A RATIONAL ALTERNATIVE FARE STRUCTURE FOR BRITISH RAIL'S LONDON AND SOUTH-EAST COMMUTER PASSENGERS

By J. G. Gibson*

INTRODUCTION

Consider the following hypothetical situation. A commuter train takes on its full complement of passengers (100% utilisation) and transports them 50 miles to a terminus in London, where they all detrain. This idyllic state of affairs is disturbed when a lone would-be commuter requests that the train be despatched a further 5 miles down the line so that he too may enjoy the same privilege as the other passengers and be regularly transported into the City. The initial reaction is a flat refusal. However, on reflection it is agreed that his request will be granted, provided he is prepared to pay for the additional costs involved. Needless to say the price proposed for his ticket bears no resemblance to the price paid by the other commuters.

This hypothetical situation is a limiting case of a reality which abounds on the London and South East Region of British Rail— with one critical difference: in the example the lone commuter is expected to pay the proper rate for the service he desires, whereas in reality, on B.R., the opposite is normal practice. Thus a commuter travelling from a fringe station not only enjoys the certainty of a seat but has the satisfaction of knowing that the rate/mile he is paying is less than that paid by some of his travelling companions who are standing shoulder to shoulder like sardines. This state of affairs is considered irrational and unfair, and it is the purpose of this paper to derive a fare structure which is both rational and equitable and thus obviates these defects of the present system.

The impetus for doing this work was given by the recent report of the Monopolies and Mergers Commission (1980). In this report the Commission says "It is important that BRB should investigate the relationship between costs and fares with respect to distance." In its conclusions the Commission also says that "BRB should make further investigations to attempt to establish the relationship between cost and distance and to establish the justification for the distance taper." It is surprising and disconcerting that the Commission fails to comment on the habit of equating units which are dimensionally different. Thus, in the first quotation above, and elsewhere in the Commission's report, fares are incorrectly equated to cost: revenue may be equated to cost (£ = £) but fares, which have the dimensions of revenue/passenger,

* Railway Technical Centre, British Rail. I thank my colleague Mr. A. W. Denham for helpful discussion. I am solely responsible for the ideas and views expressed.
may not (there can be no question about this). Any argument based on this incorrect comparison is bound to be illogical; this will lead to false conclusions, and to improper recommendations which may have highly damaging results for the whole of British Rail.

DERIVATION OF RATIONAL FARE STRUCTURE

Before proceeding with the derivation of a rational and equitable fare structure, two mandatory requirements must first be stated in order that the words “rational and equitable” should be applicable:

(1) The total cost of the particular train journey should be met by the sum of fares.
(2) All passengers over any particular section of the journey should pay the same rate/mile; that is, at any instant in the journey no passenger may be able to claim of a fellow passenger that he is paying more (or less) per mile at that instant.

In addition to these mandatory requirements some arbitrary assumption must be made concerning the distribution of passengers along the route. It will be assumed that all passengers are picked up and finally deposited at a central terminus, and that this pick-up is hyperbolic. Alternative forms (such as exponential, logarithmic, etc.) may equally well be used (see discussion following) but a hyperbolic form will be used to illustrate the basic point. The analysis is performed with continuous mathematical functions, which are more tractable than discrete step functions.

We commence with the situation shown in Figure 1.

The loading of the train is represented by the continuous hyperbolic function

$$n \left( m + \frac{k}{N} \right) = k$$

where
- $m$ = journey distance to the central terminus ($m = 0$)
- $n$ = number of passengers at $m$
- $N$ = total passengers picked up
- $k$ = constant.

If $n_1$ is the number of passengers boarding at the first station, distance $M$ miles from the centre terminus, then

$$n_1 \left( M + \frac{k}{N} \right) = k$$

The total distance travelled by the train is $M$ miles, and, if $R$ is the revenue required to meet costs plus profit, then

$$nf = \frac{R}{M}$$
where \( f \) = instantaneous fare rate (pence/pasenger mile)

\[
i.e. f = \frac{R}{Mn}
\]

(4)

