Localised modes due to defects in high contrast periodic media via homogenization

Ilia V. Kamotski and Valery P. Smyshlyaev

Bath Institute For Complex Systems

Preprint 3/06 (2006)

http://www.bath.ac.uk/math-sci/BICS

Localised modes due to defects in high contrast periodic media via homogenization

I.V. Kamotski[1,2] and V. P. Smyshlyaev[1]

15th February 2006

[1] Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, UK.

[2] Corresponding author: tel. +44 1225 386224 fax. +44 1225 386492
e-mail: I.Kamotski@maths.bath.ac.uk

Abstract

The spectral problem for an infinite periodic medium perturbed by a compact defect is considered. This may be seen for example as a simplified scalar cross-sectional model of the problem for localised modes in photonic crystal fibers. For a high contrast small size periodicity and a finite size defect we consider the critical (the so called double porosity type) scaling. We employ (high contrast) homogenization for deriving asymptotically explicit limit equations for the localised modes (exponentially decaying eigenfunctions) and associated eigenvalues. Those are expressed in terms of the eigenvalues and eigenfunctions of a perturbed version of a "two-scale" limit operator introduced by V.V. Zhikov, with an emergent explicit nonlinear dependence on the spectral parameter for the spectral problem at the macroscale. Using the method of asymptotic expansions supplemented by a "high contrast" boundary layer analysis we establish the existence of the actual eigenvalues near the eigenvalues of the limit operator, with tight "ε square root" error bounds (ε is the small parameter). An example for circular or spherical defects in a periodic medium with isotropic homogenized properties is given and displays explicit limit eigenvalues and eigenfunctions. Further results on improved convergence of eigenfunctions via the technique of strong two-scale resolvent convergence and associated two-scale compactness properties are discussed.

Keywords: localised modes, defects in periodic media, high-contrast homogenization, double porosity scaling, eigenvalue problem, boundary layer analysis, error bounds

1 Introduction

Studying the spectral properties of operators with periodic coefficients, with and without "defects", has recently received considerable attention in the mathematical literature, see e.g. recent review [22]. The interest is at present mostly motivated by problems in physics and engineering associated with photonic or phononic crystals and photonic crystal fibers, see e.g. [37], [30]. A photonic crystal fiber (PCF) for example represents geometrically a periodic medium (whose physical properties vary across the fiber but not along it), with the defect being its "core", which is a propagating "channel" or a waveguide: electromagnetic waves of certain frequencies (the "band gap" frequencies) fail to propagate in the surrounding periodic medium and hence remain localised inside the PCF, which allows for them to propagate along the core for long distances with little loss [30]. Mathematically, the problem reduces to an appropriate spectral problem at the cross-section of the PCF, cf. Figure 1. This is that of characterisation of localised modes or eigenstates (whenever such exist) in the band gaps in the Floquet-Bloch spectrum for the Maxwell's operator in the surrounding periodic medium with a fixed "propagation constant" (the wave vector along the fiber). The latter cross-sectional geometry is a periodic medium "perturbed" by a finite size heterogeneity (domain Ω_2 in Fig. 1). The problem is hence first in detecting the band gaps in the periodic medium without defects and then in finding, in the presence of a defect, the "extra point spectrum" in the gaps as well as the associated eigenfunctions, the localised states. In the present work we aim at detecting such localised modes in an asymptotically explicit way due to defects in high contrast periodic medium using the tools of (high contrast) homogenization theory. In physical terms, we consider a simplified model with scalar rather than Maxwell's equations and with in effect zero propagation constant. We expect that this nevertheless captures the essence of the underlying effects, making thereby the proposed methodology more transparent avoiding at the same time additional technical complications.

Substantial literature is devoted to problems from the above described general class. Apart from numerous computational approaches (e.g., [26, 19, 24, 25]), most of the mathematical treatments have been "qualitative", establishing the existence of the band gaps, of the point spectrum in the gaps in the presence of defects, some bounds on the number of the eigenvalues in the gap, on the pattern of the (exponential) decay of the eigenmodes, etc, see [22] and further references therein. If however the problem contains one or more small parameters, e.g. *high contrast*, often in the presence of other small parameters, e.g. thickness of thin periodic (high contrast) structures, the asymptotic methods become potentially applicable for more explicit answers to the above questions, cf. [33]. In mathematical literature, various results on the existence and the asymptotic description of the band gaps in high contrast periodic media have been obtained on this way, [16, 17, 20, 29]. In particular, methods of homogenization theory, including those of high contrast homogenization, have proven to be particularly fruitful for problems from the above general class, see e.g. [39, 4, 12, 6, 18].

Hempel and Lienau [20] studied the spectral problem for a matrix-inclusion high contrast periodic medium and established asymptotically explicit band gaps using min-max variational methods. Zhikov [39, 40] has independently used for this case techniques of high-contrast homogenization of double-porosity type. Related periodic medium has periodicity cell size ε and the contrast between the "inclusion" and the "matrix" of order ε^2 (ε is small), which is equivalent to the scaling of [20]. As a result the spectrum converges in the sense of Hausdorff to an explicitly described "limiting" spectrum which contains gaps. Zhikov, using the techniques of two-scale convergence [23, 3], has made significant further advances having additionally described an associated ("two-scale") limiting operator and clarified further the convergence of the spectra in terms of the strong two-scale resolvent convergence, the associated convergence of the spectral projectors and certain additional compactness properties. Zhikov has also shown that at the "macroscale" the spectral problem displays an emergent explicit nonlinear dependence on the spectral parameter, see (3.17) below.

In this paper, using the (high contrast) homogenization theory methods, we show that if such a rapidly oscillating high-contrast medium is perturbed by a compact "defect" of size of order one, asymptotically explicit eigenvalues and eigenfunctions can emerge in the gap when $\varepsilon \to 0$. The essential spectrum is known to remain unchanged under such a perturbation [10, 2, 14], and the existence of the point spectrum in the gaps of the unperturbed operator has also been established on some accounts together with some estimates on the number of eigenvalues in the gap, see [2, 14, 15] and further references in review [22]. We argue that the homogenization techniques allow to substantially refine this information, providing an explicit asymptotic description and tight bounds on the (convergent) eigenvalues and eigenfunctions, and ultimately on their number via some kind of "asymptotic completeness" of the spectrum in the gap as described by an "explicit" limit operator.

We employ in this work the method of asymptotic expansions supplemented by its rigorous justification, in the case of "regular" boundaries for both the periodic inclusions and the defect. This allows to obtain not only an explicit description of the "limit" equations and the "convergence" results, but also to establish the rate of convergence: the main technical result is the error estimate (4.1), with the "ε-square-root" bound being typical for classical homogenization with boundary-layer effects, see e.g. [21] and further references therein. The method of asymptotic expansions in the "moderate contrast" classical homogenization as well as its rigorous justification are well developed, see e.g. [9, 31, 7, 21]. Applications of asymptotic methods for high contrast (double porosity-type) homogenization can be found e.g. in [32]. On the other hand, error estimates in homogenization can be obtained by other methods, including the so called spectral method, e.g. [8, 38, 5, 11, 42], as well as by modifications of the asymptotic expansions method with no assumptions on the regularity of the coefficients, e.g. [41]. One novel technical ingredient in the present work is perhaps the execution of a delicate boundary layer asymptotic analysis in the high contrast case in conjunction with the need of explicitly accounting for up to the second-order corrector in the asymptotic expansion (Appendix B).

The structure of the paper is following. We first formulate the problem (Section 2), then give an explicit description of the "limit problem" in Section 3. Section 4 establishes the convergence and error bounds for the eigenvalues and the eigenfunctions (Theorem 4.1). An explicit example illustrating the existence of the point spectrum for the limit operator for spherical defects in a surrounding periodic medium with isotropic effective properties is given in Section 5. Finally we discuss the improved convergence of the eigenfunctions and their "asymptotic completeness" via the methods of two-scale resolvent convergence and associated compactness properties, Section 6. Appendix A gives formal derivation of the limit problem, as well as of the first and second order correctors required for the rigorous justification. We give full proof of Theorem 4.1, whose most technical part (Theorem 4.2, including a number of accompanying technical lemmas and propositions) is given in the Appendix B.

2 Formulation of the problem

We consider a high contrast two-phase periodic medium with a small periodicity size and with a "finite size" defect filled by a third phase. The geometric configuration is displayed on Figure 1.

The precise mathematical formulation is following. Let $Q := [0,1)^n$ be the reference periodicity cell in $\mathbb{R}^n$, $n \geq 2$, and let $Q_0 \subset\subset Q$ be a connected domain (a "reference inclusion") in Q with infinitely smooth boundary. Denote by Q_1 be the complement of Q_0 in Q, $Q_1 := Q \backslash \overline{Q_0}$ (the overbar denotes the closure of the set). Let $\hat{Q}_0^\varepsilon$ be the corresponding contracted set, i.e. $\hat{Q}_0^\varepsilon := \{x : x/\varepsilon \in Q_0\}$, where $\varepsilon > 0$ is a small positive parameter. We denote by Q_0^ε the ε-periodic cloning of $\hat{Q}_0^\varepsilon$, i.e. $Q_0^\varepsilon := \hat{Q}_0^\varepsilon + \varepsilon\mathbb{Z}^n$. Let the "defect domain" Ω_2 be an ε-independent bounded domain with infinitely smooth boundary. We denote by $\tilde{Q}_0^\varepsilon$ the set of all the inclusions in Q_0^ε which intersect with the boundary $\partial\Omega_2$ of Ω_2, and by $\tilde{\Omega}_0^\varepsilon$ the union of all the parts from $\tilde{Q}_0^\varepsilon$ outside Ω_2, *i.e.* $\tilde{\Omega}_0^\varepsilon := \tilde{Q}_0^\varepsilon \backslash \overline{\Omega_2}$, see Figure 1.

One phase, the "inclusions phase" of the resulting composite medium, denoted Ω_0^ε, is the collection of all the small inclusions lying entirely outside the

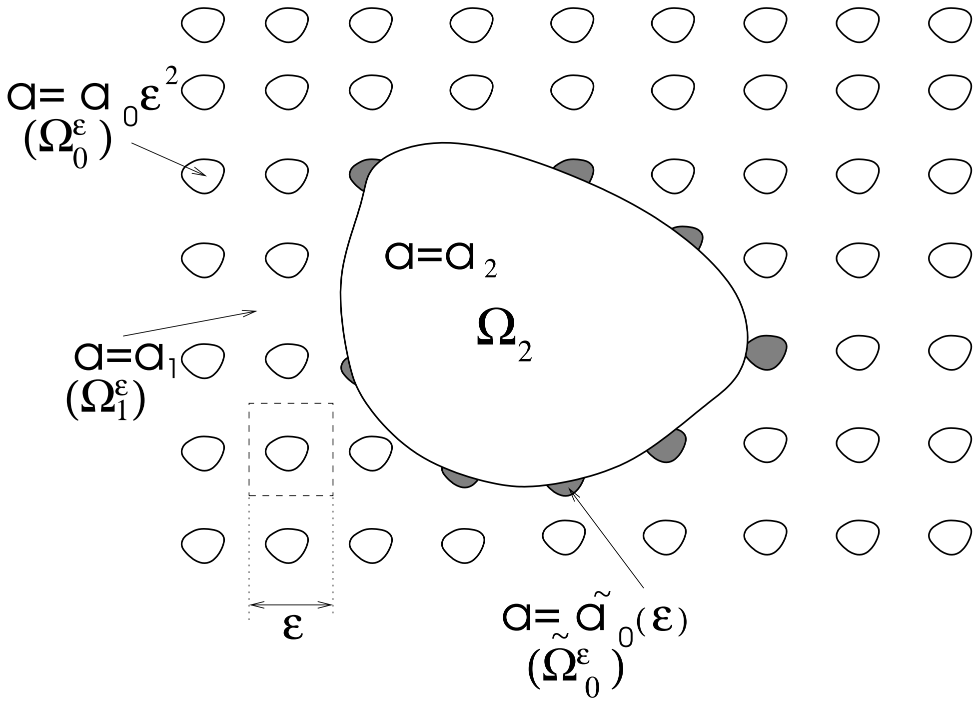

Figure 1: Geometric configuration: a defect in a rapidly oscillating high contrast medium

defect Ω_2, i.e. $\Omega_0^\varepsilon := Q_0^\varepsilon \setminus \left(\Omega_2 \cup \tilde{Q}_0^\varepsilon\right)$. The "matrix phase", denoted Ω_1^ε, is the supplement to the inclusions outside the defect, i.e. $\Omega_1^\varepsilon := \mathbb{R}^n \setminus \overline{\left(Q_0^\varepsilon \cup \Omega_2\right)}$.

