THE ECONOMICS OF TRAVEL PASSES

Non-uniform Pricing in Transport

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1. INTRODUCTION

In recent years many transport operators have introduced a differentiated price structure which offers the user a choice between an ordinary ticket and a travel pass. Travel passes may either allow unlimited travel during a certain period (travelcard) or provide a discount from the full fare (railcards). They are seen to provide benefits to the operator (for example, an improvement in cash flow) and to passengers (for example, a reduction in boarding and queuing times). When travel passes are introduced it is expected that patronage will be maintained or increased without loss in revenue. This crucially depends on how the market is segmented between buyers of travel passes and users of ordinary tickets. Travel passes thus represent a price discriminating strategy that involves self-selection: different prices per journey are paid because consumers choose different levels of consumption, though they face the same opportunities. Users who make many trips tend to choose the travel pass, while users who make few trips tend to choose the ordinary ticket. It is important to realise that the option chosen also influences the travelling pattern: once the consumer has obtained the travel pass the number of trips made depends on that choice, since it influences the marginal cost of travel.

Some attempts have been made on empirical grounds to identify the factors that induce an individual user to buy a travel pass (for example, Doxey, 1984). But nobody has tried to integrate travel passes into the optimal pricing decision of the firm. As a rule of thumb, transport operators tend to set travel pass prices in relation to single journey fares. For instance, for the travelcard, an average number of trips is assumed and a discretionary discount is applied to the result of multiplying that number by the existing single cash fare. In this way, no account is taken of the fact that with the travelcard option two substitute goods, trips with the travelcard and trips with ordinary tickets, have to be simultaneously

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priced. The aim of this paper is to examine the properties of the optimal price of a travel pass and its cash fare alternative under different corporate objectives. We discuss the travelcard plus cash fare and the railcard plus full fare schemes. We also consider what information about the distribution of consumers in terms of their trip behaviour is needed by those who implement the optimal pricing policies.

A transport operator who is to introduce a travel pass has to decide on the validation period and the price. In this paper we take the period as given and concentrate on the price. In section 2, where the travelcard is discussed, a framework is defined to analyse different strategies for ticket types. We argue that a price scheme consisting of a travelcard and a cash fare alternative is an example of the basic non-uniform price structure known as a two-part tariff. In the following sections this idea is developed along different lines. The simplest set-up is examined in section 3. In this an operator offers a transport service by a single mode to a group of heterogeneous consumers; he has to choose two prices, the travelcard price and the cash fare. Optimal pricing rules are derived under different corporate objectives. In section 4 we look at the railcard as a particular example of the travelcard. In section 5 we examine how the results are modified by some extensions to the basic model. The first is a consumption externality which arises through the use of the travelcard. As another extension we study a different scenario, which might be thought of as that of London, where the operator offers two alternative modes (bus and underground) to a heterogeneous population of users; he has to choose three prices, two for the cash fare alternatives and one for the travelcard, which is assumed to be valid on both modes. Section 6 reports the results of some numerical evaluations intended to illustrate the theoretical arguments discussed. The paper ends with a summary and conclusions in section 7. Throughout most of the paper the term "trips" may also be interpreted as "passenger miles" without loss of generality.

2. THE MICROECONOMICS OF TRAVELCARDS

There are two basic features of travelcards: they have a limited period of validity but there is no limit, within that period, to the number of trips that can be made; once the travelcard is bought the user faces a zero marginal fare for any trip. It is that, precisely, which makes a price scheme consisting of a travelcard and the ordinary ticket alternative relate to a two-part tariff structure. A two-part tariff is the simplest example of a non-uniform price schedule. In general, with a non-uniform price schedule the consumer's total expenditure is not proportional to the level of consumption incurred; quantity discounts and quantity premiums are allowed. A two-part tariff consists of a fixed fee plus a uniform unit price. In the classical two-part tariff applications, consumers who want to participate in the system (telephone, electricity, recreational parks etc.) pay both the fixed fee, which allows consumption, and the unit price when consumption is positive. When introducing a travelcard the operator is, in fact, offering a choice between, on the one hand, a two-part tariff made of a positive fee but a zero unit price (travelcard) and, on the other hand, a two-part tariff with a zero fee but a positive
unit price (ordinary ticket). Thus, the problem of finding the optimal travelcard price and its optimal cash fare alternative may be seen as that of finding a two-part tariff with the condition that the fixed fee and the uniform unit price will be paid by different consumers. Consumers must not be identical since, if they were, there would be no reason for a differentiated price scheme.

Once we realise the non-uniform nature of the strategy of the travelcard plus ordinary ticket, it is easy to evaluate its economic effects. A well known finding in the non-uniform pricing literature is that an appropriately designed non-uniform price schedule can make all consumers and the firm better off than under a uniform price regime (see Brown and Sibley, 1986). In the case of the travelcard this can easily be seen in a diagramatic example (Figure 1).

Let us assume two types of users: the "intensive" user with demand $A_2B_2$ and the "less intensive" user with demand $A_1B_1$. Let there be a constant marginal cost per trip $OC$. In the absence of travelcards the operator charges the profit-maximising uniform price, say $OP$, and $OE_1$ plus $OE_2$ trips are made. Profits are then represented by the areas $a + (a + b + c)$.

Let us introduce a travelcard option at a price $T = a + b + c + d + e + f + g + h$ without modifying the ordinary ticket fare. The less intensive user will compare the surplus obtainable with the ordinary ticket, $s_1$, with that obtainable with the
travelcard, $s_1 = (c + d + g + h)$. Thus he will opt for the ordinary ticket. In the same way, the intensive user will compare surpluses (that is, $s_1 + s_2$ versus $(s_1 + s_2 + a + b + c + d + e + f + g + h) - T$) and will on the margin choose the travelcard, increasing his number of trips to $OB_3$. In these circumstances the travelcard strategy will be profitable to the operator as long as $d > i$; that is, when the extra revenue from the travelcard net of costs is positive.