If \( dn_m \) passengers board the train \( m \) miles from the terminus, the total fare \( dF_m \) collected from these passengers for the journey of \( m \) miles is

\[
dF_m = dn_m \int_0^m -f \, dm
\]

\[
= dn_m \int_0^m \frac{R}{Mn} \, dm
\]

\[
= dn_m \int_0^m \frac{R}{Mk} \left( m + \frac{k}{N} \right) \, dm
\]

(from 1)

\[
= dn_m \frac{R}{Mk} \left( \frac{m^2}{2} + \frac{km}{N} \right)
\]

that is,
\[
\frac{dF}{dn} = \frac{R}{Mk} \left( \frac{m^2}{2} + \frac{km}{N} \right)
\]  

Equation (5), which is parabolic through the origin, shows the fare to be charged on each passenger for a journey of \( m \) miles to the terminus.

This is the rational and equitable fare structure which is required for the given arbitrary assumption. To show that it is consistent it must be shown that if all passengers are charged according to equation (5) the total sum of their fares is equal to \( R \), the required revenue.

Thus

\[
F_T = \int_{F_0}^{F} dF = \int_{n_1}^{N} \frac{R}{Mk} \left( \frac{m^2}{2} + \frac{km}{N} \right) dn + n_1 \frac{R}{Mk} \left( \frac{M^2}{2} + \frac{kM}{N} \right)
\]

The second term on the r.h.s. which is not under the integral is the contribution from the \( n_1 \) passengers who board at the first station.

Substituting into (6) from (1) gives, for the integral only:

\[
I = \frac{R}{Mk} \int_{n_1}^{N} \left( \frac{k^2(N-n)^2}{2n^2N^2} + \frac{k^2(N-n)}{nN^2} \right) dn
\]

\[
= \frac{Rk}{2MN^2} \int_{n_1}^{N} \left( \frac{N^2}{n^2} - 1 \right) dn
\]

\[
= -\frac{Rk}{2MN^2} \left( 2N - \frac{N^2}{n_1} - n_1 \right)
\]

Repeated substitution from equation (1) gives eventually

\[
I = \frac{RMn_1}{2k}
\]

Thus

\[
F_T = \frac{RMn_1}{2k} + n_1R \left( \frac{M^2}{2} + \frac{kM}{N} \right)
\]

\[
= \frac{RMn_1}{k} + \frac{Rn_1}{N}
\]

\[
= \frac{Rn_1}{k} \left( M + \frac{k}{N} \right)
\]

\[
= R
\]

This is the desired result which shows that the total fares based on the fare structure given by equation (5) does indeed yield the required revenue.
A RATIONAL RAIL FARE STRUCTURE

J. G. Gibson

DISCUSSION

A fare structure has been derived which is both rational and fair. It is rational because (a) it achieves the desired revenue and (b) it charges according to the value and quality of the service at any particular stage of the journey. It is fair because no passenger is paying a different rate from another for any particular stage. It might be argued that it is unfair for passengers to pay a higher rate for earlier parts of a journey. This would be refuted as follows. Consider equation (4) and multiply numerator and denominator by $S$, where $S$ is the total number of seats on the train.

Then

$$f = \frac{R}{MS} \frac{S}{n}$$

(7)

$$= \frac{f_0}{u}$$

where $f_0 \equiv R/MS$ is effectively a basic rate/seat mile

$u \equiv n/s$ is the utilisation.

($u$ could, alternatively, be defined as $n/N$.)

Thus, the instantaneous rate/passenger mile is inversely proportional to the utilisation. This is in nice agreement with the point made in the hypothetical situation with which this paper started.

There are at least two points in the foregoing analysis which need further comment. First is the arbitrary assumption concerning the form of the passenger loading (assumed hyperbolic in the analysis). Two remarks are worth noting here. Firstly, provided the utilisation is some inverse function of the distance from the terminus, the instantaneous rate will always be some increasing function of this distance, analogous to equation (5). Secondly, it is immaterial to know the passenger turnover along the route—the analysis does not need to distinguish individual passengers. Provided the mandatory conditions (stated above) are satisfied, all that is required is a mathematical function to describe the net passenger loading profile along the route. In principle, any experimentally observed loading profile may be described by an empirically determined mathematical function.