We assume that the matrix and the defect are "filled" with materials with ε-independent uniform properties a_1 and a_2 respectively, and the inclusions are filled with materials with ε-dependent (uniform) properties: $a_0(\varepsilon)$ and $\tilde{a}_0(\varepsilon)$ for the "full" (domain Ω_0^ε) and the "cut" (domain $\tilde{\Omega}_0^\varepsilon$)) inclusions respectively. Mathematically, for every positive (small enough) ε we consider the spectral problem

$$A_\varepsilon u^\varepsilon \;=\; \lambda(\varepsilon) u^\varepsilon \tag{2.1}$$

for operator A_ε, self-adjoint in $L^2(\mathbb{R}^n)$,

$$A_\varepsilon u^\varepsilon \;:=\; -\nabla \cdot \Big(a(x,\varepsilon) \nabla u^\varepsilon(x) \Big), \;\; x \in \mathbb{R}^n, \tag{2.2}$$

where

$$a(x,\varepsilon) = \begin{cases} a_0(\varepsilon), & x \in \Omega_0^\varepsilon, \\ \tilde{a}_0(\varepsilon), & x \in \tilde{\Omega}_0^\varepsilon, \\ a_1, & x \in \Omega_1^\varepsilon, \\ a_2, & x \in \Omega_2. \end{cases} \tag{2.3}$$

We assume that

$$a_0(\varepsilon) = a_0 \varepsilon^2 \tag{2.4}$$

(which is the so-called double porosity scaling), with a_0, as well as a_1 and a_2 being arbitrary positive constants. There is a degree of freedom in this work for selecting the scaling for $\tilde{a}_0(\varepsilon)$ in the "boundary layer", so we only require that

$$0 < \tilde{a}_0(\varepsilon) \le \tilde{A}_0 \qquad (2.5)$$

with some ε-independent positive $\tilde{A}_0$. Hence the results of the present paper remain valid, for example, both for the double porosity scaling where $\tilde{a}_0(\varepsilon) = \tilde{a}_0\varepsilon^2$ with some ε-independent positive $\tilde{a}_0$, in particular for $\tilde{a}_0 = a_0$ (physically, the "cut" inclusions outside the defect being kept), as well for $\tilde{a}_0(\varepsilon)$ being "of order one", e.g. $\tilde{a}_0(\varepsilon) = A_0$, in particular for $A_0 = a_1$ or $A_0 = a_2$ (the cut inclusions being replaced by the matrix or defect material). We also remark that the results presented in this paper will remain equally valid for the "boundary" inclusions $\tilde{Q}_0^\varepsilon$ also cutting out of the "defect" part Ω_2 (the maximal generality has not been pursued to avoid unnecessary further technical complications). Notice however that for our subsequent stronger results on the Hausdorff convergence of the spectra and the convergence of the eigenfunctions (see the Discussion section, Section 6, below) an "order one" choice for $\tilde{a}_0(\varepsilon)$ becomes necessary.

The equation (2.1) is understood in the usual weak sense, implying the continuity of u^ε and of the conormal derivatives at the boundaries of $\Omega_0^\varepsilon, \Omega_1^\varepsilon, \Omega_2$ and $\tilde{\Omega}_0^\varepsilon$.

Hence for every fixed positive ε (2.1)–(2.2) represents a spectral problem for an operator with periodic coefficients "perturbed" by a localised defect. We are mostly interested in the existence and asymptotics for the eigenvalues $\lambda(\varepsilon)$ and the associated localised solutions (eigenfunctions) $u^\varepsilon(x)$ of (2.1) when $\varepsilon \to 0$.

3 Homogenization and the limit problem

We describe in this section the formal asymptotic procedure for solving (2.1)–(2.2) when $\varepsilon \to 0$. It is rigorously justified in the subsequent sections.

One can seek a formal solution to the spectral problem (2.1)–(2.2) in the form of a standard two-scale ansatz:

$$u^\varepsilon(x) = u^{(0)}(x, x/\varepsilon) + \varepsilon u^{(1)}(x, x/\varepsilon) + \varepsilon^2 u^{(2)}(x, x/\varepsilon) + r^\varepsilon(x), \qquad (3.1)$$

$$\lambda(\varepsilon) = \lambda_0 + o(1), \qquad (3.2)$$

where $u^{(0)}(x,y)$, $u^{(1)}(x,y)$ and $u^{(2)}(x,y)$ are functions to be determined which are Q-periodic in y, the remainders $r^\varepsilon(x)$ and $o(1)$ are expected to be small when $\varepsilon \to 0$, with λ_0 and $u^{(0)}(x,y)$ subsequently having the meaning of the eigenvalues and eigenfunctions of a "limit problem". [We subsequently show that the remainder $o(1)$ in (3.2) is in fact "of order $\varepsilon^{1/2}$", i.e. $O(\varepsilon^{1/2})$, see (4.1).]

A formal substitution of (3.1)-(3.2) into (2.1) results upon straightforward calculation in the following structure of the main-order term $u^{(0)}(x,y)$, see Appendix A:

$$u^{(0)}(x,y) = \begin{cases} u_0(x), & x \in \Omega_2 \text{ or } x \in \mathbb{R}^n \backslash \Omega_2,\, y \notin Q_0, \\ u_0(x) + v(x,y), & x \in \mathbb{R}^n \backslash \Omega_2,\, y \in Q_0, \end{cases} \tag{3.3}$$

which highlights the fact that $u^{(0)}$ varies only at the "slow" scale x (as is the case in the "classical", i.e. not high-contrast, homogenization) everywhere outside the domain of "soft" inclusions Ω_0^ε, however may depend on the fast variable $y = x/\varepsilon$ in Ω_0^ε. Further, the pair of functions $(u_0(x), v(x,y))$ must solve the following "limit" coupled spectral problem:

$$-\nabla \cdot a_2 \nabla u_0(x) = \lambda_0 u_0(x), \;\; x \in \Omega_2, \tag{3.4}$$

$$-\nabla \cdot A^{\text{hom}} \nabla u_0(x) = \lambda_0 \left(u_0 + \langle v \rangle_y \right), \;\; x \in \mathbb{R}^n \backslash \Omega_2, \tag{3.5}$$

$$-a_0 \Delta_y v = \lambda_0 \left(u_0 + v \right), y \in Q_0; \; v = 0,\, y \in \partial Q_0 \;\; (x \in \mathbb{R}^n \backslash \Omega_2), \tag{3.6}$$

$$(u_0)_- = (u_0)_+, \; a_2 \left(\frac{\partial u_0}{\partial n} \right)_- = \left(A_{ij}^{\text{hom}} \frac{\partial u_0}{\partial x_j} n_i \right)_+, \; x \in \partial\Omega_2. \tag{3.7}$$

Here

$$\langle v \rangle_y(x) := |Q|^{-1} \int_Q v(x,y)\, dy \tag{3.8}$$

denotes the averaging with respect to y over the periodicity cell Q (extending v by zero outside Q_0); $(\cdot)_-$ and $(\cdot)_+$ denote respectively the interior and exterior limit values of the appropriate entities at the boundary $\partial\Omega_2$ of Ω_2, n is the interior unit normal to $\partial\Omega_2$, summation is henceforth implied with respect to repeated indices. In (3.5) $A^{\text{hom}} = \left(A_{ij}^{\text{hom}} \right)$ is the standard "porous" homogenized matrix for the above described periodic medium with $a_0 = 0$, see e.g. [21] §3.1:

$$A_{ij}^{\text{hom}} \xi_i \xi_j = \inf_{w \in C_{\text{per}}^\infty(Q)} \int_{Q \backslash Q_0} a_1 |\xi + \nabla w|^2\, dy \quad (\xi \in \mathbb{R}^n). \tag{3.9}$$

Notice in passing that, following the pattern of Zhikov [40], the above "limit problem" (3.4)–(3.7) can be interpreted[1] as a spectral problem for a "limit" operator (which we will denote A_0) acting in the following Hilbert space $\mathcal{H}_0$:

$$\mathcal{H}_0 = \Big\{ u(x,y) \in L^2(\mathbb{R}^n \times Q) \;\Big|\; u(x,y) = u_0(x) + v(x,y),\, u_0 \in L^2(\mathbb{R}^n), \; v \in L^2\Big(\mathbb{R}^n \backslash \Omega_2,\, L^2(Q_0)\Big) \Big\}. \tag{3.10}$$

[1]See discussion in Section 5 on rigorous definition of the limit operator.

Operator A_0 is generated by the (closed) symmetric and bounded from below quadratic form B_0 on dense domain

$$\mathcal{V} \;=\; H^1\left(\mathbb{R}^n\right) \;+\; L^2\left(\mathbb{R}^n\backslash\Omega_2, H^1_0(Q_0)\right) \tag{3.11}$$

of $\mathcal{H}_0$ defined as follows: for $u = u_0 + v \in \mathcal{V}$,

$$B_0(u,u) \;=\; a_2\int_{\Omega_2}|\nabla u_0|^2dx + \int_{\mathbb{R}^n\backslash\Omega_2}A^{\text{hom}}\nabla u_0\cdot\nabla u_0\,dx \;+\; \int_{\mathbb{R}^n\backslash\Omega_2}\int_{Q_0}|\nabla_y v|^2dydx. \tag{3.12}$$

The resulting (self-adjoint) operator A_0 is defined in standard way in a dense domain $D(A_0)\subset\mathcal{V}$. (The latter fact is not of a direct use in the present paper and hence is not elaborated upon here.)

On the other hand, the limit problem (3.4)–(3.7) can be simplified as follows, cf. [39, 40]. Assume $\lambda_0\neq\lambda_j$ for all $j\geq 1$, and let λ_j and $\varphi_j(y)$ be the eigenvalues and the (normalized) eigenfunctions respectively of $-a_0\Delta_y$ in Q_0 with Dirichlet boundary conditions on ∂Q_0 (Δ_y is the Laplace operator). Then, applying the spectral decomposition to (3.6), one can eliminate $v(x,y)$, as a result:

$$v(x,y) \;=\; \lambda_0 u_0(x)\sum_{j=1}^{\infty}\frac{\langle\varphi_j\rangle_y}{\lambda_j-\lambda_0}\varphi_j(y). \tag{3.13}$$

Substituting (3.13) back into (3.5) we arrive at the following spectral problem for u_0 with a nonlinear dependence on the spectral parameter λ:

$$-\nabla\cdot a_2\nabla u_0(x) = \lambda_0 u_0(x),\;\; x\in\Omega_2, \tag{3.14}$$

$$-\nabla\cdot A^{\text{hom}}\nabla u_0(x) = \beta(\lambda_0)u_0(x),\;\; x\in\mathbb{R}^n\backslash\Omega_2, \tag{3.15}$$

$$\left(u_0\right)_- \;=\; \left(u_0\right)_+,\; a_2\left(\frac{\partial u_0}{\partial n}\right)_- \;=\; \left(A^{\text{hom}}_{ij}\frac{\partial u_0}{\partial x_j}n_i\right)_+,\; x\in\partial\Omega_2. \tag{3.16}$$

Here

$$\beta(\lambda) := \lambda \;+\; \lambda^2\sum_{j:\,\langle\varphi_j\rangle_y\neq 0}\frac{\langle\varphi_j\rangle_y^2}{\lambda_j-\lambda} \tag{3.17}$$

is the function introduced by Zhikov [39, 40], see Figure 2. (Strictly speaking, λ_1, λ_2, λ_3, ... on Fig. 2 denote only the eigenvalues corresponding to φ_j with non-zero mean, ordered by increasing values.)