An economically efficient aspect of non-uniform prices is that they induce consumers to sort themselves according to their taste for the firm's output. In this way the travelcard strategy may be beneficial, since profits may be increased and the intensive user may be induced to make some additional trips he was willing to make. However, for the private operator there is still scope to benefit further from the introduction of the travelcard option. We have assumed for expositional purposes that the ordinary ticket price stays at the pre-travelcard level; but, in fact, both fares have to be chosen simultaneously. In so far as the increase in revenue from a higher travelcard price offsets the loss in revenue from a higher ordinary ticket price, both fares may be increased proportionately, to the benefit of the operator. On the other hand, a public firm which is more interested in social surplus, though financially constrained, may have the chance to set a low price for the travelcard, making it available to both types of users. All this depends on how users are distributed according to their intensity of use; in other words, on how much consumers differ in their demands for transport services.

3. THE BASIC MODEL

Let us assume a population of consumers who differ among themselves in terms of a variable, $\theta$, which may be thought of as income or as a vector of characteristics that cause consumers to have different preferences in trip frequency. This variable may be the result of a combination of circumstances, such as distance of workplace from home, employment status, car ownership, etc. Thus, the variable $\theta$ represents tastes. Usually, the transport operator will have some disaggregated information about demands, but not complete information about individuals. We may suppose that the operator knows the distribution of tastes across the population, though not the tastes of any individual consumer. Thus, we assume the operator knows $h(\theta)$, the density of $\theta$, and its cumulative distribution, $H(\theta)$, for $\theta \in (\theta, \theta')$. We will have $H(\theta) = 0$ if $\theta < \theta$ and $H(\theta) = 1$ if $\theta \geq \theta$.

The individual demands $q(p, \theta)$ are the result of maximising the consumer surplus to user $\theta$ of consuming $q$, at the expenditure of $pq$ for ordinary ticket users and $T$ for travelcard buyers. $T$ is the travelcard price and $p$ is the uniform fare. Finally, let us assume that demand strictly increases with the taste parameter $\theta$. Thus a consumer with a high value of $\theta$ has a demand curve which lies to the right of that of a consumer with a low value of $\theta$.

The simplest situation may be that of an operator offering a transport service in a single mode. Since we have assumed a continuous distribution of consumers in terms of $\theta$, there will be a consumer (or a group of them) $\bar{\theta}$ who will be indifferent between buying the travelcard and buying the ordinary tickets. This critical consumer type is characterised by the following condition:
\[ \int_{p}^{\bar{\theta}} q(p', \bar{\theta}) \text{dp}' - T = \int_{p}^{\bar{\theta}} q(p', \bar{\theta}) \text{dp}' \]  

(1)

where, from now on, \( \theta' \) and \( p' \) will define the variables of integration which correspond to \( \theta \) and \( p \) respectively.

Equation (1) says that, for the critical consumer, the consumer surplus obtainable is the same for both fare alternatives. It defines an implicit function \( \bar{\theta} = \bar{\theta}(p, T) \) with the property that

\[ \frac{\partial \bar{\theta}}{\partial p} = -q(p, \bar{\theta}) \frac{\partial \bar{\theta}}{\partial T} \]  

(2)

and

\[ \frac{dT}{dp} \bigg|_{\bar{\theta} = \text{const}} = q(p, \bar{\theta}) \]  

(2')

This result relates the critical level \( \bar{\theta} \) with changes in the prices of the tickets. It says that, to maintain the critical consumer level \( \bar{\theta} \) constant, any change in the travelcard price must be accompanied by a change in the same direction in the cash fare, so as to allow the marginal consumer to make the same number of trips as he would have made at the cash fare before the change in prices.

Note that as soon as the critical consumer is identified the demands for travelcards and ordinary tickets are defined. Any user with a value of \( \theta \) such that \( \theta \geq \bar{\theta} \) will buy the travelcard, whereas those with a value of \( \theta \leq \bar{\theta} \) will choose the cash fare alternative.

**Profit maximisation**

Profits in this model are defined as

\[ \int_{\bar{\theta}}^{\bar{\theta}} (p - c) q(p, \theta') h(\theta') \text{d}\theta' + \int_{\bar{\theta}}^{\bar{\theta}} [T - cq(0, \theta')] h(\theta') \text{d}\theta' \]  

(3)

where \( c \) is a constant average cost per trip. The first integrand is the net revenue obtained from the ordinary tickets; the second integrand is the revenue obtained from the travelcard holder, \( T \), minus the cost incurred in providing the number of trips made at zero fare.

Differentiating (3) with respect to \( p \) and \( T \), we get the following system of first order conditions:

\[
\begin{align*}
(p - c)Q_p + Q + Ah(\bar{\theta}) \frac{\partial \bar{\theta}}{\partial p} &= 0 \\
[1 - H(\bar{\theta})] + Ah(\bar{\theta}) \frac{\partial \bar{\theta}}{\partial T} &= 0
\end{align*}
\]

(4)

where

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\[ Q_p = \int_{\theta'} (\partial q / \partial p) h (\theta') d\theta', \]
\[ Q = \int_{\theta'} q (p, \theta') h (\theta') d\theta' \]
and
\[ A = (p - c) q (p, \theta') - [T - cq(0, \theta')]. \]

After (4) is solved for \( p \) and \( T \) the following equilibrium results are obtained:

\[
\begin{align*}
\frac{p - c}{p} &= \left\{ 1 + \frac{[1 - H(\theta)]\overline{q}}{Q} \right\} \frac{1}{\epsilon} \quad (5a) \\
\frac{T - [p\overline{q} + c(\overline{q} - \overline{q})]}{T} &= \frac{1}{\eta} \quad (5b)
\end{align*}
\]

where
\[ \overline{q} = q(p, \theta'), \]
\[ \overline{q} = q(0, \theta'), \]
\[ \epsilon = Q_p p / Q \text{ and} \]
\[ \eta = [h(\theta) (\partial \theta / \partial T) T] / [1 - H(\theta)]. \]

