{ being identical)}
\]

\[
= \eta_j \text{ for each } i, j.
\]

This means that \(\eta_j\) added to each mode-split elasticity in column \(j\) of matrix \(M\) gives the corresponding ordinary elasticity in column \(j\) of matrix \(E\). It can conveniently be called a second stage elasticity. By definition, \(\eta_j\) is the elasticity of demand for all trips \((T)\) on the \(n\) modes with respect to the fare or cost of travelling on mode \(j\),

\[
\eta_j = \sum_k s_k E_{kj} = \left( \frac{\partial T_1}{\partial p_j} + \ldots + \frac{\partial T_n}{\partial p_j} \right) \frac{p_j}{T_1 + \ldots + T_n} = \frac{\partial T}{\partial p_j} \frac{p_j}{T}.
\]

When the second stage elasticities are summed.
\[
\sum_j \eta_j = \sum_j \sum_k s_{kj} E_{kj} \\
= \frac{\partial T}{\partial p_1} \frac{p_1}{T} + \ldots + \frac{\partial T}{\partial p_n} \frac{p_n}{T}.
\]

This is the aggregate elasticity of demand for all trips with respect to a uniform proportional travel cost change in all \( n \) modes.

THE SPECIAL CASE

If the total number of trips is fixed, total travel does not respond to any individual travel cost, so that

\[ \eta_1 = \eta_2 = \ldots = \eta_n = 0. \]

It follows that, in this case only, the matrix of ordinary elasticities, \( E = M \), the matrix of mode-split elasticities.

This special case is of interest because it indicates that where total trips tend to be fixed, as with work trips in the very short run, it may be reasonable to treat mode-split elasticity estimates, which conform to the constraints, as approximations to the ordinary elasticities. In any longer planning period this would not be an acceptable assumption.

IMPLICATIONS FOR PRACTICAL ESTIMATION

If it is decided that two-stage estimation is an appropriate method (cf. Gaudry and Wills, 1978), ordinary elasticities could be obtained from mode-split elasticities by also estimating each second stage elasticity of demand for total travel with respect to the price on mode \( j \) (i.e., \( \eta_j \)).

Even when these second stage estimates are lacking, it will usually be possible to make some inferences. This follows from the fact that the demand system is heavily constrained:

(a) Matrix \( M \) has the property that, for any column, \( j \),

\[ \sum_{k=1}^{n} s_k m_{kj} = 0. \]

This means that whatever share of trips is diverted from the \( j \)th mode by a rise in the \( j \)th price will exactly equal the increase in the shares of the other modes.

(b) The mode-split and ordinary elasticities in any column are related by the single constant, \( \eta_j \).
INFERRING ORDINARY ELASTICITIES

(c) The ordinary cross-elasticities in matrix $E$ must satisfy the symmetry condition:

$$\varepsilon_{ij} = \frac{\text{relative expenditure on } j}{\text{relative expenditure on } i}.$$

(d) Because we are dealing with competing transport alternatives, they are gross substitutes, so that all the cross-elasticities in $E$ are positive.\(^3\)

Clearly, it would be pointless to attempt the transformation if constraint (a) were not satisfied. The estimating procedure should constrain the mode-split elasticity estimates to satisfy this relation. In the two-mode case, if the own-price mode-split elasticity is known, the cross-price mode-split elasticity in the same column is determined exactly.

Together, (b) and the symmetry condition (c) give the following set of relationships for the three-mode case:

\[
\begin{align*}
    m_{21} + \eta_1 &= \frac{s_1 p_1}{s_2 p_2} (m_{12} + \eta_2) \\
    m_{31} + \eta_1 &= \frac{s_1 p_1}{s_3 p_3} (m_{13} + \eta_3) \\
    m_{32} + \eta_2 &= \frac{s_2 p_2}{s_3 p_3} (m_{23} + \eta_3)
\end{align*}
\]

where $p_i$ is the actual or relative cost or price of using mode $i$. However, this system has no solution because the matrix of coefficients of $\eta_1$, $\eta_2$, and $\eta_3$ is singular. This follows from the way the $(s_i p_i)$ enter as ratios in the system. A solution would not be obtained even if an independent estimate of the aggregate elasticity were available.