The spectrum of the "unperturbed" limit operator (with no defects) consists of the union of λ where $\beta(\lambda)\geq 0$ and of λ_j, $j=1,2,...$, [40]. If $\langle\varphi_j\rangle_y = 0$ the spectrum of the limit operator also contains (infinite multiplicity) point spectrum at $\lambda = \lambda_j$. Hence the limit operator has gaps when $\beta(\lambda) < 0$, $\lambda\neq\lambda_j, j = 1,2,....$

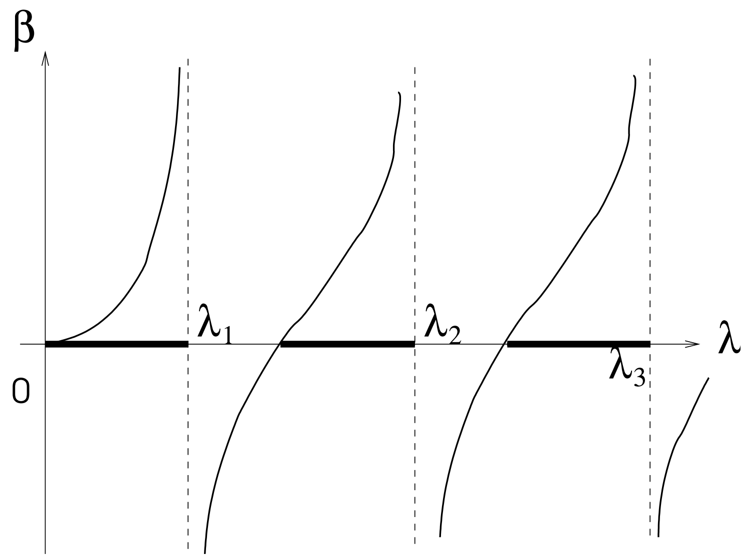

Figure 2: Function $\beta(\lambda)$

In this work, for the "perturbed" linear operator A_0, restricting ourselves only to values of λ in the gaps of the unperturbed limit operator, we define λ_0 to be an eigenvalue of A_0 if $\beta(\lambda_0) < 0$, $\lambda_0 \neq \lambda_j$, $j \geq 1$, and the system (3.14)–(3.17) admits a solution $u_0(x)$ decaying at infinity (hence, decaying exponentially since $\beta(\lambda_0) < 0$). We define as an eigenfunction (or an eigenvector) of A_0 the pair of functions (u_0, v), where $u_0(x)$ is the above solution and $v(x, y)$ is related to $u_0(x)$ via (3.6) (equivalently (3.13)) for $y \in Q_0$ and $x \in \mathbb{R}^n \backslash \Omega_2$ and extended by zero for the remaining values of x and $y \in Q$.

One can show from (3.14)–(3.16) that the "perturbed" limit operator A_0 inside the gaps of the unperturbed operator can only develop isolated eigenvalues of finite multiplicity. As we show in Section 5, the problem (3.14)–(3.16) admits in some cases an explicit calculation of its eigenvalues and eigenfunctions.

4 Convergence and error bounds for eigenvalues and eigenmodes

The main result of the present work is the following theorem establishing the closeness of the spectrum of the original operator A_ε to the spectrum of the above described limit operator A_0 as $\varepsilon \to 0$:

Theorem 4.1. *Let λ_0 be an eigenvalue of limit operator A_0, $\beta(\lambda_0) < 0$, $\lambda_0 \neq \lambda_j, j \geq 1$. Then there exists $\varepsilon_0 > 0$ and a constant $C_1 > 0$ independent of ε such that for any $0 < \varepsilon \leq \varepsilon_0$ there exists an isolated eigenvalue $\lambda(\varepsilon)$ of operator A_ε of finite* *multiplicity, such that*

$$|\lambda(\varepsilon) - \lambda_0| < C_1\varepsilon^{1/2}. \tag{4.1}$$

Moreover if (u_0, v) *is an eigenfunction of* A_0 *which corresponds to* λ_0 *then the function*

$$u^{\text{appr}}(x,\varepsilon) := \begin{cases} u_0(x) + v(x, x/\varepsilon), & x \in \Omega_0^\varepsilon, \\ u_0(x), & x \in \Omega_1^\varepsilon \cup \Omega_2 \cup \tilde{\Omega}_0^\varepsilon, \end{cases} \tag{4.2}$$

is an approximate eigenfunction for A_ε *at least in the following sense*[2]*: there exist constants* $c_j(\varepsilon)$ *such that*

$$\|u^{\text{appr}} - \sum_{j \in J_\varepsilon} c_j(\varepsilon) u_j^\varepsilon\|_{L_2(\mathbb{R}^n)} < C_2\varepsilon^{1/2}, \tag{4.3}$$

where $J_\varepsilon = \{j : |\lambda^{(j)}(\varepsilon) - \lambda_0| < C\varepsilon^{1/2}\}$ *is a finite set of indices (for each* ε*), and* $\lambda^{(j)}(\varepsilon)$, $u_j^\varepsilon(x)$ *are eigenvalues and* L_2*-normalized eigenfunctions of* A_ε*, and the constants* C_2 *and* C *are independent of* ε*.*

Proof: We first establish estimates somewhat related to the "strong resolvent" convergence of operators A_ε when $\varepsilon \to 0$. Strictly speaking, those are related to some generalization of the above: the usual resolvent convergence is not suitable for our purposes because the space where the limiting operator A_0 acts differs from the space natural for operators A_ε. One therefore has to refer to the so-called *two-scale* strong resolvent convergence, see Zhikov [39, 40].

Let $u^0(x,y) = (u_0(x), v(x,y)) \in D(A_0)$ be an eigenfunction of the operator A_0 corresponding to an eigenvalue λ_0, $\beta(\lambda_0) < 0$, $\lambda_0 \neq \lambda_j$, $j \geq 1$. Denote by U_ε the "transfer" operator, constructing from u^0 the approximate eigenfunction u^{appr} via (4.2), i.e.:

$$(U_\varepsilon u^0)(x) := \begin{cases} u_0(x) + v(x, x/\varepsilon), & x \in \Omega_0^\varepsilon, \\ u_0(x), & x \in \Omega_1^\varepsilon \bigcup \Omega_2 \cup \tilde{\Omega}_0^\varepsilon. \end{cases} \tag{4.4}$$

Notice that due to the regularity of u_0 and v (which solve (3.14)–(3.16)), decomposition (3.13) and the exponential decay of u_0 at infinity when $\beta(\lambda_0) < 0$, $U_\varepsilon u^0 \in L_2(\mathbb{R}^n)$.

We formulate next the main technical statement of this work, close to that of the two-scale resolvent convergence, cf. [39, 40]:

Theorem 4.2. *Let* $u^0(x,y) = (u_0(x), v(x,y)) \in D(A_0)$ *be an eigenfunction of the operator* A_0 *:*

$$A_0 u^0 = \lambda_0 u^0, \quad \lambda_0 \neq \lambda_j, \ j \geq 1; \ \beta(\lambda_0) < 0. \tag{4.5}$$

[2]See the discussion in Section 6 on strengthening of the result on convergence of the eigenfunctions.

Consider the following problem for the original operator A_ε:

$$(A_\varepsilon + 1)\tilde{u}^\varepsilon = (\lambda_0 + 1)U_\varepsilon u^0. \tag{4.6}$$

Then

$$\|U_\varepsilon u^0 - \tilde{u}^\varepsilon\|_{L_2(\mathbb{R}^n)} \leq C\varepsilon^{1/2}, \tag{4.7}$$

with a constant C independent of ε.

Remark 4.1. The above estimate can be rewritten in a form which somewhat clarifies the role of operator U_ε:

$$\|U_\varepsilon(A_0 + 1)^{-1}u^0 - (A_\varepsilon + 1)^{-1}U_\varepsilon u^0\| \leq C\varepsilon^{1/2}, \tag{4.8}$$

where u^0 is an eigenfunction of operator A_0, cf. [41].

The proof of Theorem 4.2 constitutes the main technical component for establishing the central result (4.1) of the present work. The proof of the main error bound (4.7) requires, among other ingredients, the development of a "high contrast" ("double porosity") version of the asymptotic analysis of the boundary layer near the boundary of the defect Ω_2, conceptually somewhat similar to e.g. [21] §1.4. The complete proof of Theorem 4.2 is given in Appendix B. We remark here that the need of executing a series of technical error estimates in Appendix B is caused by the fact that "globally" we can explicitly construct only the main order term in the asymptotic expansion, with the major obstacle being the need to control the effect of the boundary layer near the defect's border $\partial\Omega_2$. Constructing or analyzing the boundary layer in homogenization is still an open problem in general, even in the classical ("moderate contrast") case, see [27, 28] for some latest developments, although its effect can be instead somewhat "controlled" via an error bound of order $\varepsilon^{1/2}$ in the H^1 norm, see e.g. [21] §1.4 for the case of boundaries with Dirichlet or Neumann conditions. In the present work we face however the *high contrast* version of the boundary layer problem and in effect show that even in this case (for the interface conditions) the boundary layer accounts for the "ε-square root" error, but in the L_2 norm, see (4.7). The series of technical lemmas and propositions in Appendix B ensure that the approximation $U_\varepsilon u^0$ is "nearly" the exact solution $\tilde{u}^\varepsilon$ in the sense of appropriate quadratic forms, see (B.3). Namely, the "main-order" contributions of the quadratic forms correspoding to $U_\varepsilon u^0$ and $\tilde{u}^\varepsilon$ coincide and the errors are controllably small, including those due to the boundary layer.

Denote by $\sigma_{\text{ess}}(A_\varepsilon)$ the essential spectrum of operator A_ε. The next step is, partly, a specialization of a more general methodology, see e.g. [21] §11.1, to the present context:

Lemma 4.3. *Let λ_0 be an eigenvalue of operator A_0, $\beta(\lambda_0) < 0$, $\lambda_0 \neq \lambda_j$, $j \geq 1$, and let $u^0 = (u_0, v)$ be associated eigenfunction. Then:*

(i) For sufficiently small ε there exists $c > 0$ independent of ε,

$$(\lambda_0 - c, \lambda_0 + c) \bigcap \sigma_{\mathrm{ess}}(A_\varepsilon) = \emptyset. \tag{4.9}$$

(ii) There exists, for sufficiently small ε, an isolated eigenvalue (hence of finite multiplicity) $\lambda(\varepsilon)$ of operator A_ε, such that

$$|\lambda(\varepsilon) - \lambda_0| < C_1 \varepsilon^{1/2}, \tag{4.10}$$

with constant C_1 independent of ε.

(iii) There exist constants $c_j(\varepsilon)$ such that

$$\|U_\varepsilon u^0 - \sum_{j \in J_\varepsilon} c_j(\varepsilon) u_j^\varepsilon \|_{L_2} < C_2 \varepsilon^{1/2}, \tag{4.11}$$

where $J_\varepsilon = \{j : |\lambda^{(j)}(\varepsilon) - \lambda_0| < C\varepsilon^{1/2}\}$, and $\lambda^{(j)}(\varepsilon), u_j^\varepsilon(x)$ are eigenvalues and (L_2-normalized) eigenfunctions of A_ε, and the constants C_1 and C_2 are independent of ε.

Proof. (i) The assertion (4.9) follows from the Hausdorff convergence of the spectra of the "unperturbed" (i.e. with no defects) operators with periodic coefficients [20, 40], and the stability of the essential spectrum due to localized defects, e.g. [14].