The results in (5) provide the optimal conditions for the travelcard and the ordinary ticket fares under profit maximisation. Condition (5a) may be interpreted as saying that the proportionate deviation of the cash fare above marginal cost should be inversely proportional to the absolute price elasticity of demand averaged over the cash fare users (\( \epsilon \)), weighted by a certain coefficient. This coefficient is important, since it incorporates the characteristics of the distribution of consumers in which we are interested, and also appears repeatedly in the formulations below. It depends on the market share of travelcards, \[ 1 - H(\theta) \], that is, the proportion of the total consumer population who opt for the travelcard, the number of trips the marginal consumer would make at the cash fare, \( \overline{q} \), and the average number of trips made by the ordinary ticket users, \( Q \). According to condition (5a), given \( \epsilon \), the cash fare profit margin would be higher the higher the travelcard market share, the higher the marginal number of trips, or the lower the average consumption of the ordinary ticket users. This profit margin is higher than in the absence of travelcards (where \( 1 - H(\theta) = 0 \) and \( (p - c)/p = 1/\epsilon \)) and increases with the disparity in the number of trips made between the marginal and the average cash fare user: we have assumed that \( \partial q / \partial \theta > 0 \), so that \( \overline{q} > Q \) for any combination of prices and market shares.

On the other hand, condition (5b) indicates that the travelcard price should be such that its proportionate deviation above the opportunity cost of providing the
travelcard to the marginal consumer is inversely proportional to the elasticity of
travelcard participation. This opportunity cost is the revenue that would have
been gained if the marginal consumer had chosen the ordinary ticket alternative,
$p\bar{q}$, plus the cost of providing the additional trips made with the travelcard,
c($\bar{q} - q$). The elasticity of travelcard participation measures the proportional
change in the travelcard market share brought about by a unit change in the
travelcard price. Note that condition (5b) may be rearranged to yield the more
familiar expression of

$$T(1 - \frac{1}{\eta}) = c\bar{q} + (p - c)\bar{q}$$

That is, marginal revenue equals marginal cost.

Conditions (5a) and (5b) show the interaction between the cash and travelcard
fares and their relation to the characteristics of the distribution of consumers.
The general way to interpret these results may be that, given the distribution of
consumer tastes, the cash and travelcard fares have to be chosen so as to fulfil
conditions (5a) and (5b) through a process of correctly identifying the marginal
type of consumer.

Economic efficiency

We may think of a public operator whose objective function includes consumer
surplus plus producer surplus. To keep the analysis general, we further assume
that the operator is financially constrained so that the net revenue he obtains
must exceed a certain value, which can be negative.

In order to simplify notation, let us define

$$s(p, \theta) = \int_p^\infty q(p', \theta) \ dp'$$

and

$$s(0, \theta) = \int_0^\infty q(p', \theta) \ dp'.$$

The problem faced by the public operator would then be:

Maximise $W = \int_{\theta}^\infty \frac{\partial}{\partial \theta} s(p, \theta') h(\theta') d\theta' + \int_{\theta}^\infty \frac{\partial}{\partial \theta} s(0, \theta') h(\theta') d\theta'$

$$+ \int_{\theta}^\infty (p - c) q(p, \theta') h(\theta') d\theta' - \int_{\theta}^\infty cq(0, \theta') h(\theta') d\theta'$$

subject to

$$\int_{\theta}^\infty (p - c) q(p, \theta') h(\theta') d\theta' + \int_{\theta}^\infty [T - cq(0, \theta')] h(\theta') d\theta' \geq \Pi$$

That is, choose $p$ and $T$ so as to maximise the consumer surplus plus net producer
surplus for both categories of users, subject to the constraint that the net pro-
ducer surplus must be higher than a given value.

The necessary conditions for the problem are
\[
(p - c)(1 + \lambda)Q_p + \lambda Q + (1 + \lambda) Ah (\bar{q}) \frac{\partial \bar{q}}{\partial p} = 0
\]

\[
\lambda [1 - H (\bar{q})] + (1 + \lambda) Ah (\bar{q}) \frac{\partial \bar{q}}{\partial T} = 0
\]

(7)

where \( \lambda \) is the Lagrangean multiplier and \( A, Q_p \) and \( Q \) are the same expressions as in (4).

The solution for (7) in terms of \( p \) and \( T \) yields

\[
\frac{p - c}{p} = \frac{\lambda}{1 + \lambda} \left\{ 1 + \frac{[1 - H(\bar{q})]q}{Q} \right\} \frac{1}{\epsilon}
\]

(8a)

\[
\frac{T - [p\bar{q} + c(q - \bar{q})]}{T} = \frac{\lambda}{1 + \lambda} \frac{1}{\eta}
\]

(8b)

which are the price equilibrium conditions for the public operator. Note that \( \epsilon \) and \( \eta \) are defined as in (5).

If we compare this with (5), it will be noted that the only difference lies in the shadow price ratio \( \lambda/(1 + \lambda) \) which multiplies both equilibrium conditions. In the extreme case in which the financial constraint did not bind, that is, \( \lambda = 0 \), we would have \( p = c \) and \( T = c\bar{q} \) as equilibrium prices; that is to say, set cash fare equal to unit cost and travelpass price equal to the cost of providing the number of trips made by the marginal consumer at zero fare. This is in accordance with our intuition on how to maximise total surplus. On the other hand, when \( \lambda \rightarrow \infty \) the results approach the profit-maximising conditions.

**Passenger miles**

About a decade ago the idea started to gain some ground in the transport economic literature that a public transport firm, subject to a profit (or subsidy) constraint, might follow a pricing policy of maximising the total number of passenger miles sold (for example, Nash, 1978; Glaister and Collings, 1978). Indeed, London Transport adopted this strategy in 1975. The pricing rules derived to pursue that objective are modified if the ticket scheme includes a travelpass.

