Testing for near I(2) trends when the signal to noise ratio is small*

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Abstract
Researchers seldom find evidence of I(2) in exchange rates, prices, and other macroeconomics time series when they test the order of integration using univariate Dickey-Fuller tests. In contrast, the multivariate ML trace test frequently find double unit roots in the data. Our paper demonstrates by simulations that this is frequently the case when the signal-to-noise-ratio is small.

1 Introduction
The global financial and economic crisis that began in 2007 has led to an increased interest in the mechanisms that cause asset prices to undergo wide swings around estimates of long-run benchmark values. One important implication of the recent theory of Imperfect Knowledge Economics (IKE) as developed in Frydman and Goldberg (2007, 2011) is that asset prices should show a tendency to undergo persistent swings away and towards long-run benchmark values. While such swings are typical of most prices subject to speculation, such as nominal exchange rates and stock prices, they also characterize many variables in the real economy, suggesting a close two-way interrelationship between the financial market and the real economy.

The focus of this paper is on the econometric characterization of these persistent swings, in particular considering nominal exchange rates in periods of currency float. Figure 1, upper panel, illustrates how the German mark-US dollar nominal exchange rate has shown a tendency to move in long persistent swings around its long-run purchasing power.

*I owe Andreas Noack Jensen a deep thanks for having carried out all simulations in Section 3 and for many useful comments.
Figure 1: The graphs of the (mean and range adjusted) German-US price differential, $pp$, and the nominal exchange rate, $s_{12}$ (upper panel), and the $ppp = pp - s_{12}$ and the real bond rate differential (lower panel).

parity (PPP) value given by the prices differential between the two countries. As can be seen from the figure, the longest swing takes place from approximately 1976 to 1988, followed by shorter swings. Figure 1, lower panel, shows the real exchange rate (measured as $p - p^* - s$) together with the long bond rate differential. It appears that the long swings in the real exchange rates (inherited from the nominal exchange rate) move almost in parallel with the swings in the bond rate differential.

Under the assumption that purchasing power parity holds as a stationary condition one would a priori expect relative prices and nominal exchange rates to be $I(1)$ and the real exchange rate to be $I(0)$. However, econometric tests often find that many real and nominal exchange rates are even more persistent than an $I(1)$ process. For example, Engel and Hamilton (1990) found that the random walk model is strongly rejected in favor of a segmented-trends model for nominal exchange rates.\(^1\) Shocks to a segmented-trends process display a greater degree of persistence because the segmented trends have a long-lasting impact on both the level and the first difference of the variable, whereas shocks to a

\(^1\)Other studies that reject the random walk in favor of a segmented-trends model include Engel (1994) and Cheung and Erlandsson (2005).
random walk series are persistent only in the level. Also an I(2) process has a tendency to generate longer-lasting swings than an I(1) processes because shocks to an I(2) process, like a segmented-trends process, have a persistent impact both on the levels and the first differences (Johansen, 1997, 2006a, Paruolo and Rahbek, 1999).

Consistent with this, nominal and real exchange rates have frequently been found to be well approximated by an I(2) process (Juselius, 1995, 2006, Johansen et al. 2011) but only when using the multivariate unit root test. When using univariate Dickey-Fuller tests they are often found to be at most I(1). To explore why this is the case, we have simulated data designed to replicate the characteristic features of relative prices and nominal and real exchange rates. To achieve this we rely on the result in Frydman and Goldberg (2007) that real exchange rates will have a small signal-to-noise ratio under the assumption that economic actors make forecasts under imperfect information. When testing the order of integration the results show that the univariate DF test tends to reject the second (near) unit root in almost all cases, whereas the multivariate test almost always finds it. This inability to detect a second (near) unit root is shown to be due to the low power of the univariate DF tests to detect a second unit root when the shocks to the drift term of the differenced process are small compared to the shocks to the differenced process itself, i.e. when the signal-to-noise ratio is small.

Finally, we use the German Mark/US dollar exchange rate, goods prices, inflation rates, and interest rates (1975-1999) as an illustration. Based on the multivariate tests we could not reject the I(2) type characterization of the nominal and real exchange rate at high significance levels against the null of an I(1) characterization. Based on the univariate Dickey-Fuller tests the double unit root was rejected.

2 Testing for a double unit root when the signal-to-noise ratio is small

We first introduce the baseline autoregressive model of order $k$ and discuss the augmented D-F model, the Engle-Hamilton segmented trends model and the Imperfect Knowledge model. We then simulate time series with double (near) unit roots and a small signal-to-noise ratio using parameter values that closely replicate the characteristic features of actual prices and exchange rates. To start with, just one single case is used to illustrate in detail why it is often difficult to discover a second unit root when the signal-to-noise ratio is small. We also illustrate the difference between a random walk, a near I(2) and an I(2) process and then proceed to test the series using univariate versus multivariate test procedures.
2.1 Univariate models

The pronounced persistence of many real exchange rates is often associated with a unit root in the data and mostly tested based on univariate Dickey-Fuller type models such as:

$$\Delta q_t = \mu - \rho_1 q_{t-1} + \rho_{11} \Delta q_{t-1} + \ldots + \rho_{1k} \Delta q_{t-k} + \varepsilon_{q_t}$$ (1)

where $\Delta$ is the first difference operator, $q_t$ stands for the log of the real exchange rate in a period of currency float, and $\varepsilon_{q_t}$ is an error term. The null of a unit root in $q_t$ is formulated as the composite null hypothesis ($\rho_1 = 0$ and $\mu = 0$) where the second condition reflects the fact that one would not consider deterministic linear trends in real exchange rates to be economically plausible. When the null cannot be rejected we conclude that there is at least one unit root in the real exchange rates. To test whether there is a double unit root, (1) is respecified as:

$$\Delta^2 q_t = -\rho_2 \Delta q_{t-1} + \rho_{21} \Delta^2 q_{t-1} + \ldots + \rho_{2,k-1} \Delta^2 q_{t-k+1} + \varepsilon_{q_t}$$ (2)

where $\rho_2 = 0$ is the null. In this case, the real exchange rate would exhibit a very pronounced persistence as its change is a unit root process and the real exchange rates would be likely to drift off from its long-run value for extended periods of time. If, instead, $\rho_2$ deviates from zero with a very small amount, then $\Delta q_t$ would be mean-reverting, but the mean-reversion would be very slow. In this case, moderately sized realizations of $q_t$ could easily exhibit a similar persistence as an I(2) process and while being a near I(2) process, it would for practical purposes be difficult to distinguish it from an I(2) process.

