An adaptive method for PDEs which preserves scaling symmetry

Mike Baines

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Matthew Hubbard, Peter Jimack, Stefan King
Outline

1. PDEs, scale invariance, and similarity
2. Preservation of best $L_2$ fits to similarity solutions
3. A velocity-based approach
4. Moving mesh methods
5. Example and Discussion
The aim of the talk is to construct a discrete algorithm which preserves PDE scaling symmetry, and to generalise it so that it can be applied in approximate form to PDE solutions which have this symmetry as an attractor.
Why move meshes?

- For adaptivity - as an alternative to, or in addition to, AMR.
- As a way of following special features of a solution, such as shocks, singularities, moving boundaries.
- As a way of satisfying geometric constraints/properties, such as scale-invariance, orderings, asymptotics.

In the latter case all the variables are interdependent. This requires solution-adaptive methods which move the spatial mesh in time.
We consider scalar time-dependent PDE problems of the form

\[ u_t = \mathcal{L}u \]

where \( \mathcal{L} \) is a purely spatial operator, posed on a (possibly moving) interval \( a(t), b(t) \), which are

- scale-invariant
- mass-conserving
A PDE problem is **scale-invariant** if there exists a parameter \( \lambda \) and scaling indices \( \beta, \gamma \) such that the scalings

\[
t \rightarrow \lambda t, \quad x \rightarrow \lambda^\beta x, \quad u \rightarrow \lambda^\gamma u
\]

leave the problem unaltered.

It is **mass-conserving** if the boundary conditions are such that the total ’mass’

\[
\int_{a(t)}^{b(t)} u\,dx
\]

is conserved (in which case \( \beta + \gamma = 0 \)).
A self-similar solution of our mass-conserving scale-invariant PDE problem is a relationship between the similarity variables

$$x/t^\beta, \quad u/t^{-\beta}$$

of the form

$$\frac{u}{t^{-\beta}} = f\left(\frac{x}{t^\beta}\right),$$

(thus exhibiting a scaling symmetry) where $f$ satisfies an ODE.
For example, the nonlinear porous medium equation (PME)

\[ u_t = (u^n u_x)_x \]

\((n \geq 1)\), with \(u = 0\) at the boundaries, has the following compact self-similar source solution for \(u\):

\[
\begin{array}{c}
\text{Porous Medium Equation: } n=1 \\
\text{Porous Medium Equation: } n=2
\end{array}
\]
A Theorem

For any two interior points \( x_1(t), x_2(t) \) that vary as \( t^\beta \) the local mass of the self-similar solution \( u^{ss} \),

\[
\int_{x_1(t)}^{x_2(t)} u^{ss} \, dx,
\]

is constant in time

iff

the self-similar solution \( u^{ss} \) is preserved in time.
Let $U$ lie in the space spanned by the finite set of scale-invariant basis functions $\{W_i\}$ (varying as $t^\beta$ and forming a partition of unity) and let $U_{bf}^{ss}$ be the best $L_2$ fit to $u^{ss}$ in this space.

Then, using the normal equations

$$\int_{a(t)}^{b(t)} W_i U_{bf}^{ss} \, dx = \int_{a(t)}^{b(t)} W_i u^{ss} \, dx$$

we can prove...
Theorem 2

When all points on the x-axis vary as $t^\beta$, the **distributed** mass of the self-similar solution,

$$\int_{a(t)}^{b(t)} W_i u^{ss} \, dx,$$

is constant in time

**iff**

the **best** $L_2$ fit $U_{bf}^{ss}$ to a self-similar solution $u^{ss}$ is preserved in time.
A symmetry-preserving algorithm

Hence, for a scale-invariant mass-conserving PDE problem with mass conservation, the scale-invariant algorithm:

1. Find the best $L_2$ fit to the (self-similar) initial data $u^{ss}$ in the space spanned by the $W_i$.
2. Move points on the $x$-axis by a factor $t^\beta$.
3. Recover the approximation $U$ by inverting the distributed conservation principle

$$\int_{a(t)}^{b(t)} W_i U dx = \text{its initial value}$$

carries the best $L_2$ fit to a self-similar solution in time, thus preserving a scaling symmetry in a discrete setting.
Piecewise linear best fits

The result holds in particular for

- best piecewise linear $L_2$ fits, and
- best linear discontinuous $L_2$ fits with adjustable nodes,
  (giving an optimal mesh, one for which the $L_2$ norm of the error is minimised over mesh points as well as function values).