The problem faced by the operator is now defined as:

\[
\text{Maximise } \int_0^\infty q(p, \theta') h(\theta') \, d\theta' + \int_0^\infty q(0, \theta') h(\theta') \, d\theta'
\]

\[
\text{subject to } \int_0^\infty (p - c) q(p, \theta') h(\theta') \, d\theta' + \int_0^\infty [T - cq(0, \theta')] h(\theta') \, d\theta' \geq \Pi
\]

(9)

That is, maximise the number of passenger miles travelled by both categories of users, subject to a given financial constraint.
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The first order conditions for the problem are

\[
\begin{align*}
[1 + \lambda(p - c)]Q_p + \lambda Q + \lambda B h(\bar{\eta}) \frac{\partial \bar{\eta}}{\partial p} &= 0 \\
\lambda[1 - H(\bar{\eta})] + \lambda B h(\bar{\eta}) \frac{\partial \bar{\eta}}{\partial T} &= 0
\end{align*}
\]  

(10)

where \( B = p\bar{q} - T + (c - 1/\lambda)(\bar{q} - \bar{q}) \) and \( \lambda \) is the shadow price of the profit constraint. It measures the additional passenger miles that could be provided at equilibrium if the profit constraint were relaxed by a money unit. Its dimensions are passenger miles per money unit, and it is known as the "pass-mark".

The equilibrium conditions to be obtained from (10) are:

\[
\begin{align*}
1 - p \left( 1 - \frac{[1 - H(\bar{\eta})]\bar{q}}{\bar{Q}} \right) &\frac{1}{\lambda} = 1 \\
T - \frac{[p\bar{q} + (c - 1/\lambda)(\bar{q} - \bar{q})]}{\eta} &= 1
\end{align*}
\]  

(11)

Condition (11a) is saying that services should be provided up to the point where the marginal cost minus the marginal revenue generated in the cash fare user market for the marginal consumer is just equal to the inverse pass-mark. The marginal revenue is modified from the no-travelcard situation by the coefficient \((1 + [1 - H(\bar{\eta})]\bar{q}/\bar{Q})\); that is to say, assuming that the price elasticity of total demand averaged over the cash fare users is the same as the price elasticity of total demand in the absence of travelcards. Again it is easy to see that if the financial constraint did not bind, that is, if \( \lambda = 0 \), services would be provided to meet the demand originated by a situation of zero fares, since, from (11a), the difference between marginal cost and marginal revenue for the critical consumer could be as large as required. In this case there would be no rationale for a travelcard, since all travel would be free. If, on the other hand, \( \lambda \to \infty \) the conditions in (11) would collapse to those derived under profit maximisation.

4. THE RAILCARD

A railcard, as implemented for instance by British Rail, allows its holder a discount on the full fare for all, or certain, trips, depending sometimes on time of day. In the two-part tariff framework we have developed, a railcard may be considered as a particular type of travelcard. The alternatives faced by the consumer are a two-part tariff made of a zero fee but a uniform unit price (full fare), and a two-part tariff consisting of a positive fee (railcard) plus a uniform unit price lower than the full fare. This may be better understood with the help of Figure 2, which represents the demand for trips, \( AB \), of an individual consumer.

The user may either pay the full fare \( OP_h \), in which case he would make \( OQ_h \) trips, or buy the railcard at a price \( R \), which will reduce the fare to be paid to \( OP_I \) and increase the trips to \( OQ_I \). If we assume, again, that the consumer
population form a continuous distribution in terms of the taste variable \( \theta \), it is
easy to realise that the critical consumer(s) will be that (those) for whom \( R \) will
amount to precisely the value represented by the shaded area in Figure 3. For the
critical consumer \( \theta \) is defined by the following condition:

\[
\int_{P_h}^{P_l} q(P', \theta) dP' = \int_{P_l}^{P_h} q(P', \theta) dP' - R
\]

(12)

which says that the consumer surplus enjoyed with the full fare is equal to that
enjoyed with the lower fare minus the cost of the railcard. Equation (12) defines
an implicit function \( \theta = \theta (P_h, P_l, R) \) with the following properties:

\[
\frac{\partial \theta}{\partial P_h} = -q(P_h, \theta) \frac{\partial \theta}{\partial R}
\]

\[
\frac{\partial \theta}{\partial P_l} = q(P_l, \theta) \frac{\partial \theta}{\partial R}
\]

(13)

or

\[
\frac{dR}{dP_h} \bigg|_{\theta = \text{const}} = q(P_h, \theta)
\]

\[
\frac{dR}{dP_l} \bigg|_{\theta = \text{const}} = -q(P_l, \theta)
\]

(14)

The equations in (14) relate the critical level \( \theta \) with changes in the full fare
level, its discount or the price of the railcard. The first equation says that to keep
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constant a change in the railcard cost would have to be accompanied by a change, in the same direction, in the full fare, so as to allow the critical consumer to make as many trips as he would have made before the change in fares if he had chosen to pay the full fare. Similarly, in order to maintain the level of \( \bar{\theta} \) constant, any change in the discount applied to railcard users will have to be combined with a change in the same direction in the railcard price, so as to allow the marginal consumer to make as many trips with the railcard as he would have made before the discount change.

As with the travelcard, those consumers with a value of \( \theta > \bar{\theta} \) will become railcard holders, whereas those for whom \( \theta < \bar{\theta} \) will choose to pay the full fare. The difference between the travelcard and the railcard is that the marginal fare with the railcard is not zero but the full fare minus a discount. Thus, the transport operator faces a problem with three decision variables: railcard price, full fare and discount.

Let us analyse the optimal pricing conditions under profit maximisation. Profits are now defined as

\[
\int_{\bar{\theta}}^{\theta} (P_h - c) q(P_h, \theta') h(\theta') d\theta' + \int_{\bar{\theta}}^{\theta} [R + (P_i - c) q(P_i, \theta')] h(\theta') d\theta' \tag{15}
\]

where \( c \) is a constant average cost per trip, \( P_h \) is the full fare, \( P_i \) is the discounted fare, and \( R \) is the railcard price. The first integrand is the net revenue obtained from the full fare travellers; the second integrand is the revenue obtained from the railcard holders. It has two components: the railcard price and the net revenue from the trips made with the railcard.