Another possibility is to model the long swings in real exchange rate as a combination of unit roots and piecewise linear trends as in Engle and Hamilton (1990) implying the following change in the specification of (1):

$$\Delta q_t = \mu_t + \rho_{11} \Delta q_{t-1} + \ldots + \rho_{1k} \Delta q_{t-k} + \varepsilon_{q_t}, \quad \varepsilon_{q_t} \sim N(0, \sigma^2_{\varepsilon_q})$$ (3)

where $\mu_t = \gamma_i, i = 1, 2, 3, \ldots$. In this case, the real exchange rate would be described by piecewise linear trends with shifting slope parameters $\gamma_i$. If the shifts in the secular trends take place at ever smaller intervals, $\mu_t$ could instead be modelled as a random walk

$$\mu_t = \mu_{t-1} + \varepsilon_{\mu_t}, \quad \varepsilon_{\mu_t} \sim N(0, \sigma^2_{\varepsilon_{\mu_t}})$$ (4)

where $\varepsilon_{\mu_t}$ would describe the change in slope parameter from time $t$ to $t + 1$ and the piece-wise linear trend specification (3) would converge to a double unit root process.
The Imperfect Knowledge Economics (IKE) model described in Friedman and Goldberg (2007, 2011) resembles the secular trends model but differs with respect to the specification of $\mu_t$:

$$\mu_t = \rho_t \mu_{t-1} + \varepsilon_{\mu_t}, \quad \varepsilon_{\mu_t} \sim N(0, \sigma_{\varepsilon_{\mu}}^2) \quad (5)$$

where $\mu_t$ is a drift term measuring the change in the real exchange rate due to a change in individuals’ forecasting strategies and in the underlying fundamentals. When $q_t$ is in the neighborhood of the long-run benchmark value individuals change their forecasts conservatively and we would expect $\rho_t \approx 1$, whereas when $q_t$ is far away from the benchmark, we would expect $\rho_t < 1$. Since the far-from equilibrium period is likely to be much shorter than the "close-to-neighborhood" period, the average value, $\bar{\rho}$, of $\rho_t$ for $t = 1, \ldots, T$ is likely to be fairly close to unity. This is in particular so when $T$ is sufficiently large to cover several long swings in the real exchange rate.\(^2\)

Another important feature of the IKE model to be subsequently discussed is that the signal-to-noise ratio $\sigma_{\varepsilon_{\mu}}^2 / \sigma_{\varepsilon_q}^2$ can be very small implying that the second near unit root associated with (5) may not be easily detectable.

### 2.2 Illustrating different degrees of persistence

Because the drift term (5) is unobservable in actual time-series, it is useful to simulate the persistency properties of processes described by (3) - (5) under different assumptions of the drift term. For this purpose, we have generated time-series according to:

$$\Delta q_t = \mu_t + \varepsilon_{q_t}, \quad \varepsilon_{q_t} \sim N(0, 1) \quad (6)$$

and

$$\mu_t = \bar{\rho}\mu_{t-1} + \varepsilon_{\mu_t}, \quad \varepsilon_{\mu_t} \sim N(0, 0.15^2) \quad (7)$$

where $\bar{\rho}$ is \{0.0, 0.95, 1.0\} and lagged differences have been set to zero without loss of generality. The length of the simulated sample is set to 500 corresponding to roughly 40 years of monthly observations.

Figure 2, upper panel, illustrates swings that have been generated by a random walk ($\rho = 0$) and a near-I(2) process ($\rho = 0.95$). The lower panel compares the same near-I(2) process ($\rho = 0.95$) with an I(2) process ($\rho = 1.00$). The range of variation of the near-I(2) process is 50 compared to 220 for the I(2), which explains the difference in appearance of the identical near I(2) variables in the two panels. To isolate the effect

\(^2\)For varying sub-sample periods, the average value of $\rho_t$ may of course vary to some extent.
A near I(2) variable with $\rho = 0.95$ and signal-to-noise ratio = 0.15 (thick line) together with a random walk (thin line).

The same near I(2) variable (thick line) together with an I(2) variable (thin line). Signal-to-noise ratio = 0.15

Figure 2: The graph of a near-I(2) variable together with a random walk (upper panel) and the same near-I(2) variable together with an I(2) variable of the persistency parameter $\rho$, all three series have been generated from an identical realization of the random shocks $\varepsilon_{qt}$ and $\varepsilon_{\mu t}$ and the signal-to-noise ratio is 0.15 for the near I(2) and the I(2) series. While both series in the upper panel exhibit persistent swings, they are much more pronounced for the near-I(2) series compared to the random walk. The two series in the lower panel exhibit persistent swings, but the swings of the I(2) series are less bounded, signifying the absence of significant mean reversion in the changes of an I(2) process. This becomes even more apparent at the end of the sample illustrating that we often need a long sample period to distinguish a near I(2) from an I(2) process.