(If the $L_2$ error of the total mass is conserved (as it is here), such optimal best linear $L_2$ fits approximately equidistribute the $L_2$ error, a property therefore also preserved in time in the self-similar case.)
PART 2

We now seek a moving mesh method for general initial data which is scale-invariant and carries the best fit in the event of self-similar data.

We shall first construct a scale-invariant deformation procedure which reduces to the $t^\beta$ spatial deformation in the self-similar case.
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- Show that the self-similar form $v^{ss}$ is $\beta x/t$ and hence $x(t)$ moves as $t^\beta$ in that case.
- Show also that for a special class of PDEs $v^{ss}$ is preserved in time.
Strategy

Our strategy is the following:

- Find a velocity $v$ which is equivalent to the conservation principle.
- Show that the self-similar form $v^{ss}$ is $\beta x/t$ and hence $x(t)$ moves as $t^\beta$ in that case.
- Show also that for a special class of PDEs $v^{ss}$ is preserved in time.
- Invoke the first Theorem to deduce that $u^{ss}$ is preserved in time.
A mass-conserving velocity

By Leibnitz’ Rule, in a frame moving with velocity \( v \),

\[
\frac{d}{dt} \int_{x_1(t)}^{x_2(t)} u \, dx = \int_{x_1(t)}^{x_2(t)} u_t \, dx + [uv]_{x_1(t)}^{x_2(t)} = \int_{x_1(t)}^{x_2(t)} (u_t + (uv)_x) \, dx
\]

for all \( x_1(t), x_2(t) \). Hence the conservation principle

\[
\int_{x_1(t)}^{x_2(t)} u \, dx = \text{constant in time}
\]

for all \( x_1(t), x_2(t) \) is equivalent to \( u, v \) satisfying the PDE

\[
\frac{du}{dt} + (uv)_x = 0
\]
The self-similar velocity which satisfies this equation, for any $u^{ss}$, is

$$v^{ss} = \frac{\beta x}{t}$$

It then follows from $v^{ss} = \frac{dx^{ss}(t)}{dt}$ that $x^{ss}(t) \sim t^\beta$, as required.
A particular PDE

For the special class of PDEs

\[ u_t = A \left( u^{1+n_0} (u_x)^{n_1} (u_{xx})^{n_2} \ldots \right)_x \]

where \( A \) is a constant, \( \nu \) satisfies the PDE

\[ \nu_t + \beta^{-1} \nu \nu_x = 0 \]

except for terms involving derivatives of \( \nu \) higher than second. Hence, with initial data \( \nu^{ss} = \beta x \) at \( t = 1 \) (zero at the origin) this equation has the unique solution \( \nu^{ss} = \beta x / t \) for all time.

It follows that all \( x^{ss}(t) \)'s vary as \( t^\beta \) and from the first Theorem that the self-similar \( u^{ss} \) is preserved in time.
Approximation strategy

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Our strategy to **approximate** this result is:

- Find an equation for a velocity $V$ which is equivalent to the distributed conservation principle
- Approximate $V$ by a piecewise linear function
- Use the second Theorem to deduce that the best fit $U_{bf}^{ss}$ to $u^{ss}$ is approximately preserved in time, recovering $U$ by inverting the distributed conservation principle

\[
\int_{a(t)}^{b(t)} W_i U \, dx = \text{its initial value}
\]
A distributed mass-conserving velocity