The necessary conditions to maximise (15) are:

\[
\begin{align*}
(P_h - c)Q_{P_h} + Q_h + Ch(\bar{\theta}) \frac{\partial \bar{\theta}}{\partial P_h} & = 0 \\
(R - c)Q_{P_i} + Q_i + Ch(\bar{\theta}) \frac{\partial \bar{\theta}}{\partial P_i} & = 0 \\
[1 - H(\bar{\theta})] + Ch(\bar{\theta}) \frac{\partial \bar{\theta}}{\partial R} & = 0
\end{align*}
\tag{16}
\]

where

\[
\begin{align*}
Q_{P_h} &= \int_{\bar{\theta}}^{\theta} (\partial q/\partial P_h) h(\theta') d\theta' \\
Q_{P_i} &= \int_{\bar{\theta}}^{\theta} (\partial q/\partial P_i) h(\theta') d\theta' \\
Q_h &= \int_{\bar{\theta}}^{\theta} q(P_h, \theta') h(\theta') d\theta' \\
Q_i &= \int_{\bar{\theta}}^{\theta} q(P_i, \theta') h(\theta') d\theta'
\end{align*}
\]

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and 

\[ C = (P_h - c) q(P_h, \bar{\theta}) - (P_l - c) q(P_l, \bar{\theta}) - R. \]

After (16) has been solved for \( P_h, P_l \) and \( R \), the following equilibrium results are obtained:

\[
\begin{align*}
\frac{P_h - c}{P_h} &= \left\{ \frac{1}{1 + \frac{[1 - H(\bar{\theta})]}{\bar{Q}_h}} \right\} \frac{1}{\epsilon_h} \quad (17a) \\
\frac{P_l - c}{P_l} &= \left\{ \frac{1}{1 - \frac{[1 - H(\bar{\theta})]}{\bar{Q}_i}} \right\} \frac{1}{\epsilon_i} \quad (17b) \\
R - \left\{ P_h \bar{q}_h - P_l \bar{q}_l + c(\bar{q}_i - \bar{q}_h) \right\} &= \frac{1}{\eta} \quad (17c)
\end{align*}
\]

where

\[
\bar{q}_i = q(P_i, \bar{\theta}), \\
\epsilon_i = \frac{Q_h P_l}{\bar{Q}_i} \quad (i = h, l), \\
\eta = \frac{h(\bar{\theta})(\partial \bar{\theta})/\partial R)}{1 - H(\bar{\theta})}.
\]

The results in (17) provide the optimal conditions for the railcard, full fare and discount under profit maximisation. They are very similar to those derived for the travelcard (see equations (5)), though there is an additional equation to consider. Condition (17a) says that the proportionate deviation of the full fare above marginal cost should be inversely proportional to the absolute price elasticity of demand averaged over the full fare users (\( \epsilon_h \)), weighted by a coefficient which depends on the market share of railcard users \( (1 - H(\bar{\theta})) \), the number of trips the marginal consumer would make paying the full fare \( (\bar{q}_h) \), and the average number of trips made by the full fare users \( Q_h \). Similarly, condition (17b) says that the discount applied to the full fare should be such that the proportionate deviation of the resulting lower fare above marginal cost should be inversely proportional to the absolute price elasticity of demand averaged over the railcard users \( (\epsilon_i) \) weighted by its corresponding coefficient, which, in this case, depends on the market share of railcard users \( (1 - H(\bar{\theta})) \), the number of trips the marginal consumer would make with the railcard \( (\bar{q}_i) \), and the average number of trips made by the railcard holders \( (\bar{Q}_i) \).

Note that the elements in the ratio \( \left\{ [1 - H(\bar{\theta})] \bar{q}_i \right\} /Q_i \) (i = h, l) in equations (17a) and (17b) play opposite roles in explaining the proportionate deviation of the fares above marginal cost. Thus, other things being equal, the higher the railcard market share the larger (smaller) the proportionate deviation of the full
fare (lower fare) above marginal cost. Also, the higher the average number of trips made by the full fare (railcard) users the smaller (larger) that proportionate deviation of the full fare (lower fare).

Finally, condition (17c) indicates that the railcard price should be such that its proportionate deviation above the opportunity cost of providing the railcard to the marginal consumer is inversely proportional to the elasticity of railcard participation. This opportunity cost is the net revenue that would have been gained if the marginal consumer had chosen the full fare alternative instead of the railcard, \( P_n q_n - P_q q_t \), plus the cost of providing the additional trips this marginal consumer would make with the railcard, \( c(q_{t+1} - q_n) \). The elasticity of railcard participation measures the proportional change in the railcard market share brought about by a unit change in the railcard price.

5. SOME EXTENSIONS

A consumption externality

One of the alleged benefits of travelcards is that they reduce boarding times. However, the administration (that is, production and renewal) of travelcards also imposes a fixed cost (for example, clerical work), which may differ from that of a cash-fare-only system. Let us ignore the fixed cost associated with the administration of travelcards (it can be easily incorporated) and concentrate on what remains to be acknowledged as a consumption externality: the fact that an additional user buys a travelcard reduces the riding times, through a reduction in boarding times (for example, on buses), or in the queuing times in the ticket office (for example, on the underground) experienced by other users. Our interest is to see how the pricing rules change when this consumption externality is considered. Littlechild (1975) examines a related consumption externality in telecommunications when a new subscriber joins the system. We assume that the demand function now depends not only on the value of \( \theta \) and the fares, but also on the proportion of users who hold a travelcard, \( 1 - H(\theta) \). If that proportion increases the overall time spent in travelling, as a proportion of the generalised cost, will be reduced, and thus demand will increase.