A small, but persistent, drift term can be almost hidden for the eye when variance of the first differences is large and may not be easily detectable in a time graph. A (sufficiently long) MA of the original series will smoothen out the highly volatile short-term movements and, therefore, can provide a first rough indication of a persistent drift term in the data. To illustrate this, Figure 3 upper panel, shows the graph of $\mu_t$ together with a 12 period MA of $\Delta q_t$ for the simulated process (6) with $\bar{\rho} = 0.95$. While not identical, the moving average component seems to provide a fairly good description of the near I(2) drift term $\mu_t$. But,
as successive moving average values share 2k-2 identical observations, a moving average component is inherently a time dependent process. It is, therefore, likely to exhibit swings also when there are no swings in the data. For example, when \( q_t \) is a random walk process, i.e. when \( \mu_t = 0 \) and \( \Delta q_t \) is temporally independent, its \( k \) length moving average, \( \Delta q_t = f(\Delta q_{t-k}, \ldots, \Delta q_{t+k}) \), is not independent. This is illustrated in Figure 3, middle panel, showing the differenced random walk of Figure 2 together with its 12 period moving average. As expected, the latter exhibits persistent fluctuations but compared to the moving average of the differenced near I(2) series they stay bounded within much more narrow bands around the mean.

To illustrate the difference between a small and a large signal-to-
noise ratio, the lower panel shows a differenced near-I(2) series that was generated with a large ratio of $\sigma_{\varepsilon_t}/\sigma_{\varepsilon_q}$. The persistence of the drift term of this series is identical to that of the near-I(2) series in the upper panel, i.e. both are generated with $\rho = 0.95$. Hence the series in the upper panel and the lowest panel differ only in terms of the signal to noise ratio, i.e. by the relative magnitude of the shocks $\varepsilon_{\mu_t}$ and $\varepsilon_{q_t}$. When the shocks to $\mu_t$ and $\Delta q_t$ are of similar magnitude the difference is striking: no moving average is needed to see the persistent drift in the data.

Finally, Figure 4 illustrate how different realizations of $\varepsilon_{\mu_t}$ and $\varepsilon_{q_t}$ can produce series that look widely different. Both panels show a near I(2) ($\rho = 0.95$) and an I(2) ($\rho = 1.00$) series with a signal-to-noise ratio of 0.15, so the difference between the panels is only due to different stochastic realizations of the process. The two series in the upper panel exhibit long swings around the zero line whereas in the lower panel they show a much more pronounced tendency to move away from the zero line. As expected, the I(2) series tend to drift away from the zero line more persistently than the near I(2) series. Also, the divergence of the near I(2) and the I(2) series tend to be much stronger at the end of the sample. As will be demonstrated in Section 3 based on a simulation study of 5000 replications, a long sample is often often need to be able to statistically discriminate between the two. The "true" underlying
process that have generated an observed economic variable is of course much more complex than all the models discussed in Section 2.1. The graphs of the simulated series suggest that a near I(2) process (with a small signal-to-noise ratio) can reproduce the long swings behavior we often see in actual real exchange rate data and that similar behavior also can be generated by an I(2) process for moderately sized samples. Whether these models are sufficiently close approximations to allow us to make inference on the broad features of the underlying data generating process will be studied by simulation in Section 3.

Another issue to discuss is how the asymptotic $I(2)$ inference is affected when data are near $I(2)$ rather than exactly $I(2)$. It is useful to distinguish between the case when a near unit root is treated as (1) stationary or (2) nonstationary. In the first case Johansen (2006b) showed by simulation that some inference (on steady-state values) became very fragile when a near unit root was treated as stationary. For example, up to 5000 observations were needed for the empirical distribution to converge to Students $t$ when the near unit root was 0.998.

In the second case, Elliot (1998) showed both analytically and by simulations that the asymptotic distribution is no longer mixed Gaussian and that standard asymptotic inference can be misleading. However, Corollary 1 in Johansen (1997) can be used to show that inference on $\beta$ and $\alpha$ in the $I(2)$ model is efficient and unbiased also in the near $I(2)$ case. Since all results discussed in the subsequent sections have been obtained by cointegration analysis in the $I(2)$ model, the corollary result allows us to attach a fair degree of confidence to our empirical findings. Nonetheless, robustness is an important issue which needs to be further studied.

2.3 Univariate Dickey-Fuller Tests

The near $I(2)$ and the $I(2)$ series were all simulated for a fairly small signal-to-noise ratio (0.15) and the drift term was not easily detectable as it was dominated by very volatile first differences. Both processes contain two large characteristic roots, one associated with a high, $\sigma^2_{\varepsilon_q}$, the other with a small error variance, $\sigma^2_{\varepsilon_\mu}$. In this case, the power of univariate unit root tests to detect the second (near) unit root is likely to be low because the estimated residual is a function of $\varepsilon_q$ and $\varepsilon_\mu$. With a small signal-to-noise ratio, $\varepsilon_q$ will completely dominate $\varepsilon_\mu$, and the small but persistent drift that is associated with the second large root becomes hard to detect. To illustrate, we test the null of a unit

\[\text{null} \quad \varepsilon_q = 0 \quad \text{and} \quad \varepsilon_\mu = 0\]

This is because the second reduced rank condition (which is associated with the $I(2)$ model property) does not affect the asymptotic efficiency of the ML estimator of $\beta$ and $\alpha$.
Table 1: Testing the order of integration with a Dickey-Fuller test

<table>
<thead>
<tr>
<th>DF tests of</th>
<th>( q_t \sim I(2), \rho = 1.0 )</th>
<th>( q_t \sim \text{near}I(2), \rho = 0.95 )</th>
<th>( q_t \sim I(1), \rho = 0.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_{\varepsilon}/\sigma_{\varepsilon} )</td>
<td>0.15</td>
<td>0.15</td>
<td>-</td>
</tr>
<tr>
<td>( I(1) : \gamma_1 = 0 )</td>
<td>0.007</td>
<td>5.57</td>
<td>-0.004</td>
</tr>
<tr>
<td>( I(2) : \gamma_2 = 0 )</td>
<td>-0.31</td>
<td>-6.48</td>
<td>-0.72</td>
</tr>
<tr>
<td>( \sigma_{\varepsilon}/\sigma_{\varepsilon} )</td>
<td>1.0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>( I(1) : \gamma_1 = 0 )</td>
<td>0.0006</td>
<td>1.96</td>
<td>0.0006</td>
</tr>
<tr>
<td>( I(2) : \gamma_2 = 0 )</td>
<td>-0.021</td>
<td>-1.90</td>
<td>-0.031</td>
</tr>
</tbody>
</table>