(ii) To prove (4.10) recall that the distance from a point μ to the spectrum of a linear self-adjoint operator B in $L_2(\mathbb{R}^n)$ can be bounded from above as

$$\mathrm{dist}(\mu, \sigma(B)) \leq \frac{\|Bu - \mu u\|_{L_2(\mathbb{R}^n)}}{\|u\|_{L_2(\mathbb{R}^n)}}, \tag{4.12}$$

with any $u \in D(B)$, $u \neq 0$. Let us take as B and μ, respectively, $B = B_\varepsilon := (A_\varepsilon + 1)^{-1}$ and $\mu = (\lambda_0 + 1)^{-1}$. Then obviously $\lambda \in \sigma(A_\varepsilon)$ if and only if $(\lambda + 1)^{-1} \in \sigma(B_\varepsilon)$. Now select $u = U_\varepsilon u^0$. Then according to Theorem 4.2, see (4.8), the numerator in (4.12) can be estimated as follows

$$\|(A_\varepsilon + 1)^{-1} U_\varepsilon u^0 - (1 + \lambda_0)^{-1} U_\varepsilon u^0 \|_{L_2(\mathbb{R}^n)} =$$

$$= \|(A_\varepsilon + 1)^{-1} U_\varepsilon u^0 - U_\varepsilon (A_0 + 1)^{-1} u^0 \|_{L_2(\mathbb{R}^n)} \leq C\varepsilon^{1/2}, \tag{4.13}$$

(where we have also used that $(1 + \lambda_0)^{-1} u^0 = (A_0 + 1)^{-1} u^0$). Obviously, the denominator in (4.12) is bounded from below (e.g. $\|U_\varepsilon u^0\|_{L_2(\mathbb{R}^n)} \geq \|u_0\|_{L_2(\Omega_2)} > 0$). As a result, $\mathrm{dist}(\mu, \sigma(B_\varepsilon)) \leq c_1 \varepsilon^{1/2}$, with some ε-independent c_1. Using then for small enough ε for example obvious inequality $\mathrm{dist}(\lambda_0, \sigma(A_\varepsilon)) \leq L(\varepsilon) \mathrm{dist}(\mu, \sigma(B))$ with $L(\varepsilon) = \left(\mu - c_1 \varepsilon^{1/2}\right)^{-2}$ we arrive at $\mathrm{dist}(\lambda_0, \sigma(A_\varepsilon)) \leq$

$c\varepsilon^{1/2}$, with some ε-independent positive constant c. Finally notice that, for sufficiently small ε, the interval $(\lambda_0 - c\varepsilon^{1/2}, \lambda_0 + c\varepsilon^{1/2})$ may contain only isolated eigenvalues of A_ε by (4.9). This proves the existence of eigenvalues $\lambda(\varepsilon)$ satisfying (4.10).

(iii) The assertion (4.11) is a consequence of (4.10) and general results, see e.g. [36] or [21] §11.1, and follows by applying spectral decomposition of $U_\varepsilon u^0$ with respect to A_ε, and using (i) and (ii). □

Now the Theorem 4.1 directly follows from Theorem 4.2 and Lemma 4.3. □

Remark 4.2. We expect it would be relatively straightforward to slightly modify the statement of Theorem 4.1 for limit eigenvalues λ_0 with "multiplicities". Namely, let there exist m, $2 \leq m < \infty$, linearly independent eigenfunctions $u_j^{(0)}(x,y)$, $j = 1, ..., m$ of the two-scale limit operator A_0. Then we claim that, for sufficiently small ε, there exist not less than m eigenvalues of A_ε (counted with their own multiplicities) such that the error bound (4.7) is valid for each of them. The above proof could be modified for this case by for example first "orthogonalizing" $u_j^{(0)}(x,y)$, $j = 1, ..., m$ (with respect to the inner product induced by the quadratic form B_0, see (3.12)), and then showing that the associated approximations $U_\varepsilon u_j^0$, $j = 1, ..., m$ are "approximately" mutually orthogonal for small ε in the "original" space $L_2(\mathbb{R}^n)$. The latter would then allow to modify the above existence and error bounds argument for the case of multiplicities. We do not elaborate on this in detail to avoid further technical complications, but also since we believe that a further strengthening of this result is possible (via a further advance of the theory), to the effect that there are not only "at least" but also "at most" as many eigenvalues: see the discussion in Section 6.

5 An example

Straightforward analysis of the limit problem (3.14)–(3.16) shows that there are explicitly calculated isolated eigenvalues of operator A_0, at least in the case if $\Omega_2 = \{x : |x| < R\}$ (i.e. the defect is a ball of some radius $R > 0$) and $A^{\text{hom}} = a^{\text{hom}} I$, i.e. the porous homogenized matrix is isotropic as is the case e.g. when Q_0 has appropriate symmetries (in particular being itself a "small" ball). We briefly sketch the details below.

Under the above assumptions a solution to the spectral problem (3.14)–(3.16) is sought by separation of variables in the spherical coordinates $x = (r, \omega)$, $r := |x|$, $\omega := x/|x| \in S^{n-1}$, $(n \geq 2)$:

$$u_0(x) = \begin{cases} r^{-(n-2)/2} J_m\left((\lambda/a_2)^{1/2} r\right) P_m(\omega), & |x| \leq R, \\ \alpha\, r^{-(n-2)/2} I_m\left(|\beta(\lambda)/a^{\text{hom}}|^{1/2} r\right) P_m(\omega), & |x| \geq R, \end{cases} \tag{5.1}$$

assuming $\lambda > 0$, $\beta(\lambda) < 0$. Parameters m, λ and α are to be found. Here $J_m(z)$ and $I_m(z)$ are the Bessel and the modified Bessel functions respectively, see e.g. [1], $P_m(\omega)$ are the spherical functions which are the eigenfunctions of the Laplace-Beltrami operator Δ_ω on S^{n-1}:

$$\left(\Delta_\omega + m^2 - (n-2)^2/4\right) P_m(\omega) = 0. \tag{5.2}$$

The spherical spectral problem (5.2) is a classical one and defines explicit eigenvalues m and eigenfunctions P_m (e.g. for $n = 3$ the spectral parameter m is half-integer and P_m, in spherical coordinates (θ, φ), are products of the Legendre polynomials of $\cos\theta$ and trigonometric functions of φ; for $n = 2$ those are trigonometric functions and m is integer). Having selected m and associated $P_m(\omega)$ the function $u_0(x)$ determined by (5.1) automatically satisfies the equations (3.14) and (3.15), and exponentially decays at infinity ($r \to \infty$), with the rate $\exp\left(-|\beta(\lambda)/a^{\text{hom}}|^{1/2} r\right)$. The remaining parameters α and λ (the eigenvalue) are determined from (3.16) at $r = R$, which specializes to:

$$J_m\left((\lambda/a_2)^{1/2} R\right) = \alpha\, I_m\left(|\beta(\lambda)/a^{\text{hom}}|^{1/2} R\right) \tag{5.3}$$

$$(\lambda a_2)^{1/2} J_m'\left((\lambda/a_2)^{1/2} R\right) - a_2 \frac{n-2}{2R} J_m\left((\lambda/a_2)^{1/2} R\right) = \tag{5.4}$$

$$= \alpha \left|\beta(\lambda) a^{\text{hom}}\right|^{1/2} I_m'\left(|\beta(\lambda)/a^{\text{hom}}|^{1/2} R\right) - \alpha a^{\text{hom}} \frac{n-2}{2R} I_m\left(|\beta(\lambda)/a^{\text{hom}}|^{1/2} R\right),$$

where J_m' and I_m' denote derivatives of the relevant functions.

All λ with $\beta(\lambda) < 0$ for which there exists a solution to (5.2), (5.3)–(5.4) describe the point spectrum of the operator A_0 in the "gaps". One can see that it is generally non-empty. Say, for $n = 3$ and $m = 1/2$, $P_{1/2}(\omega) \equiv 1$ is an eigenfunction of (5.2) (which determines the spherically symmetric solutions of (5.1)), and $J_{1/2}$ and $I_{1/2}$ are represented by explicit trigonometric and exponential functions respectively. This allows replacing the "radial" parts in the right hand sides of (5.1) by $r^{-1}\sin((\lambda/a_2)^{1/2} r)$ and $r^{-1}\exp(-|\beta(\lambda)/a^{\text{hom}}|^{1/2} r)$, transforming (5.3)–(5.4) into:

$$\sin\left((\lambda/a_2)^{1/2} R\right) = \alpha \exp\left(-|\beta(\lambda)/a^{\text{hom}}|^{1/2} R\right) \tag{5.5}$$

$$(\lambda a_2)^{1/2} \cos\left((\lambda/a_2)^{1/2} R\right) - \frac{a_2}{R} \sin\left((\lambda/a_2)^{1/2} R\right) =$$

$$-\alpha\, a^{\text{hom}} \left[|\beta(\lambda)/a^{\text{hom}}|^{1/2} + \frac{1}{R}\right] \exp\left(-|\beta(\lambda)/a^{\text{hom}}|^{1/2} R\right). \tag{5.6}$$

The condition of solvability of (5.5)–(5.6) obviously reads:

$$\text{cotan}\left((\lambda/a_2)^{1/2} R\right) + \frac{a^{\text{hom}} - a_2}{(\lambda a_2)^{1/2} R} = -\left(\frac{a^{\text{hom}}|\beta(\lambda)|}{\lambda a_2}\right)^{1/2}, \;\; \beta(\lambda) < 0. \tag{5.7}$$

Noticing that the left hand side of (5.7) is a function of $\lambda^{1/2}R$ one can easily see that by varying $R > 0$ one can obtain infinitely many solutions of (5.7) at any λ in any gap ($\beta(\lambda) < 0$) of the unperturbed linear operator.

6 Discussion: further refinement of the results.

In this section we describe a strategy for further refinement of the results formulated in the Theorem 4.1 using the techniques of two-scale convergence [23, 3, 39, 40]:

1. One can first establish the strong two-scale resolvent convergence for the original operators A_ε and the limit operator A_0, see (3.10)–(3.12), following the strategy of Zhikov [39, 40].

2. The above strong two-scale resolvent convergence implies the ("two-scale") convergence of the spectral projectors, cf. [39, 40] for the case with no defect.

3. Following further the pattern of Zhikov, one expects establishing the "two-scale" compactness property of the (normalised) eigenfunctions in the "gap": Let $\lambda_\varepsilon \in \sigma_p(A_\varepsilon)$, $\{\lambda_\varepsilon\}$ bounded, be eigenvalues of A_ε with associated normalized eigenfunctions u^ε ($\|u^\varepsilon\|_{L^2(\mathbb{R}^n)} = 1$). Then the set $\{u^\varepsilon\}$ is pre-compact in the sense of two-scale convergence, i.e. there exists a subsequence $\varepsilon_j \to 0$ such that $\lambda_\varepsilon \to \lambda_0$ and u_{ε_j} converges to $u^0(x,y)$ strongly in the sense of two-scale convergence. (The latter implies that λ_0 is in the point spectrum of the limit operator A_0 and u^0 is the associated eigenfunction.)

The above implies the convergence of the spectrum of A_ε to that of A_0 in the sense of Hausdorff, a result establishing a converse part of the Theorem 4.1: if $\lambda_\varepsilon \to \lambda_0$ then $\lambda_0 \in \sigma_p(A_0)$. The above also implies a refined convergence of the eigenfunctions compared to (4.3): for any eigenvalue λ_0 of A_0 and the associated eigenfunction $u^0(x,y)$ there exist sequences of eigenvalues $\lambda(\varepsilon)$ of A_ε with associated eigenfunctions $u^\varepsilon(x)$ such that

$$\|u^\varepsilon(x) - u^0(x, x/\varepsilon)\| \le C\,\varepsilon^{1/2}. \tag{6.1}$$

This all can be interpreted as an asymptotic "one-to-one correspondence" as $\varepsilon \to 0$ between the gap spectrum of A_ε and that ("explicit") for the limit operator A_0. A detailed account will be given elsewhere.

Finally, we expect the conditions on the regularity of the boundaries of domains Q_0 and Ω_2 could also be relaxed, using the technique of "translational averaging", cf. [41, 34], [21] §8.3.

A Formal derivation of the limit problem (3.4)–(3.7).

This appendix formally derives the limit equations (3.4)–(3.7) via two-scale asymptotic expansions in the form (3.1). It also establishes the structure of the "first" and "second" corrector terms in (3.1) as required for subsequent rigorous justification.

The asymptotic solution to (3.4)–(3.7) is sought in the form of two-scale ansatz (3.1)–(3.2), where $u^{(0)}(x,y)$, $u^{(1)}(x,y)$ and $u^{(2)}(x,y)$ are assumed to depend periodically on the "fast" variable y only outside the defect Ω_2. The exact solution $u^\varepsilon(x)$ satisfies the standard continuity conditions at the boundary $\partial\Omega_0^\varepsilon$ of the small inclusions away from the defect:

$$\left(u^\varepsilon(x)\right)_0 \;=\; \left(u^\varepsilon(x)\right)_1, \;\; x \in \partial\Omega_0^\varepsilon, \tag{A.1}$$

and

$$\left(a_0\varepsilon^2\frac{\partial u^\varepsilon}{\partial n}(x)\right)_0 \;=\; \left(a_1\frac{\partial u^\varepsilon}{\partial n}(x)\right)_1, \;\; x \in \partial\Omega_0^\varepsilon, \tag{A.2}$$

where the subscripts "0" and "1" denote the limit values at the boundary evaluated in Ω_0^ε and Ω_1^ε, respectively; n stands for unit normal to $\partial\Omega_0^\varepsilon$ which we select as outward for the matrix phase Ω_1^ε (and hence inward for the inclusions Ω_0^ε). Similar boundary conditions are also satisfied at the boundaries of the defect $\partial\Omega_2$ and of the "boundary layer" inclusions $\tilde\Omega_0^\varepsilon$.