Therefore, we now have

\[
q = q[\theta, p, 1 - H(\theta)] \quad q_1, q_3 > 0, q_2 < 0.
\]

where \( q_t \) is the partial derivative of \( q \) with respect to the \( t \)th component in the demand function. The marginal consumer level of \( \theta \) has the same properties as those derived when the consumption externality was not considered.

Let us analyse profit maximisation. Profits are now defined as

\[
\int_\theta (p - c)q[\theta', p, 1 - H(\theta)]h(\theta')d\theta' + \int_\theta [T - cq(\theta')0, 1 - H(\theta)]h(\theta')d\theta' \quad (18)
\]

where all the elements are the same as those in (3) except for the new argument in the demand function.
If we differentiate (18) with respect to \( p \) and \( T \), the following system of first order conditions is obtained:

\[
\begin{align*}
(p - c)Q_p + Q + Dh(\theta) \frac{\partial \theta}{\partial p} &= 0 \\
[1 - H(\theta)] + Dh(\theta) \frac{\partial \theta}{\partial T} &= 0
\end{align*}
\]

(19)

where

\[
D = ((p - c)\bar{q} - (T - c\bar{q}) - (p - c)\bar{Q}_h + c\bar{Q}_h).
\]

Note that \( \bar{q} \) and \( \bar{q} \) are defined as in (4), but there are two new elements in \( D \), namely,

\[
\bar{Q}_h = \int_{\theta} \left\{ \frac{\partial q}{\partial \theta} [1 - H(\theta)] \right\} h(\theta', \theta) d\theta' \quad \text{and} \quad \bar{Q}_h = \int_{\theta} \left\{ \frac{\partial q}{\partial \theta} [1 - H(\theta)] \right\} h(\theta', \theta) d\theta'.
\]

After solving (19) for \( p \) and \( T \), we obtain the following equilibrium results:

\[
\begin{align*}
\frac{p - c}{p} &= \left\{ 1 + \frac{[1 - H(\theta)]\bar{Q}}{\bar{Q}} \right\} \frac{1}{\epsilon} \\
T &= \frac{((p - c)\bar{q} + c\bar{q}) - (p - c)\bar{Q}_e - c\bar{Q}_e \epsilon_t)/[1 - H(\theta)]}{\eta} = \frac{1}{\eta}
\end{align*}
\]

(20)

where

\[
\bar{Q} = \int_{\theta} \left\{ \frac{\partial q}{\partial \theta} [p, \theta', 1 - H(\theta)] h(\theta', \theta) d\theta',
\]

\[
\epsilon_e = \frac{Q_h [1 - H(\theta)]}{\bar{Q}},
\]

\[
\epsilon_t = \frac{\bar{Q}_h [1 - H(\theta)]}{\bar{Q}}.
\]

The rest of the elements are defined as in (5).

The results are very similar to those in (5). In fact, condition (20a) is the same as (5a). When we compare the travelcard price with the result derived without considering the consumption externality, the opportunity cost of providing the travelcard to the marginal consumer has now two components. The first is the net revenue that would have been gained if the marginal consumer had chosen the ordinary ticket alternative, \((p - c)\bar{q}\), plus the cost of providing the number of zero fare trips made with the travelcard, \(c\bar{q}\). The second component is the additional net revenue from the demand generated on average in both sides of the market, cash fare, \((p - c)\bar{Q}_e\), and travelcard, \(c\bar{Q}_e\), weighted by the inverse of
the travelcard market share. Note that $\epsilon_c$ and $\epsilon_t$ are the demand elasticities with respect to the proportion of travelcard holders averaged over payers of the cash fare and over travelcard users, respectively. The second component captures the effect of the consumption externality on demand, and thus affects the optimal travelcard price. The proportionate deviation of the travelcard above the difference between these two components will obviously be higher the higher the first component and the smaller the second component. Note, however, that the larger the travelcard market share the smaller will be the effect of the consumption externality, because not only the denominator of the second component will increase, but the numerator will decrease through a reduction in $Q$ and an increase in $Q$.

Two alternative modes

Another scenario we want to consider may be related to London. Let us assume London Regional Transport offers two alternative modes, bus and underground, to a consumer population who, as in the basic model, form a continuous distribution according to their tastes $\theta$. We assume that, in general, the two modes may be substitutes or complements, and the travelcard is valid in both modes. Our purpose is to illustrate the information necessary for the optimal implementation of a travelcard option. Instead of two modes we might consider services in the peak and the off-peak periods, and the model would equally apply. We only discuss the case of a profit maximiser.

In this model the critical consumer type is no longer characterised by equation (1), but instead by the following condition:

$$\int_{p_1}^{\infty} q_1(p_1', p_2, \theta)dp_1' + \int_{p_2}^{\infty} q_2(p_1, p_2', \theta)dp_2' =$$

$$\int_{0}^{\infty} q_1(p_1', p_2, \theta)dp_1' + \int_{0}^{\infty} q_2(p_1, p_2', \theta)dp_2' - T$$

where $q_1, q_2, p_1, p_2$ are respectively the demands for transport in modes 1 and 2 and their cash fares.