The results indicate that the univariate Dickey-Fuller test has great difficulty detecting the large root associated with the persistent drift term, \( \mu_t \), when the signal-to-noise ratio is small. The null of a double unit root was strongly rejected both when \( q_t \sim I(2) \) and when it was a near-I(2) series with \( \bar{\rho} = 0.95 \). When \( q_t \) was a random walk it was also rejected but now correctly so. In all cases the (correct) null of (at least) one unit root could not be rejected. Section 3 reports a more comprehensive simulation study.

2.4 Multivariate I(2) trace tests

In a univariate model, it is straightforward to determine whether a variable has two large roots (with a modulus that is large but less than one) whereas in a multivariate model we can only determine the number of common stochastic trends in the system and whether they are of first order or second order. The classification of variables according to their order of integration is, therefore, more involved in the latter case.
which might suggest the use of univariate tests. But, as demonstrated in Section 3, univariate Dickey-Fuller type of tests have a low power to detect a second unit root, in particular when the signal to noise ratio is small. Since the multivariate model accounts for all information in the data a multivariate model is more powerful in this respect. Section 4.3 illustrates that the order of integration of individual series can also be determined within this model.

To illustrate the main issues we first analyze a three-dimensional VAR model in more detail and then check the generality of our findings using a much larger simulation study.

The unrestricted VAR with two lags can conveniently be expressed in acceleration rates, changes and levels:

\[
\Delta^2 x_t = \Gamma \Delta x_{t-1} + \Pi x_{t-1} + \mu_0 + \mu_1 t + \varepsilon_t
\]

(8)

where \(\{\Gamma, \Pi\}\) are variation free but \(\mu_0, \mu_1\) are restricted so as to exclude the possibility of quadratic trends. It is useful first to define the following: \(p\) is the number of variables in the VAR model, \(r\) is the cointegration rank, \(p-r=s_1+s_2\) is the number of common stochastic trends of which \(s_1\) are integrated of order one and \(s_2\) of order two.

The hypothesis that \(x_t \sim I(2)\) is formulated as two reduced rank hypothesis:

\[
\Pi = \alpha \beta'
\]

(9)

where \(\alpha, \beta\) are \(p \times r\) and \(r < p\), and

\[
\alpha' \Gamma \beta' = \xi \eta'
\]

(10)

where \(\beta'\), \(\alpha'\) are the \(p \times p-r\) orthogonal complements to \(\beta, \alpha\), and \(\xi, \eta\) are \(p-r \times p-r-s_2\). If \(s_2 = 0\), (10) is a full rank matrix and \(x_t \sim I(1)\). Testing whether \(x_t \sim I(2)\) amounts to testing \(s_2 > 0\) for \(r < p\).

The three variables have been simulated to reflect the real exchange rate properties in Section 4 and are reflecting the time series properties of the real exchange rate, \(q_t\), the interest rate differential between the two countries, \(b_t\), and the long-term drift term, \(\mu_t\) (for example measured by the change in nominal exchange rate, \(\Delta s_t\)). They are simulated in

\[5\]For an extensive discussion and analysis of CVAR models, see Juselius (2006, 2013).
accompanying with the IKE models (3) and (5):

\[ q_t = q_{t-1} + \mu_t + \varepsilon_{q,t} = \sum_{i=1}^{t} \sum_{s=1}^{i} \bar{\rho}^{i-s} \varepsilon_{\mu,s} + \sum_{i=1}^{t} \varepsilon_{q,i} + \bar{\rho} \mu_0 \sum_{i=1}^{t} \bar{\rho}^i + q_0 \]

\[ b_t = b_{t-1} + \mu_t + \varepsilon_{b,t} = \sum_{i=1}^{t} \sum_{s=1}^{i} \bar{\rho}^{i-s} \varepsilon_{\mu,s} + \sum_{i=1}^{t} \varepsilon_{b,i} + \bar{\rho} \mu_0 \sum_{i=1}^{t} \bar{\rho}^i + b_0 \]

\[ \mu_t = \bar{\rho} \mu_{t-1} + \varepsilon_{\mu,t} = \sum_{i=1}^{t} \bar{\rho}^{i-1} \varepsilon_{\mu,i} + \bar{\rho}^i \mu_0 \]

where \( \sigma_{\varepsilon_q} = 1.0, \sigma_{\varepsilon_b} = 0.2, q_t, \sigma_{\varepsilon_{\mu}} = 0.15, \) and \( \bar{\rho} = 0.95. \) Thus, the variables \( q_t \) and \( b_t \) share the same realization of \( \mu_t. \) For an application to a problem with similar but more realistic characteristics, see Johansen et al. (2010). As in the univariate case, we have generated 500 observations.