The second point which needs comment is related to the point just discussed, and is not so easy to deal with; it is the perturbation of the loading profile which will occur if a new fare structure (such as equation (5)) is introduced. Intuitively one would expect the first effect to be a reduction in the number of long-distance travellers, and this effect should feed back to increase, overall, the fare rate. This feed-back might be, in turn, offset by truncating the route at an obvious point. Thus, for example, consider the hyperbolic distribution shown in Figure 1; if the tail of this curve fell as indicated by the dashed line the distribution could be considered, for practical purposes, to be linear. It could then be extrapolated to produce a truncation of the route at $M_r$. If the passenger loading is then described by an equation of the form

$$n = km + N,$$

(8)
this leads to a fare structure (analogous to equation (5)) given by the equation

\[
\frac{dF}{dn} = \frac{R}{kM_t} \ln \left(1 + \frac{km}{N_t}\right)
\]

which again is the fare to be charged to each passenger for a journey of \(m\) miles to the terminus. Again, the total revenue from the fares sums to \(R\).

In a previous study by the author it was argued that railway patronage might follow Boltzmann's equation, in which the perceived rate/mile is the variable exponential index. It is proposed to examine the possibility of unifying that idea with the idea presented here.

CONCLUSIONS

It has been argued that the present fare structure on the London and South East rail network is irrational and unfair, and a fare structure has been derived which eliminates these defects. The ideas presented are radically different from the historic policy pursued by the British Railways Board up to the present. They also go some way towards answering some of the doubts hinted at by the Monopolies Commission (1980) and may be said to represent a contribution to the current discussion about that report.

The introduction of such a fare structure would be expected to have a profound effect on the financial state of BR and its customers and, also, on the social and economic state of London and a large part of southern England. Accordingly the author offers the ideas presented for wider discussion and comment.

APPENDIX

Calculation of overall average fare rates from annual aggregates

Assume that the annual sum of travellers is carried on one journey on a hypothetical train. The cost of the operation is the annual cost of running London and South East Region.

Assume 60 miles is a representative journey and allow the passengers to be picked up in 6 equal increments, as shown in Figure 2.

Average trip = 35 miles. Hence \(N_t = 8 \times 10^9/35\) where \(8 \times 10^9\) is taken from BR Facts and Figures.

\[k = -N_t/70\]

Assume cost of L & S.E. is £500m/annum

\[R = 1.2 \times 500 = £600m\]

(The factor 1.2 allows for the deviation from linearity caused by picking up passengers in increments.)

Substituting in equation (9) gives the fares to the central terminus shown in Table 1. Total fare from all passengers comes to £510m. The rate/mile seems remarkably modest.

274
A RATIONAL RAIL FARE STRUCTURE

J. G. Gibson

\[ N_r \]

\[ m \text{ (miles)} \]

FIGURE 2

<table>
<thead>
<tr>
<th>Distance from T (miles)</th>
<th>Fare £</th>
<th>Rate p/mile</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.40</td>
<td>4.0</td>
</tr>
<tr>
<td>20</td>
<td>0.88</td>
<td>4.4</td>
</tr>
<tr>
<td>30</td>
<td>1.47</td>
<td>4.9</td>
</tr>
<tr>
<td>40</td>
<td>2.22</td>
<td>5.6</td>
</tr>
<tr>
<td>50</td>
<td>3.29</td>
<td>6.7</td>
</tr>
<tr>
<td>60</td>
<td>5.12</td>
<td>8.5</td>
</tr>
</tbody>
</table>

Fare from A → B = [(A → T) − (B → T)]

<table>
<thead>
<tr>
<th>Rate p/mile</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>30</td>
</tr>
<tr>
<td>40</td>
</tr>
<tr>
<td>50</td>
</tr>
<tr>
<td>60</td>
</tr>
<tr>
<td>70</td>
</tr>
</tbody>
</table>

If the given fares are multiplied by 1.2\(X/600\) where \(X\) is any alternative assumed for the cost of running L & S.E., this will give the revised fare; for example, if \(X = £250\)m the fares are exactly halved.

An interesting comparison with the figures in Table 1 is provided by the figures in Table 2. These figures were deduced from Figure 10.1 of Monopolies and Mergers Commission (1980) and were the rates charged at 6 January 1980 for long-period season tickets on the L & S.E.

REFERENCE