The ansatz (3.1)–(3.2) is first substituted into the equation (2.1)–(2.2) and the "interface conditions" (A.1)-(A.2), away from the defect.

By equating first the terms of order ε^{-2} in (2.1) and of order ε^{-1} in (A.2),

$$-\nabla_y\cdot a_1\nabla_y u^{(0)}(x,y) = 0 \;\; y\in Q_1;\;\; a_1\frac{\partial}{\partial n_y}u^{(0)}(x,y) = 0, \;\; y\in\partial Q_0,$$

(with n_y being the outward unit normal for Q_1).

This is a homogeneous Neumann problem in Q_1 with periodic boundary conditions, whose solution is an arbitrary constant in Q_1 (i.e. independent of y) which implies (3.3). The balance of the terms of order ε^0 in (A.1) implies (cf. (3.6))

$$v(x,y) = 0, \;\; y\in\partial Q_0. \tag{A.3}$$

Equating next the terms of order ε^{-1} in (2.1) and of order ε^0 in (A.2), we arrive at:

$$-\nabla_y\cdot a_1\nabla_y u^{(1)}(x,y) = 0, \;\; y\in Q_1;\;\; a_1\frac{\partial}{\partial n_y}u^{(1)}(x,y) = -a_1\frac{\partial}{\partial n_x}u_0(x), \;\; y\in\partial Q_0,$$

together with the periodicity conditions in y. This is a standard "corrector" problem for "porous" (or "perforated") periodic domains. As a result,

$$u^{(1)}(x,y) \;=\; N_j(y)\frac{\partial u_0(x)}{\partial x_j}, \tag{A.4}$$

where N_j is the solution of the "porous" unit cell problem with periodic boundary conditions (e.g. [21] §3.1):

$$a_1 \triangle N_j = 0, \; y \in Q_1; \quad a_1 \frac{\partial}{\partial n_y} N_j = -a_1 n_j, \; y \in \partial Q_0. \tag{A.5}$$

It is convenient to treat the functions $N_j(y)$ as defined in the whole of the periodicity cell Q: with this aim they are extended into the "inclusion" domain Q_0 by "harmonic continuation": $\triangle N_j = 0$, in Q_0, $N_j \in C(Q)$. The above procedure determines N_j up to a constant which is specified by the condition that the average of N_j is zero: $\langle N_j(y) \rangle_y = 0$.

Equate now the terms of order ε^0 in (2.1). As a result, in Q_0,

$$-a_0 \Delta_y v(x,y) \; = \; \lambda_0 (u_0(x) + v(x,y)), \tag{A.6}$$

which together with (A.3) fully recovers (3.6). In particular, this implies that $v(x,y)$ can be uniquely presented as

$$v(x,y) = u_0(x) V(y), \tag{A.7}$$

where V is a solution of the problem

$$-a_0 \triangle_y V = \lambda_0 V + \lambda_0, \; y \in Q_0; \; V = 0, \; y \in \partial Q_0. \tag{A.8}$$

It is further assumed that V is extended by zero to Q and is then periodically extended on the whole $\mathbb{R}^n$.

In turn, in Q_1, taking into account (3.3) and (A.4):

$$-a_1 \Delta_y u^{(2)}(x,y) \; = \; 2a_1 N_{j,k}(y) u_{0,jk}(x) \; + \; a_1 \Delta_x u_0(x) \; + \; \lambda_0 u_0(x), \; y \in Q_1. \tag{A.9}$$

(Henceforth comma in subscript denotes differentiation with respect to variables with following indices.)

This equation has to be supplemented by boundary conditions which result from equating in (A.2) the terms of order ε^1, which results in:

$$a_1 \frac{\partial}{\partial n_y} u^{(2)} = -a_1 \, N_j \frac{\partial}{\partial n_x} u_{0,j} + a_0 \frac{\partial}{\partial n_y} v. \tag{A.10}$$

Treating (A.9)–(A.10) as boundary value problem for $u^{(2)}$ in y for any fixed x, the Green's formula together with the periodicity boundary conditions in y imply:

$$a_1 \int_{Q_1} \Delta_y u^{(2)} = a_1 \int_{\partial Q_0} \frac{\partial}{\partial n_y} u^{(2)},$$

which yields:

$$\left(-a_1 \, \Delta_x u_0(x) - \lambda_0 u_0(x)\right) |Q_1| - 2a_1 u_{0,jk}(x) \int_{Q_1} N_{j,k}(y) dy =$$

$$\int_{\partial Q_0} \left(-a_1 \, N_j(y) u_{0,jk}(x) n_k(y) + a_0 \frac{\partial}{\partial n_y} v(x,y) \right) dy.$$

Applying the integration by parts to the right hand side surface integrals and using (A.6) we arrive at (3.5) where

$$A_{ij}^{\text{hom}} := \left\langle \mu(y) a_1 \left(\delta_{ij} + N_{j,i}(y) \right) \right\rangle_y, \tag{A.11}$$

and $\mu(y)$ is the characteristic function of Q_1 ($\mu(y) = 1$, $y \in Q_1$; $\mu(y) = 0$, $y \in Q_0$). This is a well-known representation of the entries of the porous homogenized matrix A^{hom}, equivalent to (3.9), see e.g. [21] §3.1. Hence the limit equation (3.5) in recovered.

To complete the formal derivation of the limit problem (3.4)–(3.7) we notice that the natural equation (3.4) is simply "postulated" within the (homogeneous) defect Ω_2. The limit interface conditions (3.7) at the boundary of the defect are also postulated: they have the meaning of the continuity of the fields and the flows "to main order" and the proof of the Theorem 4.2 ensures that those produce a "controllably small" boundary layer, eventually ensuring the main result (4.1) of the paper.

Finally, assuming (u_0, v) solves the limit problem (3.4)–(3.7) implies by the above construction the solvability of the boundary value problem (A.9)-(A.10) for $u^{(2)}$, up to an arbitrary constant. By a straightforward manipulation, splitting the right hand sides of (A.9)–(A.10) and using (A.7), (A.11) and (3.15) we arrive at a representation for $u^{(2)}$ as follows:

$$u^{(2)}(x,y) = M_{jk}(y) u_{0,jk}(x) + W(y) u_0(x) + L_{jk}(y) u_{0,jk}(x), \tag{A.12}$$

where $M_{ji}(y)$, $L_{ji}(y)$ and $W(y)$ are solutions of the following problems:

$$a_1 \triangle M_{jk} = -a_1 N_{j,k}, \text{in } Q_1, \quad a_1 \partial_n M_{jk} = -a_1 n_k N_j, \text{on } \partial Q_0 \tag{A.13}$$

$$a_1 \triangle L_{jk} = -a_1 (N_{j,k} - |Q_1|^{-1} \int_{Q_1} N_{j,k} dy), \text{in } Q_1, \quad a_1 \partial_n L_{jk} = 0, \text{on } \partial Q_0. \tag{A.14}$$

$$a_1 \triangle W = b(\lambda_0), \text{in } Q_1, \quad a_1 \partial_n W = a_0 \partial_n V, \text{on } \partial Q_0. \tag{A.15}$$

Here $b(\lambda_0) = |Q_1|^{-1} \lambda_0 (\langle V \rangle + |Q_0|)$, $\langle V \rangle := \int_{Q_0} V(y) dy$ and usual periodicity conditions are assumed. Notice that the solvability of (A.13), (A.14) obviously holds. Solvability of the (A.15) follows from (A.8). Finally clarify the connection between $b(\lambda)$ and $\beta(\lambda)$ given by (3.17). Using the spectral decomposition, it is easy to see that

$$b(\lambda) = |Q_1|^{-1} \beta(\lambda) - \lambda. \tag{A.16}$$

[We remark that the split in (A.12) of the terms containing $u_{0,jk}$ into two groups is motivated by the need for subsequent rigorous justification: while we do need explicitly accounting for the term containing M_{jk}, the term containing L_{jk} is not required, see Appendix B below.]

B Proof of Theorem 4.2

We give in this appendix a full proof of the main technical Theorem 4.2, whose central error bound (4.7) establishes the closeness of the "exact" solution $\tilde{u}^\varepsilon$ of (4.6) to the "approximate" solution $U_\varepsilon u^0$ constructed via (4.4) in terms of the solution $u^0 := \big(u_0(x), v(x,y)\big)$ of the limit problem (3.4)–(3.7).

The proof of the Theorem 4.2 will be divided into a number of stages. The plan is roughly as follows. The "closeness" of $U_\varepsilon u^0$ and $\tilde{u}$ is established employing the associated quadratic form b_ε, see (B.3) below, where $U_\varepsilon u^0$ is replaced by its modification $U_1^\varepsilon(x)$ incorporating some higher-order terms in the asymptotic expansion (3.1), see (B.1). Namely, we show that it is sufficient for our purposes to establish the closeness in the sense of (B.6), see Lemma B.1. A technical proof of Lemma B.1 itself then follows by first splitting the quadratic form in the left hand side of (B.6) into those corresponding to $U_1^\varepsilon(x)$ and $\tilde{u}$, and then splitting the former one further into a number of "components" (corresponding to the various domains of the integration) examining those separately, see Propositions B.2–B.5. For each "component" the main-order parts are explicitly evaluated and the "errors" are bounded. Eventually everything is assembled together and the main-order terms cancel each other as expected, whereas the errors are shown to be "at worst" of order $\varepsilon^{1/2}$. (The latter $\varepsilon^{1/2}$-errors correspond in a sense to the effect of the boundary-layer near the defect's border, and those of order ε or higher to the truncation of the asymptotic ansatz away from it.) An essential specific technical ingredient used in the course of implementing the above strategy is the employment of the so-called "extension lemma" in Proposition B.4. The extension lemma has been successfully used for homogenization problems in "perforated" domains before, see e.g. [21] and further references therein, and is reviewed by us below too, see (B.30) and preceeding discussion.

To proceed, notice first that the above defined approximation $U_\varepsilon u^0$ lies in an appropriate functional space. Observe to this end that u_0 is infinitely smooth in Ω_2 and $\mathbb{R}^n\backslash\overline{\Omega_2}$ as a solution of elliptic equations with constant coefficients (3.14) and (3.15) respectively. Next u_0 decays exponentially at infinity, as a decaying solution of equation with constant coefficients (3.15) outside Ω_2, since $\beta(\lambda_0) < 0$ and hence the fundamental solution of (3.15) in the whole $\mathbb{R}^n$ is exponentially decaying. Further, since, by (A.7), $v(x, x/\varepsilon) = u_0(x)V(x/\varepsilon)$, where $V(y)$ specified by (A.8) is an H^1 periodic function and its restrictions to Q_0 and Q_1 are infinitely smooth, we conclude that $v(x, x/\varepsilon)$ is an exponentially decaying function belonging to $H^1(\mathbb{R}^n\backslash\Omega_2)$.