Provided the near unit root \( \bar{\rho} = 0.95 \) is approximated with a unit root in an empirical CVAR application, then the results would be consistent with \( r = 1, s_1 = 1, \) and \( s_2 = 1. \) In this case, \( q_t \) and \( b_t \) would share one common stochastic near I(2) trend and

\[ q_t - b_t = \sum_{i=1}^{t} \varepsilon_{q,i} - \sum_{i=1}^{t} \varepsilon_{b,i} + q_0 - b_0 \]  \( (11) \)

would be \( CI(2,1). \) Since the two I(1) trends in (11) cannot be canceled by a linear combination of \( \Delta x_t = [\Delta q_t, \Delta b_t, \Delta \mu_t], \) (11) corresponds to the \( \beta_{11} x_t \) relation \( (s_1 = 1) \) which can only become stationary by differencing. The polynomially cointegrated relation \( (r = 1) \) corresponds to \( \mu_t = \delta_1 \Delta q_t - \delta_2 \Delta b_t \) where \( \delta_1 + \delta_2 = 1. \)

Table 2 reports the multivariate rank test results where the first row shows the trace test for \( s_2 = 3, 2, 1, 0, \) given \( r = 0 \) and the second row for \( s_2 = 2, 1, 0, \) given \( r = 1. \) The hypothesis \( \{r = 1, s_2 = 1\} \) cannot be rejected based on a p-value of 0.23. It implies three unit roots in the characteristic polynomial. The lower part of Table 2 reports the modulus of the four largest roots in the characteristic polynomial. It appears that the unrestricted VAR contains two roots almost exactly equal to one and a third root, 0.96, which is very close to the simulated value of 0.95. If we approximate the latter with a unit root then the highest unrestricted root is 0.07 for the choice of \( \{r = 1, s_1 = 1, s_2 = 1\}. \) But if we ignore the possibility of I(2) then the model would contain an unrestricted root of 0.96 for the choice of \( \{r = 1, s_1 = 2, s_2 = 0\}. \) Such a large root is likely to jeopardize standard inference on stationarity at least for some hypotheses (Johansen, 2006b).

Paruolo and Rahbek (1999) suggested the following parameterization of (8):
Table 2: Determination of the two rank indices in the bivariate model

<table>
<thead>
<tr>
<th>Rank Test Statistics</th>
<th>( p - r )</th>
<th>( r )</th>
<th>( s_2 = 3 )</th>
<th>( s_2 = 2 )</th>
<th>( s_2 = 1 )</th>
<th>( s_2 = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>1169.66</td>
<td>727.76</td>
<td>375.24</td>
<td>334.49</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>[0.00]</td>
<td>[0.00]</td>
<td>[0.00]</td>
<td>[0.00]</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>366.12</td>
<td>\textbf{28.42}</td>
<td>17.33</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>[0.00]</td>
<td>[0.23]</td>
<td>[0.40]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>7.92</td>
<td>3.69</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>[0.84]</td>
<td>[0.78]</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The characteristic roots:
Unrestricted VAR: 0.99 0.99 0.96 0.07
\( r = 1 \); \( s_1 = 2 \); \( s_2 = 0 \)
\( r = 1 \); \( s_1 = 1 \); \( s_2 = 1 \)

Table 3: The estimated values of \( \beta, \beta_{\perp 1}, \beta_{\perp 2} \) for the simulated I(2) process

<table>
<thead>
<tr>
<th>( q_t )</th>
<th>( b_t )</th>
<th>( \mu_t )</th>
<th>( t )</th>
<th>( \Delta q_t )</th>
<th>( \Delta b_t )</th>
<th>( \Delta \mu_t )</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta' )</td>
<td>-0.00</td>
<td>-0.00</td>
<td>1.00</td>
<td>0.00</td>
<td>( \delta' ) -0.49</td>
<td>-0.46</td>
<td>-0.00</td>
</tr>
<tr>
<td>( \beta_{\perp 1} )</td>
<td>-0.93</td>
<td>1.00</td>
<td>0.00</td>
<td>0.02</td>
<td>( \beta_{\perp 2} )</td>
<td>1.00</td>
<td>0.93</td>
</tr>
</tbody>
</table>

The polynomially cointegrated relation: \( \beta' x_t + \delta' \Delta x_t \)

The medium run cointegrated relation: \( \beta_{\perp 1} x_t \)

The non-cointegrating relation: \( \beta_{\perp 2} x_t \)

\[ \Delta^2 x_t = \alpha(\beta' x_{t-1} + \delta' \Delta x_{t-1}) + \zeta \tau' \Delta x_{t-1} + \mu_0 + \mu_t t + \varepsilon_t \]

where \( \tau = [\beta, \beta_{\perp 1}] \). Table 3 report the estimates of \( \beta, \beta_{\perp 1}, \beta_{\perp 2} \).

The test of the hypothesis:

\[
H_T = \begin{bmatrix}
1 & 0 \\
-1 & 0 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\beta' x_t \\
\beta_{\perp 1} x_t \\
\end{bmatrix}
\]

could not be rejected based on \( \chi^2(2) = 1.60[0.45] \) consistent with the true data generating process.

To conclude the main difference between testing the order of integration based on a univariate versus a multivariate model is that the former is generally unable to detect a large root in \( \mu_t \) when the signal to noise ratio is small, whereas the multivariate trace test is able to detect the large root. Evidence of the second large root can also be found by
calculating the characteristic roots of the multivariate model which is not the case in the univariate model when the signal-to-noise ratio is small. Approximating a root of 0.95 with a unit root allows us in addition to structure the data in terms of cointegration and common trends exploiting the different persistency profiles of the DGP. For sample sizes typical of economic data the tests would generally fail to reject the unit root hypothesis when there is a near unit root in the DGP.

3 A simulation study

We consider now the IKE model (6) - (7)

\[
\begin{align*}
    x_{1,t} &= x_{1,t-1} + \mu_t + \varepsilon_{1,t} \\
    x_{2,t} &= x_{2,t-1} + \mu_t + 0.2\varepsilon_{2,t} \\
    \mu_t &= 0.95\mu_{t-1} + 0.15\varepsilon_{u,t}.
\end{align*}
\]

where \(x_{1,t}\) represents the real exchange rate and \(x_{2,t}\) the real bond rate differential. They share the same drift term but differs in terms of the short-term volatility with \(x_{1,t}\) fluctuating much more than \(x_{2,t}\). The drift term \(\mu_t\) is highly persistent and the signal-to-noise ratio between \(\varepsilon_{1,t}\) and \(\varepsilon_{u,t}\) is 1/0.15. The model (12)-(14) represents the baseline model. Since \(\mu_t\) is generally not directly observable we have simulated three versions of the three-dimensional process \(x_t = [x_{1,t}, x_{2,t}, x_{3,t}]\) which differs with respect to the choice of \(x_{3,t}\):

- **Case 1**: \(x_{3,t}^{(1)} = \mu_t\)
- **Case 2**: \(x_{3,t}^{(2)} = \mu_t + 0.2\varepsilon_{4,t}\)
- **Case 3**: \(x_{3,t}^{(3)} = \mu_t + \varepsilon_{5,t}\)

where \(\varepsilon_{i,t} \sim NID(0, \sigma_i^2), i = 1, ..., 5\).