Nevertheless, the following example is used to show that symmetry and non-negativity make it feasible to infer from mode-split to ordinary elasticities.

A DISCRETIONARY TRAVEL EXAMPLE

Although the relationships presented in this paper are applicable to other types of choice, such as freight shipment, they are likely to be of greatest importance in the study of discretionary passenger travel. Substantial differences between ordinary and mode-split elasticities can be expected in this case.

The study by Taplin (1980) used available estimates and various theoretical constraints, including symmetry, to synthesize a complete system of ordinary demand

\(^2\) This is the simple (Hotelling-Jureen) form. See Taplin (1980).

\(^3\) The modes are almost certainly Hicks-Allen substitutes as well, but gross substitutes are all that is required. See Layard and Walters (1978), p. 141.
elastocities for vacation travel by Australians. The following hypothetical example has been adapted from the results of that study. It refers to internal vacation travel by air, car and bus, the shares of trips and the expenditure being:

<table>
<thead>
<tr>
<th>Proportion of Trips</th>
<th>Total Expenditure</th>
<th>(money units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air (mode 1)</td>
<td>0.2</td>
<td>40</td>
</tr>
<tr>
<td>Car (mode 2)</td>
<td>0.7</td>
<td>60</td>
</tr>
<tr>
<td>Bus (mode 3)</td>
<td>0.1</td>
<td>10</td>
</tr>
</tbody>
</table>

The matrix of ordinary elasticities is:

$$E = \begin{bmatrix} -1.5 & 0.3 & 0.1 \\ 0.2 & -1.7 & 0.1 \\ 0.4 & 0.6 & -1.2 \end{bmatrix}.$$ 

The resulting matrix of mode-split elasticities is:

$$M = \begin{bmatrix} -1.38 & 1.37 & 0.13 \\ 0.32 & -0.63 & 0.13 \\ 0.52 & 1.67 & -1.17 \end{bmatrix}.$$ 

The second stage elasticities are:

$$\eta_1 = -0.12$$
$$\eta_2 = -1.07$$
$$\eta_3 = -0.03$$

The aggregate elasticity, \(\sum_{i=1}^{3} \sum_{j=1}^{3} s_j e_{ij} = -1.22 = \eta_1 + \eta_2 + \eta_3\). The mode-split elasticities with respect to the price of a mode having a small share (column 3) are very similar to the corresponding ordinary elasticities, but the mode-split elasticities with respect to the price of a mode having a large share (column 2) differ greatly from the corresponding ordinary elasticities. The difference between the elasticities is greater for a mode with a large share, \(s_j\), because the (large) own price elasticity \(e_{ij}\) is multiplied by a small \((1 - s_j)\) in forming \(\eta_j\).

**INFERRING THE ORDINARY ELASTICITIES**

If an ordinary elasticity estimate is available, the corresponding mode-split elasticity is subtracted from it to give the \(\eta\) for that column; the other \(\eta\)'s follow by symmetry. This method can only be used with confidence if the ordinary elasticity is with respect to the price of a mode having a substantial share. Otherwise errors will be magnified (the effect of symmetry). To illustrate, using the example, it is assumed that the mode-split elasticities (matrix \(M\)) are known, but not the ordinary elasticities (matrix \(E\)).

60
INFERRING ORDINARY ELASTICITIES

If the only ordinary elasticity estimate available is an own-price elasticity of \(-1.8\) (deviating by \(-0.1\) from the true) with respect to costs by car (mode 2), which has a 70% share of the traffic:

\[
\text{estimated } E = \begin{bmatrix} -1.567 & 0.20 & 0.083 \\ 0.113 & -1.80 & 0.083 \\ 0.333 & 0.50 & -1.217 \end{bmatrix}, \quad \text{true } E = \begin{bmatrix} -1.5 & 0.3 & 0.1 \\ 0.2 & -1.7 & 0.1 \\ 0.4 & 0.6 & -1.2 \end{bmatrix}.
\]

The inferred elasticities with respect to the prices of the modes having the relatively small shares (columns 1 and 3) are even better approximations to the true elasticities than the elasticity estimate of \(-1.8\) on which they are based.