Equation (21) says that the consumer surplus obtained by consuming the desired quantities of services in both modes under both ticket alternatives must be the same for the critical consumer type $\theta$. It defines an implicit function $\theta = \theta(p_1, p_2, T)$ such that

$$\frac{\partial \theta}{\partial p_i} = - [\bar{q}_i + \bar{q}_{ij}] \frac{\partial \theta}{\partial T}$$

(22)

where

$$\bar{q}_{ij} = \int_{0}^{p_i} (\partial q_i / \partial p_i) dp_i \quad \text{for } i \neq j \quad \text{and } i, j = 1, 2.$$
\[
\frac{dT}{dp_i} = \text{const} = [\bar{q}_i + \bar{q}_j]
\]
\[(22')\]

\((i \neq j, i, j = 1, 2)\)

The interpretation of this result is similar to that of \((2')\). It shows how far the cash fare in mode \(i\) would have to be modified after a change in the travelcard fare in order to maintain constant the level of \(\bar{q}\). This degree of modification includes not only the number of trips made with the cash fare in mode \(i\) by the marginal consumer, \(\bar{q}_i\), but also the variation in the consumer surplus enjoyed by her from making trips in mode \(j\) after the change in \(p_i\), \(\bar{q}_j\). For instance, if we assume that \(i\) and \(j\) are substitute modes, then, after a marginal increase in \(T\), \(p_i\) will have to be increased so as to keep \(\bar{q}\) constant. Thus the demand for \(q_i\) will shift, say, from \(AB\) to \(A'\overline{B'}\) (see Figure 3), and \(\bar{q}_j\) will be given by the shaded area. Note that a similar shift in demand and change in consumer surplus have to be considered for \(q_i\) after the change that would take place in \(p_j\).

The profit maximising operator faces the following objective function:

\[
\int \bar{\theta} \left( (p_1 - c_1)q_1(p_1, p_2, \theta') + (p_2 - c_2)q_2(p_1, p_2, \theta') \right) h(\theta')d\theta' + \int \bar{\theta} \left( T - c_1 q_1(0, 0, \theta') - c_2 q_2(0, 0, \theta') \right) h(\theta')d\theta'
\]

\[(23)\]

where \(c_1\) and \(c_2\) are the assumed constant average costs of providing services in modes 1 and 2, respectively. Equation \((14)\) is maximised by choosing appropriate values for \(p_1\), \(p_2\), and \(T\). The first order conditions yield the following system of equations:

\[
\begin{align*}
(p_1 - c_1)Q_1' + (p_2 - c_2)Q_2' + Q_1 + \frac{Eh(\bar{\theta})}{dp_1} \frac{\partial \bar{\theta}}{\partial p_1} &= 0 \\
(p_1 - c_1)Q_1' + (p_2 - c_2)Q_2' + Q_2 + \frac{Eh(\bar{\theta})}{dp_2} \frac{\partial \bar{\theta}}{\partial p_2} &= 0 \\
[1 - H(\bar{\theta})] + \frac{Eh(\bar{\theta})}{dT} \frac{\partial \bar{\theta}}{\partial T} &= 0
\end{align*}
\]

\[(24)\]

where

\[
Q_i' = \int \bar{\theta} (\frac{\partial q_i}{\partial p_j}) h(\theta')d\theta',
\]

\[
Q_i' = \int q_i(p_i, p_j, \theta') h(\theta')d\theta' \text{ for } i \neq j, i, j = 1, 2
\]

and

\[
E = (p_1 - c_1)\bar{q}_1 + (p_2 - c_2)\bar{q}_2 - [T - (c_1 \bar{q}_1 + c_2 \bar{q}_2)].
\]

The solution to \((24)\) in terms of \(p_1\), \(p_2\) and \(T\) is the following:
\begin{align}
\frac{p_1 - c_1}{p_1} &= -\frac{\alpha_1 \varepsilon_{22} - \kappa \alpha_2 \varepsilon_{21}}{\varepsilon_{11} \varepsilon_{22} - \varepsilon_{12} \varepsilon_{21}} \\
\frac{p_2 - c_2}{p_2} &= -\frac{\alpha_2 \varepsilon_{11} - (1/\kappa) \alpha_1 \varepsilon_{12}}{\varepsilon_{11} \varepsilon_{22} - \varepsilon_{12} \varepsilon_{21}} \\
T - \left[ (p_1 \bar{q}_1 + p_2 \bar{q}_2) + c_1 (\bar{q}_1 - \bar{q}_1) + c_2 (\bar{q}_2 - \bar{q}_2) \right] &= \frac{1}{\eta} 
\end{align}
(25a)

(25b)

(25c)

where

\[ \epsilon_{ij} = \frac{Q_j p_j}{Q_i} \quad (i, j = 1, 2), \]

\[ \alpha_i = 1 + \frac{[1 - H(\bar{q})](\bar{q}_i + \bar{q}_j)}{Q_i} \quad (i \neq j, i, j = 1, 2) \]

and

\[ \kappa = \frac{p_2 Q^2}{p_1 Q^1}. \]

These equilibrium results are similar to those derived with a single mode. They are more complicated because the presence of an alternative mode is
captured by the cross elasticities. In particular, the conditions in (25a) and (25b) resemble those which Rohlfis first named "superelasticities" (Rohlfis, 1974). They comprise a vast amount of information which may, in practice, be impossible to gather. If we were to assume that the cross-price effects were negligible (that is, \( \epsilon_{12} = \epsilon_{21} = 0 \)), then conditions (25a) and (25b) would collapse to that in (5a) for each mode, since \( \widetilde{q}_{12} = \widetilde{q}_{21} = 0 \) too. In that case the price-fixing problem could become more manageable. However, experience in London Regional Transport after the introduction of the travelcards, provides no evidence, in principle, to suggest that the cross-price effects can be ignored (see Fairhurst et al., 1987).

6. A NUMERICAL EXAMPLE

In this section we illustrate the theoretical arguments discussed by reporting the results of some numerical simulations. We concentrate on comparing a uniform price schedule with a price scheme consisting of a travelcard and its cash fare alternative, under profit maximisation. The data used are related to actual figures from London Regional Transport, but they are subject to some uncertainty and there are problems of aggregation. In addition, the generality of the conclusions is limited by the assumption of certain functional specifications for the demand in terms of passenger miles and for the distribution of consumers in terms of \( \theta \). Thus the results of this exercise should be regarded as illustrative only.

We consider the following linear demand function:

\[
q(p, \theta, \beta) = \theta - \beta p \quad q \geq 0, p \geq 0
\]  

(26)

where \( \theta \) is distributed uniformly over the interval \( (0, M) \) and \( \beta \) is a constant. Without loss of generality we may assume that the consumer population has total measure \( N = 1 \), so that equation (26) may be interpreted as demand from consumers of different values of \( \theta \) (see, for example, Leland and Meyer, 1976).