Table 4 reports the simulated results of testing the hypotheses \((r, s_2)\) for \(r = 0, 1, 2, 3\) and \(s_2 = p - r, p - r - 1, ..., 0\) based on the multivariate rank test in a CVAR model. The latter is estimated for the three cases \(S_1 = (x_{1,t}, x_{2,t}, x_{3,t}^{(1)})\), \(S_2 = (x_{1,t}, x_{2,t}, x_{3,t}^{(2)})\), \(S_3 = (x_{1,t}, x_{2,t}, x_{3,t}^{(3)})\) using two lags. \(S_1\) represents the case when the drift term, \(\mu_t\), is known, \(S_2\) when it can be measured with a small error, \(S_3\) when it is very imprecisely measured.

The results show generally that the multivariate I(2) test:

- rarely rejects cointegration, except when \(\mu_t\) is imprecisely measured (\(S_3\)),
- seldom accepts two or three cointegration relations and no I(2) trends,
Table 4: Simulated frequences of testing the cases \((r = i \text{ and } s_2 = 3 - i - j), i = 0, 1, 2, 3 \text{ and } j = 3 - i, ..., 0\)

<table>
<thead>
<tr>
<th>(H(r, s_2))</th>
<th>(T = 50)</th>
<th>(T = 100)</th>
<th>(T = 250)</th>
<th>(T = 500)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_1)</td>
<td>(S_2)</td>
<td>(S_3)</td>
<td>(S_1)</td>
<td>(S_2)</td>
</tr>
</tbody>
</table>

An \(I(2)\) process \(\rho = 1.0\):

- rarely rejects \(I(2)\) when \(T = 50\) or 100 provided \(\mu_t\) is either known \((S_1)\) or measured with a small error \((S_2)\),
- rarely rejects the preferred hypothesis \(H(1,1)\) unless \(T\) is large (500) and \(\mu_t\) is imprecisely measured \((S_3)\),
- frequently fails to reject the hypothesis \(H(1,0)\) when \(T\) is large and \(\mu_t\) is imprecisely measured \((S_3)\), i.e. \(\bar{\rho} = 0.95\) is found to be significantly different from a unit root, and
- seldom accepts that the process can be stationary.

In general, the larger the sample the smaller the standard errors and the higher the power of the test to discriminate between a root

\[
\begin{array}{cccccccccc}
\text{An } I(2) \text{ process } \rho = 1.0 \\
H(0,3) & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
H(0,2) & 0.1 & 0.3 & 0.6 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
H(0,1) & 9.0 & 16.4 & 24.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
H(0,0) & 2.6 & 3.4 & 4.2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
H(1,2) & 7.9 & 8.1 & 5.0 & 0.0 & 0.0 & 0.1 & 0.0 & 0.0 & 0.0 \\
H(1,1) & 72.1 & 62.8 & 48.6 & 93.4 & 89.6 & 94.4 & 90.0 & 48.7 & 94.8 & 89.2 & 45.1 \\
H(1,0) & 0.9 & 1.5 & 4.0 & 0.6 & 1.6 & 9.3 & 0.7 & 1.5 & 11.8 & 0.4 & 1.6 & 12.3 \\
H(2,1) & 5.0 & 5.3 & 9.0 & 4.1 & 5.5 & 17.6 & 3.2 & 5.2 & 23.1 & 3.0 & 5.4 & 24.0 \\
H(2,0) & 1.6 & 1.4 & 3.3 & 1.2 & 2.4 & 8.7 & 1.2 & 2.7 & 13.7 & 1.3 & 2.8 & 15.4 \\
H(3,0) & 0.6 & 0.8 & 1.4 & 0.7 & 0.9 & 2.5 & 0.5 & 0.7 & 2.6 & 0.5 & 1.1 & 3.2 \\
\end{array}
\]

A near \(I(2)\) process \(\rho = 0.95\):

- rarely rejects \(I(2)\) when \(T = 50\) or 100 provided \(\mu_t\) is either known \((S_1)\) or measured with a small error \((S_2)\),
- rarely rejects the preferred hypothesis \(H(1,1)\) unless \(T\) is large (500) and \(\mu_t\) is imprecisely measured \((S_3)\),
- frequently fails to reject the hypothesis \(H(1,0)\) when \(T\) is large and \(\mu_t\) is imprecisely measured \((S_3)\), i.e. \(\bar{\rho} = 0.95\) is found to be significantly different from a unit root, and
- seldom accepts that the process can be stationary.