In contrast, an unacceptable result would follow from using an elasticity estimate of \(-1.3\) (also deviating by \(-0.1\) from the true) with respect to the fare by bus (mode 3), which has only a 10% share of the traffic:

\[
\text{estimated } E = \begin{bmatrix} -1.90 & -0.30 & 0 \\ -0.20 & -2.30 & 0 \\ 0 & 0 & -1.30 \end{bmatrix}.
\]

This estimate of \(E\) would be rejected because it contains a pair of negative cross-elasticities. Applying the non-negativity constraint would result in a far better estimate:

\[
\text{estimated } E = \begin{bmatrix} -1.70 & 0 & 0.05 \\ 0 & -2.00 & 0.05 \\ 0.20 & 0.30 & -1.25 \end{bmatrix}.
\]

If the homogeneity condition, that all the elasticities in a row sum to zero, is applied to this result (cf. Taplin, 1980), it is found to imply the following:

<table>
<thead>
<tr>
<th>Sum of income elasticity and other cross-elasticities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
</tr>
<tr>
<td>Car</td>
</tr>
<tr>
<td>Bus</td>
</tr>
</tbody>
</table>

It can be assumed that the income elasticity makes up most of the sum of elasticities in each case. These estimates are high (see Taplin, 1980) and could be reduced by making an assumed second-stage elasticity, \(\eta\), less negative (and so reducing the others as well).

The implied aggregate elasticity of \(-1.77\) could also be compared with other information. It would probably be concluded that demand for internal vacation travel, in aggregate, is not as elastic as this; the appropriate change would be the same as indicated on the basis of the implied income elasticities.
CONCLUSIONS

This paper has shown the relation between mode-split elasticities and ordinary elasticities for \( n \) transport modes. While mode-split elasticities can be derived from ordinary elasticities, the converse is not true. An additional second stage elasticity, reflecting the responsivenes of total travel to the fare or cost on one mode, is needed to link mode-split elasticities to ordinary elasticities.

Even when estimates of the second stage elasticities are unavailable, the system of demand elasticities is so constrained that the \( n \) second stage elasticities can be inferred from an estimate of one ordinary elasticity. If there is no such estimate, the system constraints on the two elasticity matrices will still narrow the range of possible values that the unknown ordinary elasticities can take.

APPENDIX

With the terms defined as in the body of the paper, the mode-split elasticity of demand for mode 1 with respect to the price of mode \( j \),

\[
m_{1j} = \frac{\partial s_1}{\partial p_j} \frac{p_j}{s_1} = \frac{\partial (T_i/(T_1 + T_2 + \ldots + T_n))}{\partial p_j} \frac{p_j}{T_i/(T_1 + T_2 + \ldots + T_n)}
\]

\[
= \frac{(T_1 + T_2 + \ldots + T_n) \frac{\partial T_1}{\partial p_j} - T_1 \left( \frac{\partial T_1}{\partial p_j} + \frac{\partial T_2}{\partial p_j} + \ldots + \frac{\partial T_n}{\partial p_j} \right)}{(T_1 + T_2 + \ldots + T_n)^2} \frac{p_j}{T_i/(T_1 + T_2 + \ldots + T_n)}
\]

\[
= \frac{(T_2 + T_3 + \ldots + T_n) \left( \frac{\partial T_1}{\partial p_j} \frac{p_j}{T_1} - T_2 \left( \frac{\partial T_2}{\partial p_j} \frac{p_j}{T_2} - T_3 \left( \frac{\partial T_3}{\partial p_j} \frac{p_j}{T_3} - \ldots - T_n \left( \frac{\partial T_n}{\partial p_j} \frac{p_j}{T_n} \right) \right) \right) \right)}{T_1 + T_2 + \ldots + T_n}
\]

\[
= (1 - s_1)e_{1j} - s_2 e_{2j} - s_3 e_{3j} - \ldots - s_n e_{nj}.
\]

The other mode-split elasticities are obtained similarly. In matrix form, the mode-split elasticities can be expressed as

\[
(I - S) E
\]

\[
n \times n \quad n \times n
\]

where \( I \) is the identity matrix and \( S \) and \( E \) are defined as in the body of the paper.

62
INFERRING ORDINARY ELASTICITIES

REFERENCES