We further aim at establishing error bounds in the energy norms, i.e. with the quadratic forms (B.3) associated with the equation (4.6). We slightly alter for this the approximation $U_\varepsilon u^0$ by adding to it some higher-order "correctors" in the asymptotic expansion (3.1), introducing the following corrected approximation:

$$U_1^\varepsilon(x) = \begin{cases} U_\varepsilon u^0(x) \ = u_0(x), & x \in \Omega_2, \\ U_\varepsilon u^0(x) + \varepsilon N_j(x/\varepsilon) u_{0,j}(x), & x \in \Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon, \\ U_\varepsilon u^0(x) + \varepsilon N_j(x/\varepsilon) u_{0,j}(x) + \varepsilon^2 \tilde{u}^{(2)}(x, x/\varepsilon), & x \in \Omega_1^\varepsilon. \end{cases} \quad \text{(B.1)}$$

Here $\varepsilon N_j(x/\varepsilon)u_{0,j}(x)$ is the first order corrector constructed everywhere outside the defect, see (A.4)-(A.5), and $\tilde{u}^{(2)}$ is a part of the second-order corrector $u^{(2)}$constructed in Ω_1^ε, see (A.12), namely

$$\tilde{u}^{(2)}(x,y) = M_{jk}(y) u_{0,jk}(x) + W(y) u_0(x). \quad \text{(B.2)}$$

Consider now for any $\varepsilon > 0$ the quadratic form b_ε generated by operator $A_\varepsilon + 1$:

$$b_\varepsilon(w,u) := \sum_{j=0}^{2} \int_{\Omega_j^\varepsilon} a_j(\varepsilon) \nabla w \cdot \nabla u dx + \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0(\varepsilon) \nabla w \cdot \nabla u dx + \int_{\mathbb{R}^n} w\, u dx, \quad \text{(B.3)}$$

(for brevity of notation, $\Omega_2^\varepsilon := \Omega_2$, $a_j(\varepsilon) := a_j, j = 1,2$). In particular, for the actual solution $\tilde{u}^\varepsilon$ of (4.6),

$$b_\varepsilon(\tilde{u}^\varepsilon, w) = (f^\varepsilon, w)_{\mathbb{R}^n}, \ \forall w \in H^1(\mathbb{R}^n), \quad \text{(B.4)}$$

where

$$(f^\varepsilon, w)_{\mathbb{R}^n} := \int_{\mathbb{R}^n} f^\varepsilon w dx,$$

and

$$f^\varepsilon := (\lambda_0 + 1) U_\varepsilon u^0 = \begin{cases} (\lambda_0 + 1) u_0(x) \Big(1 + V(x/\varepsilon) \Big), & x \in \Omega_0^\varepsilon \\ (\lambda_0 + 1) u_0(x), & x \notin \Omega_0^\varepsilon, \end{cases} \quad \text{(B.5)}$$

via (4.6), (4.4) and (A.7).

The domain of the form (B.5) is $H^1(\mathbb{R}^n)$, however we extend it to all "piecewise H^1" functions w, i.e. such that $w \in H^1(\Omega_j^\varepsilon)$, $j = 1,2,3$, $w \in H^1(\tilde{\Omega}_0^\varepsilon)$, for which $b_\varepsilon(w,w)$, as directly defined by the right hand side of (B.3), is bounded. In particular, U_1^ε is in the "extended" domain.

The proof of Theorem 4.2 will be based on the following technical lemma. We first state the lemma, then prove the theorem assuming it is valid, and then prove the lemma itself (which will fall in turn into several technical stages).

Lemma B.1. *There exists an ε-independent $C > 0$ such that for any sufficiently small $\varepsilon > 0$ and for any $w \in H^1(\mathbb{R}^n)$*

$$\left| b_\varepsilon \left(w, U_1^\varepsilon - \tilde{u}^\varepsilon \right) \right| \le C \varepsilon^{1/2} b_\varepsilon(w,w)^{1/2}. \quad \text{(B.6)}$$

Proof of Theorem 4.2:

Assume Lemma B.1 as valid. Select for any $\varepsilon > 0$ as $w = w^\varepsilon(x) := U_2^\varepsilon(x) - \tilde{u}^\varepsilon(x)$, where U_2^ε is another "corrected" approximation constructed as follows:

$$U_2^\varepsilon(x) = \begin{cases} U_\varepsilon u^0(x) = u_0(x), & x \in \Omega_2, \\ U_\varepsilon u^0(x) + \varepsilon \chi_\varepsilon(x) N_j(x/\varepsilon) u_{0,j}(x), & x \in \Omega_1^\varepsilon \cup \Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon, \end{cases} \tag{B.7}$$

where

$$\chi_\varepsilon(x) := \chi(\mathrm{dist}(x, \partial\Omega_2)\varepsilon^{-1}) \tag{B.8}$$

and $\chi(t)$ is a cut-off function: $\chi \in C^\infty(\mathbb{R})$; $\chi(t) = 0, t < 1/2$ and $\chi(t) = 1, t > 1$. Notice that both U_2^ε and $\tilde{u}^\varepsilon$ are in $H^1(\mathbb{R}^n)$. Hence by Lemma B.1,

$$|b_\varepsilon\,(U_2^\varepsilon - \tilde{u}^\varepsilon, U_1^\varepsilon - \tilde{u}^\varepsilon)| \le C\varepsilon^{1/2} b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_2^\varepsilon - \tilde{u}^\varepsilon)^{1/2}, \tag{B.9}$$

with C denoting henceforth constants C independent of ε whose precise value is insignificant.

On the other hand, by the positivity of the "extended" quadratic form, obviously,

$$b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_2^\varepsilon - \tilde{u}^\varepsilon) \le 2|b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_1^\varepsilon - \tilde{u}^\varepsilon)| + b_\varepsilon(U_1^\varepsilon - U_2^\varepsilon, U_1^\varepsilon - U_2^\varepsilon). \tag{B.10}$$

Notice next that from (B.1) and (B.7)

$$U_1^\varepsilon(x) - U_2^\varepsilon(x) = \begin{cases} 0, & x \in \Omega_2, \\ (1 - \chi_\varepsilon(x))\varepsilon N_j(x/\varepsilon) u_{0,j}(x), & x \in \Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon, \\ (1 - \chi_\varepsilon(x))\varepsilon N_j(x/\varepsilon) u_{0,j}(x) + \varepsilon^2 \tilde{u}^{(2)}(x, x/\varepsilon), & x \in \Omega_1^\varepsilon. \end{cases} \tag{B.11}$$

Then, due to the small size (of order ε near $\partial\Omega_2$) of the support of $1 - \chi(\mathrm{dist}(x, \partial\Omega_2)\varepsilon^{-1})$, and the regularity $N_j(y)$, $u_0(x)$ and $\tilde{u}^{(2)}(x, y)$ (see (B.2), (A.13), (A.15)), we conclude that

$$b_\varepsilon(U_1^\varepsilon - U_2^\varepsilon, U_1^\varepsilon - U_2^\varepsilon) \le C\varepsilon. \tag{B.12}$$

Combining (B.10) with (B.9) and (B.12) obviously implies:

$$b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_2^\varepsilon - \tilde{u}^\varepsilon) \le C\varepsilon + 2C\varepsilon^{1/2} b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_2^\varepsilon - \tilde{u}^\varepsilon)^{1/2} \le$$

$$\frac{1}{2} b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_2^\varepsilon - \tilde{u}^\varepsilon) + (2C^2 + 1)\varepsilon,$$

which implies

$$\|U_2^\varepsilon - \tilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^n)} \le b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_2^\varepsilon - \tilde{u}^\varepsilon) \le C\varepsilon. \tag{B.13}$$

Notice finally that from (B.7), the boundedness of N_j and χ_ε as well as boundedness and exponential decay of $u_{0,j}$ we conclude

$$\|U_\varepsilon u^0 - U_2^\varepsilon\|_{L^2(\mathbb{R}^n)} \le C\varepsilon.$$

This together with (B.13) implies via the triangle inequality that

$$\|U_\varepsilon u^0 - \tilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C\varepsilon^{1/2},$$

with appropriate constant C which establishes (4.7) and hence proves the theorem.□

Proof of Lemma B.1:

First, using (B.4) the entity in the left hand side of (B.1) can be evaluated as follows:

$$b_\varepsilon(w, U_1^\varepsilon - \tilde{u}^\varepsilon) = b_\varepsilon(w, U_1^\varepsilon) - (w, f^\varepsilon)_{\mathbb{R}^n} = I_1(\varepsilon) + I_2(\varepsilon), \tag{B.14}$$

where

$$I_1(\varepsilon) := b_\varepsilon(w, U_1^\varepsilon) = \sum_{j=0}^{2} \int_{\Omega_j^\varepsilon} a_j(\varepsilon)\nabla w \cdot \nabla U_1^\varepsilon dx + \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0(\varepsilon)\nabla w \cdot \nabla U_1^\varepsilon dx + \int_{\mathbb{R}^n} w U_1^\varepsilon dx, \tag{B.15}$$

and, via (B.5),

$$I_2(\varepsilon) := -(w, f^\varepsilon)_{\mathbb{R}^n} = -(\lambda_0 + 1)\left(\int_{\Omega_0^\varepsilon} w u_0(1+V)dx + \int_{\tilde{\Omega}_0^\varepsilon} w u_0 dx + \int_{\Omega_2} w u_0 dx + \int_{\Omega_1^\varepsilon} w u_0 dx\right). \tag{B.16}$$

It is further convenient to break $I_1(\varepsilon)$ into four separate terms for the four integration domains:

$$I_1(\varepsilon) = \tilde{A}_0(\varepsilon) + \sum_{j=0}^{2} A_j(\varepsilon), \tag{B.17}$$

where

$$\tilde{A}_0(\varepsilon) := \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0(\varepsilon)\nabla w \cdot \nabla U_1^\varepsilon dx + \int_{\tilde{\Omega}_0^\varepsilon} w U_1^\varepsilon dx \tag{B.18}$$

and

$$A_0(\varepsilon) := \int_{\Omega_0^\varepsilon} \varepsilon^2 a_0 \nabla w \cdot \nabla U_1^\varepsilon dx + \int_{\Omega_0^\varepsilon} w U_1^\varepsilon dx, \tag{B.19}$$

$$A_j(\varepsilon) := \int_{\Omega_j^\varepsilon} a_j \nabla w \cdot \nabla U_1^\varepsilon dx + \int_{\Omega_j^\varepsilon} w U_1^\varepsilon dx, \quad j = 1, 2. \tag{B.20}$$

We will be separately estimating $\tilde{A}_0(\varepsilon)$, $A_j(\varepsilon)$, $j = 0, 1, 2$ and then $I_2(\varepsilon)$ in the series of the following propositions, and will subsequently derive (B.6) by combining all these estimates.

Proposition B.2.

$$\left|\tilde{A}_0(\varepsilon)\right| \leq C\varepsilon^{1/2} b_\varepsilon(w,w)^{1/2}. \tag{B.21}$$

Proof. Notice that the measure of $\tilde{\Omega}_0^\varepsilon$ is bounded by $C\varepsilon$. As a result, using Hölder's inequality,

$$\left|\int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0(\varepsilon)\nabla w \cdot \nabla U_1^\varepsilon dx\right| \;\leq\; \left(\int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0(\varepsilon)\nabla w \cdot \nabla w dx\right)^{1/2} \left(\int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0(\varepsilon)\left|\nabla U_1^\varepsilon\right|^2 dx\right)^{1/2} \leq C\varepsilon^{1/2} b_\varepsilon(w,w)^{1/2},$$

where we have used (2.5) and the boundedness of ∇U_1^ε in $\tilde{\Omega}_0^\varepsilon$ via (B.1). The second term in (B.18) is bounded similarly which leads to (B.21). □

Proposition B.3.

$$A_0(\varepsilon) = \int_{\Omega_0^\varepsilon} w f^\varepsilon dx - \int_{\partial\Omega_0^\varepsilon} w\varepsilon a_0 n_i V_{,i}(x/\varepsilon)u_0 dS + \hat{A}_0(\varepsilon),\;\; |\hat{A}_0(\varepsilon)| \leq C\varepsilon b_\varepsilon(w,w)^{1/2}. \tag{B.22}$$

Proof. First we will need expressions for "flows" associated with U_1^ε in (B.19), i.e. let $p_0^\varepsilon(x) := a_0\varepsilon^2\nabla U_1^\varepsilon(x)$, $x \in \Omega_0^\varepsilon$. Using (B.1), (4.4) and (A.7),

$$(p_0^\varepsilon(x))_i = a_0\varepsilon V_{,i}(x/\varepsilon)u_0(x) + a_0\varepsilon^2(r_0^\varepsilon(x,x/\varepsilon))_i,\;\; x \in \Omega_0^\varepsilon, \tag{B.23}$$

where

$$(r_0^\varepsilon(x,y))_i := u_{0,i}(x)(1+V(y)) + N_{j,i}(y)u_{0,j}(x) + \varepsilon N_j(y)u_{0,ji}(x)\;,\;\; y = x/\varepsilon. \tag{B.24}$$

(Henceforth p_i denotes the i-th component of the appropriate vector field $p(x)$.)