Profits and consumer surplus when price is uniform (that is, in the absence of travelcards) are given by

\[
\Pi_u = \int_{\beta p}^{M} (p_u - c)(\theta' - \beta p_u)h(\theta')d\theta'
\]  

(27)

\[
CS_u = \int_{\beta p}^{M} \left\{ \frac{1}{2} [(\theta' / \beta) - p_u](\theta' - \beta p_u) \right\} h(\theta')d\theta'
\]  

(28)

where \( p_u \) is the uniform price, \( \beta p_u \) is the minimum value of \( \theta \) for which there is a positive demand, and \( h(\theta) = 1/M \) by definition.

On the other hand, when the price structure consists of a travelcard and its cash fare alternative, profits and consumer surplus are defined by

\[
\Pi_t = \int_{\beta p}^{\bar{\theta}} (p - c)(\theta' - \beta p)h(\theta')d\theta' + \int_{\bar{\theta}}^{M} (T - c\theta')(\theta')h(\theta')d\theta'
\]  

(29)

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J. C. Carbajo

\[ CS_t = \int_{\beta p}^{\bar{\beta}} \{ \frac{1}{2} (\theta' - \beta p) \} h(\theta') d\theta' + \int_{\theta^*}^{\theta} \frac{h(\theta^*)}{\theta} \} d\theta^* \] (30)

where \( \bar{\beta} \) is the marginal consumer level of \( \theta \) and is obtained from applying condition (1) to our specific functions; that is,

\[ \bar{\beta} = (T/p) + (\beta p/2) \] (31)

The analytical solutions for profit-maximising schemes of uniform price and travelcard plus cash fare alternative are:

\[ p_u = \frac{2}{3} \left( \frac{M}{2\bar{\beta}} + c \right) \]
\[ p = \frac{2}{5} \left( \frac{M}{\bar{\beta}} + c \right) \]
\[ T = p(M - \beta(p - c)) \] (32)

Table 1 contains some specific results. The parameter values chosen for illustration deserve some explanation. Our main source of information has been the London Regional Transport 1985/86 annual report. The values of 17 and 12 for the parameter \( \beta \) are very close to the average cost per passenger-mile (in pence) for bus and underground. The choice of \( M \) was induced by the values of the average passenger-miles per journey made by bus (2.25) and by underground (4.86). We simply multiplied these values by 10, approaching perhaps a weekly figure, and chose the two extreme values from a very limited range around the resulting numbers. A limitation of the functional specifications assumed is that the price elasticity averaged over the cash fare users is unity. The choice of \( \beta \) is therefore even more arbitrary, though it implies different values for the price elasticity of the biggest consumer, that is, the one with \( \theta = M \) (see column \( \epsilon_M \)).

From the results in Table 1 several properties of the optimal pricing structures can be observed:

1. Profits are strictly larger with travelcard than without it.
2. The cash fare charged when the travelcard option is available is higher than the uniform price set in the absence of a travelcard.
3. Consumer surplus is lower with travelcards. However, total benefits (profits plus consumer surplus) may increase with travelcards because of the increase in profits.

We may therefore conclude that, in this specific example, higher profits due to the introduction of a travelcard option do not necessarily imply lower total benefits when benefits are measured by the unweighted sum of consumers' surplus plus profits. Note, however, that this is an illustrative exercise. Further research is needed; it should either contemplate other distribution functions which, although parametric, are closer to reality, or try to apply the theoretical results to real data.

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### Table 1
Profit Maximising Price Schedules

<table>
<thead>
<tr>
<th>Parameter Values</th>
<th>Uniform Pricing</th>
<th>Travelcard plus Cash Fare</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>$M$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>17</td>
<td>25</td>
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</tr>
<tr>
<td>17</td>
<td>25</td>
<td>0.20</td>
</tr>
<tr>
<td>17</td>
<td>25</td>
<td>0.10</td>
</tr>
<tr>
<td>17</td>
<td>25</td>
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<td>0.25</td>
</tr>
<tr>
<td>17</td>
<td>20</td>
<td>0.20</td>
</tr>
<tr>
<td>17</td>
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<td>0.10</td>
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<tr>
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<td>0.10</td>
</tr>
<tr>
<td>12</td>
<td>45</td>
<td>0.05</td>
</tr>
</tbody>
</table>

$\theta$ is the index of the "less intensive" consumer with positive consumption.

$CS$ is consumer surplus.

$TB$ is consumer surplus ($CS$) plus profits ($\Pi$).

$\epsilon_M$ is the absolute price elasticity of the "most intensive" consumer.

$TMS$ is the travelcard market share.

All money values are expressed in pence.
7. SUMMARY AND CONCLUSIONS

We have identified a framework to analyse different non-uniform price schemes implemented by transport operators. We have derived optimal pricing rules under different objectives for a ticket type structure consisting of a travelcard and its ordinary ticket alternative. We have examined how the equilibrium conditions for the travelcard are modified in the presence of a consumption externality, or when the transport firm provides two alternative modes. As a further application of the framework, we have considered the example of the railcard plus full fare. Some illustrative calculations have been reported to show the differences in profits and consumer surplus between two price schemes, with and without travelcard, under profit maximisation.

In all the non-uniform price schemes analysed our purpose has been to determine what kind of information is needed to implement the optimal pricing policy. We emphasise that it is necessary to know the characteristics of the distribution of the consumer population in terms of their trip behaviour before we can calibrate the effect that different combinations of fares will have on revenue, through their effect on the marginal consumer type and, therefore, on the market shares of the tickets.

As a final conclusion, we would like to see transport operators depart from arbitrary pricing policies when setting fares for any type of travel pass (together with the alternative ticket) in order to attain some of the potential gains that this kind of price discrimination may produce.

REFERENCES