In general, the larger the sample the smaller the standard errors and the higher the power of the test to discriminate between a root

\[
\begin{array}{cccccccccc}
\text{A near } I(2) \text{ process } \rho = 0.95 \\
H(0,3) & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
H(0,2) & 0.1 & 0.3 & 0.8 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
H(0,1) & 9.5 & 17.2 & 26.2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
H(0,0) & 5.1 & 6.2 & 6.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
H(1,2) & 7.7 & 7.6 & 4.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
H(1,1) & 70.1 & 60.6 & 46.0 & 94.2 & 89.8 & 55.7 & 89.6 & 73.4 & 14.5 & 69.6 & 28.7 & 0.2 \\
H(1,0) & 1.0 & 2.2 & 7.0 & 0.7 & 3.9 & 26.7 & 4.0 & 18.6 & 67.3 & 22.5 & 63.5 & 82.8 \\
H(2,1) & 5.2 & 4.7 & 6.9 & 4.0 & 5.4 & 14.6 & 3.8 & 6.0 & 14.7 & 3.7 & 5.1 & 13.6 \\
H(2,0) & 0.8 & 0.8 & 1.5 & 0.8 & 0.6 & 2.1 & 2.1 & 1.6 & 2.9 & 3.8 & 2.4 & 2.9 \\
H(3,0) & 0.5 & 0.4 & 0.6 & 0.2 & 0.3 & 0.5 & 0.5 & 0.4 & 0.6 & 0.5 & 0.3 & 0.5 \\
\end{array}
\]
Table 5: Simulated frequencies for testing the order of integration based on the ADF

<table>
<thead>
<tr>
<th>Lags</th>
<th>$T = 50$</th>
<th>$T = 100$</th>
<th>$T = 250$</th>
<th>$T = 500$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>3</td>
<td>0.11</td>
<td>0.94</td>
<td>0.91</td>
<td>0.95</td>
</tr>
<tr>
<td></td>
<td>0.00</td>
<td>0.95</td>
<td>0.76</td>
<td>0.91</td>
</tr>
<tr>
<td>x_{1,t} is an I(2) process $\tilde{\rho} = 0.95$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I(2)</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>I(1)</td>
<td>0.94</td>
<td>0.95</td>
<td>0.95</td>
<td>0.95</td>
</tr>
<tr>
<td>I(0)</td>
<td>0.06</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
</tbody>
</table>

of 0.95 and 1.00. While the drift term $\mu_t$ is equally persistent with a sufficiently large sample size even a tiny deviation from unity will be found significant. Also, the more imprecisely $\mu_t$ is measured, the more difficult is it to detect the large near I(2) root in the model.

Table 5 reports simulated univariate augmented Dickey-Fuller tests of the hypothesis $x_{1,t} \sim I(2), I(1), I(0)$ based on two respective three lags for $T = 50, 100, 250$ and 500. Each case has been replicated 5,000 times. The simulations of $x_{1,t}$ are identical to the ones being analyzed in Table 3 above. The upper part of the table reports the results for $\tilde{\rho} = 1.0$ and the lower part for $\tilde{\rho} = 0.95$. The results show that the ADF test

- rejects I(2) in essentially all cases whether $\rho = 1$ or 0.95,
- fails to reject I(1) in the absolute majority of cases

Thus, the results suggest that the univariate Dickey-Fuller test will essentially never detect a double unit root when the signal-to-noise ratio is small.

4 Empirical illustration: the dollar Dmk rate

This section illustrates the same test procedures based on actual real exchange data for USA and Germany for a sample from May 1975 to December 1998 comprising the post Bretton Woods period of currency float. Section 4.1 first reports the univariate Dickey-Fuller test to determine the order of integration of the real and nominal exchange rate, US and German prices and long-term interest rates. Section 4.2 then reports the order of integration and cointegration based on the multivariate
trace test in a five-dimensional VAR model of $x'_t = [p_{1,t}, p_{2,t}, s_{12,t}, b_{1,t}, b_{2,t}]$ where $p$ stands for prices, $s$ for the nominal dollar-Dmk rate, $b$ for long-term bond rates and the subscript 1 stands for USA and 2 for Germany. Finally Section 4.3 reports tests of the order of integration of the individual series within the multivariate model.

### 4.1 Univariate Dickey-Fuller tests

Table 6 report the Dickey-Fuller univariate tests of I(1) based on (1), and I(2) based on (2) for the five variables and for the following transformations: real exchange, $p_{1,t} - p_{2,t} - s_{12,t}$, the interest rate differential, $b_{1,t} - b_{2,t}$ and relative prices, $p_{1,t} - p_{2,t}$. The I(1) tests failed to reject the null of one unit root in all cases except for the relative price for which $\hat{\rho}_1 = -0.005$ was found to be significant due to a very small standard error. The I(2) version of the Dickey-Fuller test failed to detect a second (near) unit root in almost all cases except for US long-term bond rate and US prices (borderline). For both the nominal and real exchange rate a double unit root was strongly rejected with almost identical test statistics signifying the fact that the two variables have moved almost in parallel. That a second large root was detected in the US bond rate is interesting and suggests that it has exhibited similar long swings as the real exchange rates while not exhibiting the same short-term volatility. That the test borderline failed to reject a double unit root in the US CPI is also interesting as consumer prices are not likely to be subject to speculation in any significant manner and, hence, should not be affected by the long swings drift term $\mu_t$. Therefore, a double root in US CPI prices suggests that CPI shocks, while not necessary large, have been persistent over this period. Thus, the result may indicate that the power of Dickey-Fuller test to detect a double unit root is reasonable provided the signal-to-noise ratio is not too small.
### Table 7: Determination of the two rank indices

<table>
<thead>
<tr>
<th>Rank Test Statistics</th>
<th>$p - r$</th>
<th>$r$</th>
<th>$s_2 = 5$</th>
<th>$s_2 = 4$</th>
<th>$s_2 = 3$</th>
<th>$s_2 = 2$</th>
<th>$s_2 = 1$</th>
<th>$s_2 = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1</td>
<td>489.2</td>
<td>291.6</td>
<td>149.5</td>
<td><strong>84.1</strong></td>
<td><strong>77.19</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2</td>
<td>140.4</td>
<td><strong>56.96</strong></td>
<td><strong>42.85</strong></td>
<td><strong>37.22</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3</td>
<td><strong>36.7</strong></td>
<td>17.4</td>
<td>15.3</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Six largest characteristic roots:

- Unrestricted VAR: $0.99$ $0.99$ $0.98$ $0.98$ $0.82$ $0.50$
- $r = 2, p - r = 3$: $1.0$ $1.0$ $1.0$ $0.96$ $0.96$ $0.50$
- $r = 2, s_1 = 2, s_2 = 1$: $1.0$ $1.0$ $1.0$ $1.0$ $0.96$ $0.51$
- $r = 2, s_1 = 1, s_2 = 2$: $1.0$ $1.0$ $1.0$ $1.0$ $1.00$ $0.51$

### 4.2 Multivariate VAR based unit root tests

We estimate the VAR model (8) for $x_t' = [p_{1t}, p_{2t}, s_{12t}, b_{1t}, b_{2t}]$ augmented with a few dummy variables, primarily to account for the German reunification as explained in Johansen et al. (2010) and in Juselius (2012).