A weak form of Leibnitz’ Rule is

\[ \frac{d}{dt} \int_{a(t)}^{b(t)} W_i U dx = \int_{a(t)}^{b(t)} W_i (U_t + (UV)_x) dx \]

If, for all \( W_i \),

\[ \int_{a(t)}^{b(t)} W_i U dx = \text{constant in time} \]

then \( V \) satisfies

\[ \int_{a(t)}^{b(t)} W_i (U_t + (UV)_x) dx = 0 \]
An equation for the velocity $V$

Using a weak form of the PDE, this equation gives

$$-\int_{a(t)}^{b(t)} W_i (UV)_x dx = \int_{a(t)}^{b(t)} W_i \mathcal{L} U dx$$

With $V$ a piecewise linear projection this leads to a matrix equation for the nodal values $V$ of the form

$$B(U)V = g$$

However, the matrix $B(U)$ is ill-conditioned (quasi-antisymmetric) and difficult to invert. Regularisation is possible but precision is lost.
Three piecewise linear projections for $V$

From the weak form Leibnitz’ Rule,

$$- \int_{a(t)}^{b(t)} W_i (UV)_x \, dx = \int_{a(t)}^{b(t)} W_i \mathcal{L} U \, dx,$$

consistent with the weak form conservation principle but ill-conditioned.
Three piecewise linear projections for $V$

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consistent with the weak form conservation principle but ill-conditioned.

2. By direct $L_2$ projection of the velocity $v$ into piecewise linears, introducing a small error into the weak form conservation principle but well-conditioned.
Three piecewise linear projections for $V$

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$$- \int_{a(t)}^{b(t)} W_i(UV)_x \, dx = \int_{a(t)}^{b(t)} W_i L U \, dx,$$

consistent with the weak form conservation principle but ill-conditioned.

2. By direct $L_2$ projection of the velocity $v$ into piecewise linears, introducing a small error into the weak form conservation principle but well-conditioned.

3. Using nodewise collocation,

$$V_i = v_i(U)$$

with upwind finite differences for the derivatives, again introducing a small error into the weak form conservation principle but respecting the boundary condition on $v$. 

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The moving mesh method implemented here consists of:

- A piecewise linear projected velocity $V$.
- First order time-stepping in $t^\beta$, which is exact for self-similar solutions.
- A piecewise linear $U$, recovered from the conservation principle

\[ \int_{a(t)}^{b(t)} W_i U dx = \text{its initial value} \]
Numerical example

The porous medium equation (PME)

$$u_t = \left( u^n u_x \right)_x$$

($n \geq 1$), with an exact self-similar source solution

$$u_{ss} = A t^{-\beta} \left\{ 1 - \left( \frac{x}{t^{\beta}} \right)^2 \right\}^{1/n}$$

($A$ a constant) will be used for comparisons.
Accuracy - against self-similar solutions

$L_2$ errors for the three methods for $n = 1$ and $3$ (with equispaced initial data).
A random perturbation and its evolution compared with two self-similar solutions, for $n = 1$. 
Waiting times - using a projected form of velocity

Waiting times with initial data $u = (1 - x)^p$.

$n=2$, with 81 nodes.
Waiting times - using collocation and upwinding

Waiting time for initial data \( u = (1 - x^2)^{4.5} \) using upwinding. (n=1 with 11, 21 and 41 nodes).
Conclusions

- A conservation principle moving with the similarity velocity is equivalent to self-similar solutions being preserved in time.
- A weak conservation principle moving with the similarity velocity is equivalent to the best $L_2$ fit to a self-similar solution being preserved in time.
- A general scale-invariant velocity-based procedure is constructed which preserves self-similar solutions for a class of PDEs.
- Scale-invariant moving mesh numerical methods are found which approximately preserve self-similar solutions in the event of a self-similar solution.
- The methods are validated on the example of the porous medium equation.
Extensions

- Non mass-conserving problems.
- Scale invariant conservation principles using monitor functions.
References