Then, upon "partial" integration by parts, $A_0(\varepsilon)$ can be evaluated as follows:

$$\begin{aligned} A_0(\varepsilon) = & -a_0\int_{\Omega_0^\varepsilon} w\Big(u_0\triangle V + \varepsilon V_{,i}u_{0,i}\Big)dx + a_0\int_{\Omega_0^\varepsilon}\varepsilon^2\nabla w\cdot r_0^\varepsilon\, dx \qquad \text{(B.25)}\\ & + \int_{\Omega_0^\varepsilon} w\Big(u_0(1+V) + \varepsilon N_j u_{0,j}\Big)dx - \int_{\partial\Omega_0^\varepsilon} w\varepsilon a_0 n_i V_{,i}(x/\varepsilon)u_0 dS = \\ & \int_{\Omega_0^\varepsilon} wu_0\left(-a_0\triangle V + 1 + V\right)dx - \int_{\partial\Omega_0^\varepsilon} w\varepsilon a_0 n_i V_{,i}(x/\varepsilon)u_0 dS \\ & + \int_{\Omega_0^\varepsilon}\varepsilon w\Big(-a_0 V_{,i}u_{0,i} + N_j u_{0,j}\Big)dx + a_0\int_{\Omega_0^\varepsilon}\varepsilon^2\nabla w\cdot r_0^\varepsilon dx. \end{aligned}$$

Using (A.8) and (B.5) we observe that

$$\int_{\Omega_0^\varepsilon} wu_0(-a_0\triangle V + 1 + V)dx = \int_{\Omega_0^\varepsilon} wu_0(\lambda_0+1)(1+V)dx = \int_{\Omega_0^\varepsilon} wf^\varepsilon dx. \tag{B.26}$$

Moreover, the last two terms in the right hand side of (B.25) are small:

$$\left|\int_{\Omega_0^\varepsilon} \varepsilon w\left(-a_0 V_{,i} u_{0,i} + N_j u_{0,j}\right) dx\right| \le C\,\varepsilon\,\|w\|_{L_2(\mathbb{R}^n)} \le C\varepsilon b_\varepsilon(w,w)^{1/2}, \quad \text{(B.27)}$$

$$\left|\int_{\Omega_0^\varepsilon} \varepsilon^2 \nabla w \cdot r_0^\varepsilon dx\right| \le C\,\varepsilon\, b_\varepsilon(w,w)^{1/2}, \quad \text{(B.28)}$$

(having used boundedness of N_j, u_0, V and exponential decay of u_0). As a result we arrive at (B.22). □

Consider next the integrals over Ω_1^ε in (B.20):

$$A_1(\varepsilon) = \int_{\Omega_1^\varepsilon} a_1 \nabla w \cdot \nabla U_1^\varepsilon dx + \int_{\Omega_1^\varepsilon} w U_1^\varepsilon dx. \quad \text{(B.29)}$$

Before formulating the corresponding result for $A_1(\varepsilon)$ we need to use following technical construction. One can extend any function w from $H^1\left(\mathbb{R}^n \setminus (\Omega_0^\varepsilon \cup \tilde{Q}_0^\varepsilon)\right)$ (let us remind that $\tilde{Q}_0^\varepsilon$ is the set of all the inclusions in Q_0^ε which intersect with the boundary $\partial\Omega_2$ of Ω_2, see Section 2) into the whole $H^1(\mathbb{R}^n)$, controlling its norm "uniformly" with respect to ε. More precisely, for any ε and any $w \in H^1\left(\mathbb{R}^n \setminus (\Omega_0^\varepsilon \cup \tilde{Q}_0^\varepsilon)\right)$ there exists a function $\hat{w} \in H^1(\mathbb{R}^n)$ such that

$$\hat{w}(x) = w(x),\; x \in \mathbb{R}^n \setminus (\Omega_0^\varepsilon \cup \tilde{Q}_0^\varepsilon) \;\text{ and }\; \|\hat{w}\|_{H^1(\mathbb{R}^n)} \le C\|w\|_{H^1(\mathbb{R}^n \setminus (\Omega_0^\varepsilon \cup \tilde{Q}_0^\varepsilon))}, \quad \text{(B.30)}$$

where C does not depend on ε and w. The above follows e.g. via a straightforward modification of the so called "extension lemma", see e.g. [21] §3.1 Lemma 3.2 which uses the extension construction, see for the latter e.g. [35] §6.3.1, p.181, Theorem 5.

Proposition B.4.

$$A_1(\varepsilon) = (\lambda_0+1)|Q_1| \int_{\mathbb{R}^n\setminus\Omega_2} \hat{w} u_0 dx + \int_{\partial\Omega_2} \hat{w} n_i A_{ij}^{\text{hom}} u_{0,j} dS + \int_{\partial\Omega_0^\varepsilon} w\varepsilon a_0 n_i V_{,i} dS + \hat{A}_1(\varepsilon), \quad \text{(B.31)}$$

where

$$|\hat{A}_1(\varepsilon)| \le C\varepsilon^{1/2} b_\varepsilon(w,w)^{1/2}. \quad \text{(B.32)}$$

Proof. Let μ^ε be characteristic function of Ω_1^ε. We can rewrite the first integral in (B.29) via (B.1) and (B.2) as follows

$$\int_{\Omega_1^\varepsilon} a_1 \nabla w \cdot \nabla U_1^\varepsilon dx = \int_{\mathbb{R}^n\setminus\Omega_2} \nabla \hat{w} \cdot p_1^\varepsilon dx, \quad \text{(B.33)}$$

where

$$(p_1^\varepsilon(x))_i := \mu^\varepsilon a_1\Bigg(u_{0,i}(x) + N_{j,i}(y)u_{0,j}(x) + \varepsilon N_j(y)u_{0,ji}(x) + \tag{B.34}$$

$$\varepsilon M_{jk,i}(y)u_{0,jk}(x) + \varepsilon W_{,i}(y)u_0(x)\Bigg) + \varepsilon^2 r_1^\varepsilon(x),\ x \in \mathbb{R}^n\backslash\Omega_2, y = x/\varepsilon,$$

and

$$r_1^\varepsilon(x) := \mu^\varepsilon(x)a_1\Bigg(M_{jk}(x/\varepsilon)u_{0,jki}(x) + W(x/\varepsilon)u_{0,i}(x)\Bigg). \tag{B.35}$$

The above flow $(p_1^\varepsilon(x))_i$ can be re-written in the following form

$$(p_1^\varepsilon(x))_i = A_{ij}^{\rm hom}u_{0,j} + g_i^j(x/\varepsilon)u_{0,j} + \mu^\varepsilon a_1\Bigg(\varepsilon N_j(x/\varepsilon)u_{0,ji}(x) + \tag{B.36}$$

$$\varepsilon M_{jk,i}(y)u_{0,jk}(x) + \varepsilon W_{,i}(y)u_0(x)\Bigg) + \varepsilon^2 r_1^\varepsilon(x),\ x \in \mathbb{R}^n\backslash\Omega_2, y = x/\varepsilon.\ ,$$

Here

$$g_i^j(y) := \mu(y)a_1(\delta_{ij} + N_{j,i}(y)) - A_{ij}^{\rm hom}, y \in Q, \tag{B.37}$$

μ is the characteristic function of Q_1, and $A_{ij}^{\rm hom}$ are the entries of the homogenized matrix $A^{\rm hom}$, see (A.11), and $N_j(y)$ are here assumed extended by zero on Q_0. It follows then from (A.5) that vector field g_i^j (with fixed j)is divergence free in Q in the following (weak) sense:

$$\int_Q \partial_i\psi g_i^j = 0,\ \forall\psi \in H^1(\square), \tag{B.38}$$

($H^p(\square)$ stands for the closure in $H^p(Q)$ of all periodic C^∞ functions and $\partial_i := \partial/\partial y_i$). Since (B.37) and (A.11) also imply that g_i^j have zero mean value, they can be rewritten as "gradients" of a skew-symmetric fields $G_{ik}(y) \in H^1(\square)$:

$$g_i^j(y) = \partial_k G_{ik}^j(y),\ \ G_{ik}^j(y) = -G_{ki}^j(y), \tag{B.39}$$

see e.g. [21] §1.1.

Consequently (B.36) can be rewritten as

$$(p_1^\varepsilon(x))_i = A_{ij}^{\rm hom}u_{0,j}(x) + \varepsilon\frac{\partial}{\partial x_k}\Bigg(G_{ik}^j(x/\varepsilon)u_{0,j}(x)\Bigg) + \mu^\varepsilon a_1\Bigg(\varepsilon N_j(x/\varepsilon)u_{0,ji}(x) + \tag{B.40}$$

$$\varepsilon M_{jk,i}(x/\varepsilon)u_{0,jk}(x) + \varepsilon W_{,i}(x/\varepsilon)u_0(x)\Bigg) + \tilde r_1^\varepsilon(x)\ , x \in \mathbb{R}^n\backslash\Omega_2,\ x \in \mathbb{R}^n\backslash\Omega_2,$$

where

$$\tilde{r}_1^\varepsilon(x) := -\varepsilon G_{ik}^j(x/\varepsilon)u_{0,jk}(x) + \varepsilon^2\mu^\varepsilon(x)\Big(M_{jk}(x/\varepsilon)u_{0,jki}(x) + W(x/\varepsilon)u_{0,i}(x)\Big). \tag{B.41}$$

Function $\frac{\partial}{\partial x_k}(G_{ik}^j u_{0,j})$ from (B.40) is not divergence free in $\mathbb{R}^n\backslash\Omega_2$. We introduce its "divergence free modification" $\frac{\partial}{\partial x_k}(\chi_\varepsilon G_{ik}^j u_{0,j})$ which differs from $\frac{\partial}{\partial x_k}(G_{ik}^j u_{0,j})$ insignificantly by employing again the cut-off function $\chi_\varepsilon(x)$, see (B.8). As a result,

$$(p_1^\varepsilon(x))_i = A_{ij}^{\mathrm{hom}}u_{0,j}(x) + \varepsilon\frac{\partial}{\partial x_k}\Big(\chi_\varepsilon G_{ik}^j(x/\varepsilon)u_{0,j}(x)\Big) + \mu^\varepsilon a_1\Big(\varepsilon N_j(x/\varepsilon)u_{0,ji}(x) + \tag{B.42}$$

$$\varepsilon M_{jk,i}(x/\varepsilon)u_{0,jk}(x) + \varepsilon W_{,i}(x/\varepsilon)u_0(x)\Big) + \tilde{\tilde{r}}_1^\varepsilon(x)\ , x\in\mathbb{R}^n\backslash\Omega_2,$$

where

$$\begin{aligned}(\tilde{\tilde{r}}_1^\varepsilon(x))_i \;&=\; \varepsilon\frac{\partial}{\partial x_k}\Big((1-\chi_\varepsilon(x))G_{ik}^j(x/\varepsilon)u_{0,j}(x)\Big) - \varepsilon G_{ik}^j(x/\varepsilon)u_{0,jk}(x)\ + \\ &\varepsilon^2\mu^\varepsilon(x)\Big(M_{jk}(x/\varepsilon)u_{0,jki}(x) + W(x/\varepsilon)u_{0,i}(x)\Big).\end{aligned} \tag{B.43}$$

Integrating by parts in (B.33) and using (B.42) we obtain

$$\int_{\Omega_1^\varepsilon} a_1\nabla w\cdot\nabla U_1^\varepsilon dx = \int_{\mathbb{R}^n\backslash\Omega_2}\nabla\hat{w}\cdot p_1^\varepsilon dx = -\int_{\mathbb{R}^n\backslash\Omega_2}\hat{w}A_{ij}^{\mathrm{hom}}u_{0,ij}dx \tag{B.44}$$

$$-\int_{\Omega_1^\varepsilon} wa_1\Big(N_{j,i}(y)u_{0,ji}(x) + M_{jk,ii}(y)u_{0,jk}(x) + W_{,ii}(y)u_0(x))\Big)dx$$

$$+\int_{\partial\Omega_2}\hat{w}n_iA_{ij}^{\mathrm{hom}}u_{0,j}dS + \int_{\partial\Omega_0^\varepsilon} w\varepsilon n_ia_1\Big(N_j(y)u_{0,ji}(x) + M_{jk,i}(y)u_{0,jk}(x) + W_{,i}(y)u_0(x)\Big)dS$$

$$+R_1^\varepsilon + R_2^\varepsilon + S_1^\varepsilon,\ \ y = x/\varepsilon,$$

with the "remainders"

$$R_1^\varepsilon := \int_{\mathbb{R}^n\backslash\Omega_2}\nabla\hat{w}\cdot\tilde{\tilde{r}}_1^\varepsilon dx, \tag{B.45}$$

$$R_2^\varepsilon := -\int_{\Omega_1^\varepsilon} w\,\varepsilon a_1\,\Big(N_j(y)u_{0,jii}(x) + M_{jk,i}(y)u_{0,jki}(x) + W_{,i}(y)u_{0,i}(x))\Big)dx, \tag{B.46}$$

$$S_1^\varepsilon := \int_{\partial\Omega_1^\varepsilon\backslash\partial\Omega_0^\varepsilon} w\varepsilon n_ia_1\Big(N_j(y)u_{0,ji}(x) + M_{jk,i}(y)u_{0,jk}(x) + W_{,i}(y)u_0(x)\Big)dS,\ \ y = x/\varepsilon. \tag{B.47}$$

We argue that all the three remainders are "small".