Table 7 reports the $I(2)$ trace tests as well as the characteristic roots of the model. Since all versions of $r = 0$ were strongly rejected we do not report them in the table. The estimated characteristic roots reported in Table 7 suggest a total number of five large roots in the unrestricted VAR, four of which are almost exactly on the unit circle, while the fifth, while large (0.88), is not equally close to one. Juselius (2012) shows that the case $(r = 2, s_1 = 1, s_2 = 2)$ is theoretically consistent with an IKE-based model. The first acceptable choice is $\{r = 1, s_2 = 3, s_2 = 1\}$. Jensen (2013) has shown that the correct test procedure is now to consider the next case along diagonal, i.e. $\{r = 2, s_1 = 1, s_2 = 2\}$. Both cases are consistent with five unit roots in the model. The latter case is consistent with our theoretical prior and has a p-value of 0.93 and the largest unrestricted root given this choice is 0.51. If, instead, we had chosen $s_2 = 0$ (and treated the variables in the model as $I(1)$), our model would contain two large roots with a modulus of 0.96 and any inference on stationarity would be jeopardize in the case.

To summarize: the case $\{r = 2, s_1 = 1, s_2 = 2\}$ is strongly supported by the data, is able to account for all five large roots in the unrestricted VAR, and is consistent with the economic prior.
4.3 Testing the order of integration of individual variables

As discussed above, the multivariate trace tests are informative about the order of integration of the vector process but are generally uninformative about the individual variables. It is, however, straightforward to formulate and test hypotheses of the order of integration of these variables given the choice of \( r, s_1 \) and \( s_2 \). We first define \( \tau = \{\beta, \beta_{L,1}\} \). The hypothesis that a variable is \( I(1) \) in the \( I(2) \) CVAR model can be formulated as a known vector \( b_1 \) in \( \tau = (b_1, b_{1L}\varphi) \) where \( b_{1L}\varphi \) defines the other vectors to be restricted to lie in the orthogonal space of \( b_1 \). For example, \( b_1 = [0, 0, 1, 0, 0, 0, 0] \) is a test of the hypothesis that the nominal exchange rate, \( s_{12,t} \), is a unit vector in \( \tau \). If not rejected, it can be considered \( I(1) \), otherwise \( I(2) \). See Johansen et al. (2006) for further details.

As prices generally are subject to linear deterministic (as well as stochastic) trends we need to test the hypothesis that the price as well as the trend-adjusted price is \( I(1) \). As discussed in Johansen et al. (2010) this can be formulated as a test on \( \beta = (H_1\varphi_1, H_2\varphi_2) \). For example, \( H_1' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \) would be a test of trend-adjusted US price being \( I(1) \). As the VAR model was specified to allow for a change in the slope of the linear trend at the time of the reunification we formulate the test to also allow for this possibility. Table 8 reports the Likelihood Ratio test results. Except for the German bond rate, all hypotheses were strongly rejected, indicating that the differenced processes are persistent enough to reject the \( I(1) \) hypothesis in favor of an \( I(2) \)-type characterization. \(^6\)

5 Conclusions

Macro-financial data (nominal exchange rates, stock prices, etc.) are typically characterized by very volatile short-run changes along smooth persistent trends. This implies that the drift term in the changes of the process is a very persistent process with a small variance compared to the large variance of the short-run changes, i.e. the process is (near) \( I(2) \) with a small signal-to-noise ratio. But while one would a priori expect near-\( I(2) \) trends to be prevalent in exchange rates, prices, and other variables affected by financial speculation, researchers seldom find evidence of \( I(2) \) in such data based on univariate Dickey-Fuller tests. In contrast, the multivariate ML trace test frequently find such evidence.

\(^6\)The inability to reject the \( I(1) \) hypothesis for the German bond rate with a p-value of 0.20 indicates that the German bond rate has moved in a slightly less persistent manner than the other variables.
Table 8: Testing hypotheses of I(1) versus I(2)

<table>
<thead>
<tr>
<th>p_{1,t}</th>
<th>p_{2,t}</th>
<th>s_{12,t}</th>
<th>b_{1,t}</th>
<th>b_{2,t}</th>
<th>t_{91}</th>
<th>t</th>
<th>\chi^2(v)</th>
<th>p - val</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>-1.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>64.09</td>
<td>0.00</td>
</tr>
<tr>
<td>-</td>
<td>-1.0</td>
<td>1.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>23.6</td>
<td>0.00</td>
</tr>
<tr>
<td>1.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>*</td>
<td>39.1</td>
<td>0.00</td>
</tr>
<tr>
<td>-</td>
<td>-1.0</td>
<td>1.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>*</td>
<td>48.02</td>
<td>0.00</td>
</tr>
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<td>1.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>11.2</td>
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<td>-</td>
<td>-</td>
<td>-</td>
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</tr>
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<td>-</td>
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<td>1.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>5.5(3)</td>
<td>0.24</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>-1.0</td>
<td>-1.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>10.4</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Our paper demonstrates by simulations that univariate Dickey-Fuller seldom finds a double unit root when the signal-to-noise-ratio is small.

6 References


change Rate Based On Real Interest Differentials, “American Economic Review” 69, September, 610-22.