We start with most delicate remainder, S_1^ε. First of all we divide this integral into two parts:

$$S_1^\varepsilon = S_1^1(\varepsilon) + S_1^2(\varepsilon), \tag{B.48}$$

where

$$S_1^1(\varepsilon) := \varepsilon \int_{\partial\Omega_2 \cap \partial\Omega_1^\varepsilon} wzds, \;\; S_1^2(\varepsilon) := \varepsilon \int_{\partial\Omega_1^\varepsilon \setminus (\partial\Omega_0^\varepsilon \cup \partial\Omega_2)} wzds, \tag{B.49}$$

$$z := n_i a_1 \left(N_j(y) u_{0,ji}(x) + M_{jk,i}(y) u_{0,ji}(x) + W_{,i}(y) u_0(x) \right), \; y = x/\varepsilon,$$

i.e. in the first integral we integrate over a part of $\partial\Omega_2$ and in the second one over the remaining part of boundary of small inclusions $\tilde{\Omega}_0^\varepsilon$ intersecting with $\partial\Omega_2$. Then

$$|S_1^1(\varepsilon)| \le C\varepsilon \int_{\partial\Omega_2 \cap \partial\Omega_1^\varepsilon} |w| ds$$

$$\le C\varepsilon \int_{\partial\Omega_2} |w| ds \le C\varepsilon |\partial\Omega_2|^{1/2} \left(\int_{\partial\Omega_2} |w|^2 ds \right)^{1/2} \le C\varepsilon b_\varepsilon(w,w)^{1/2},$$

where we have used boundedness of z and then the continuity of the trace operator from $H^1(\Omega_2)$ into $L_2(\partial\Omega_2)$.

The second integral $S_1^2(\varepsilon)$ can be evaluated as follows:

$$|S_1^2(\varepsilon)| \le C\varepsilon \int_{\partial\Omega_1^\varepsilon \setminus (\partial\Omega_0^\varepsilon \cup \partial\Omega_2)} |w| ds \le C\varepsilon \int_{\partial\tilde{Q}_0^\varepsilon} |w| ds \le C\varepsilon \left(\int_{\partial\tilde{Q}_0^\varepsilon} |w|^2 ds \right)^{1/2},$$

where we recall that $\tilde{Q}_0^\varepsilon$ is the set of small inclusions from Q_0^ε which intersect with $\partial\Omega_2$, see beginning of §2, and notice that the measure of their total boundary is uniformly bounded with respect to ε. Using then rescaling for the "small" (order ε size) inclusions from $\tilde{Q}_0^\varepsilon$, then the trace estimates and finally (B.30) we obtain

$$\left(\int_{\partial\tilde{Q}_0^\varepsilon} |w|^2 ds \right)^{1/2} = \left(\int_{\partial\tilde{Q}_0^\varepsilon} |\hat{w}|^2 ds \right)^{1/2} \le C\varepsilon^{-1/2} \|\hat{w}\|_{H^1(\tilde{Q}_0^\varepsilon)}$$

$$\le C\varepsilon^{-1/2} \|w\|_{H^1(\mathbb{R}^n \setminus (\Omega_0^\varepsilon \cup \tilde{Q}_0^\varepsilon))} \le C\varepsilon^{-1/2} b_\varepsilon(w,w)^{1/2}. \tag{B.50}$$

Consequently

$$|S_1^\varepsilon| \le \varepsilon^{1/2} b_\varepsilon(w,w)^{1/2}.$$

Let us next consider R_1^ε. The only term of "order one" in R_1^ε is $\varepsilon \frac{\partial}{\partial x_k} \left((1-\chi^\varepsilon) G_{ik}^j u_{0,j} \right)$, see (B.45) and (B.43). However the size of the support of $(1-\chi^\varepsilon)$ is of order ε, and consequently

$$|R_1^\varepsilon| \le \varepsilon^{1/2} b_\varepsilon(w,w)^{1/2}. \tag{B.51}$$

(Having noticed that the remaining, "order ε", terms in R_1^ε contribute only order ε terms into the right hand side of (B.51), upon straightforward application of Hölder's inequality.)

Finally, notice that all the terms of R_2^ε are of order ε with exponentially decaying u_0, which implies $|R_2^\varepsilon| \leq \varepsilon b_\varepsilon(w,w)^{1/2}$.

Summarising, we obtain following estimate on the combined smallness of the three remainders:

$$|R_1^\varepsilon + R_2^\varepsilon + S_1^\varepsilon| \leq C\varepsilon^{1/2} b_\varepsilon(w,w)^{1/2}. \tag{B.52}$$

Next consider the terms of "order zero" in (B.44). It follows from (A.13) and (A.15) that the integrals over Ω_1^ε and $\partial\Omega_0^\varepsilon$ in the right hand side of (B.44) can be evaluated in the following way:

$$-\int_{\Omega_1^\varepsilon} wa_1 \Big(N_{j,i}(x/\varepsilon)u_{0,ji}(x) + M_{ji,kk}(x/\varepsilon)u_{0,ji}(x) + W_{,ii}(x/\varepsilon)u_0(x) \Big) dx =$$

$$-\int_{\Omega_1^\varepsilon} wb(\lambda_0)u_0(x)dx,$$

and

$$\int_{\partial\Omega_0^\varepsilon} w\varepsilon n_i a_1 \Big(N_j(y)u_{0,ji}(x) + M_{jk,i}(y)u_{0,jk}(x) + W_{,i}(y)u_0(x) \Big) dS = \int_{\partial\Omega_0^\varepsilon} w\varepsilon a_0 n_i V_{,i} dS. \tag{B.53}$$

As a result we can evaluate $A_1(\varepsilon)$ (see (B.29)) as follows

$$A_1(\varepsilon) = -\int_{\mathbb{R}^n\setminus\Omega_2} \hat{w} A_{ij}^{\mathrm{hom}} u_{0,ij} dx - (b(\lambda_0) - 1)\int_{\Omega_1^\varepsilon} wu_0 dx + \tag{B.54}$$

$$+\int_{\partial\Omega_2} \hat{w} n_i A_{ij}^{\mathrm{hom}} u_{0,j} dS + \int_{\partial\Omega_0^\varepsilon} \hat{w}\varepsilon a_0 n_i V_{,i} dS + \tilde{A}_1(\varepsilon),$$

where

$$\tilde{A}_1(\varepsilon) := R_1^\varepsilon + R_2^\varepsilon + S_1^\varepsilon + \int_{\Omega_1^\varepsilon} w\Big(\varepsilon N_j u_{0,j} + \varepsilon^2 \left(M_{jk}u_{0,jk} + Wu_0 \right) \Big) dx,$$

and consequently

$$|\tilde{A}_1(\varepsilon)| \leq C\varepsilon^{1/2} b_\varepsilon(w,w)^{1/2}. \tag{B.55}$$

Let us consider the integral over Ω_1^ε in (B.54):

$$\int_{\Omega_1^\varepsilon} wu_0 dx = \int_{\mathbb{R}^n\setminus\Omega_2} \mu^\varepsilon wu_0 dx, \tag{B.56}$$

treating $\mu^\varepsilon(x) = \mu(x/\varepsilon)$ as the characteristic function of $\mathbb{R}^n \backslash Q_0^\varepsilon$. It is easy to see that the associated Q-periodic characteristic function $\mu(y)$ can be presented as follows:

$$\mu(y) = |Q_1| + \triangle_y M(y), y \in Q, \tag{B.57}$$

where $M \in H^2(\square)$. Then (B.56) can be evaluated in the following way:

$$\int_{\mathbb{R}^n\backslash\Omega_2} \mu^\varepsilon w u_0 dx = \int_{\mathbb{R}^n\backslash\Omega_2} \mu^\varepsilon \hat{w} u_0 dx = \int_{\mathbb{R}^n\backslash\Omega_2} (|Q_1| + \varepsilon^2 \triangle_x M(x/\varepsilon))\hat{w} u_0 dx = \tag{B.58}$$
$$\int_{\mathbb{R}^n\backslash\Omega_2} |Q_1|\hat{w} u_0 dx - \int_{\mathbb{R}^n\backslash\Omega_2} \varepsilon^2 \nabla_x M(x/\varepsilon) \cdot \nabla_x(\hat{w} u_0) dx + \int_{\partial\Omega_2} \varepsilon^2 n_i \frac{\partial}{\partial x_i} M(x/\varepsilon)\hat{w} u_0 dS.$$

Following the pattern of the previous estimates (i.e. again using the boundedness and the trace properties, cf. (B.50)), the last two terms can be estimated as follows:

$$\left| \int_{\partial\Omega_2} \varepsilon^2 n_i \frac{\partial}{\partial x_i} M(x/\varepsilon)\hat{w} u_0 dS - \int_{\mathbb{R}^n\backslash\Omega_2} \varepsilon^2 \nabla_x M(x/\varepsilon) \cdot \nabla_x(\hat{w} u_0) dx \right| \leq C\varepsilon b_\varepsilon(w,w)^{1/2}. \tag{B.59}$$

As a result we have

$$A_1(\varepsilon) = -\int_{\mathbb{R}^n\backslash\Omega_2} \hat{w} A_{ij}^{\text{hom}} u_{0,ij} dx - (b(\lambda_0) - 1)|Q_1| \int_{\mathbb{R}^n\backslash\Omega_2} \hat{w} u_0 dx + \tag{B.60}$$
$$+ \int_{\partial\Omega_2} \hat{w} n_i A_{ij}^{\text{hom}} u_{0,j} dS + \int_{\partial\Omega_0^\varepsilon} w\varepsilon a_0 n_i V_{,i} dS + \hat{A}_1(\varepsilon),$$

with $\hat{A}_1(\varepsilon)$ satisfying (B.32). Finally, using equation (3.15) for u_0 and (A.16) we obtain (B.31). □

Consider now the integrals $A_2(\varepsilon)$ over Ω_2, see (B.16):

Proposition B.5.

$$A_2(\varepsilon) = \int_{\Omega_2} w f^\varepsilon dx - \int_{\partial\Omega_2} \hat{w} n_i a_2 u_{0,i} dS + \hat{A}_2(\varepsilon), \tag{B.61}$$

where

$$|\hat{A}_2(\varepsilon)| \leq C\varepsilon^{1/2} b_\varepsilon(w,w)^{1/2}. \tag{B.62}$$

Proof.

$$A_2(\varepsilon) = \int_{\Omega_2} a_2 \nabla w \cdot \nabla U_1^\varepsilon dx + \int_{\Omega_2} w U_1^\varepsilon dx = \tag{B.63}$$
$$\int_{\Omega_2} a_2 \nabla \hat{w} \cdot \nabla U_1^\varepsilon dx + \int_{\Omega_2} \hat{w} U_1^\varepsilon dx + \tilde{A}_2(\varepsilon),$$

where

$$\tilde{A}_2(\varepsilon) = \int_{\Omega_2} a_2 \nabla(w - \hat{w}) \cdot \nabla U_1^\varepsilon dx + \int_{\Omega_2} (w - \hat{w}) U_1^\varepsilon dx.$$

Integrating by parts, we obtain (with the consistent choice of the normal n to $\partial\Omega_2$ being inward for Ω_2)

$$A_2(\varepsilon) = -\int_{\Omega_2} \hat{w} a_2 \triangle u_0 dx + \int_{\Omega_2} \hat{w} u_0 dx - \int_{\partial\Omega_2} \hat{w} n_i a_2 u_{0,i} dS + \tilde{A}_2(\varepsilon) = \quad \text{(B.64)}$$

$$(\lambda_0 + 1) \int_{\Omega_2} \hat{w} u_0 dx - \int_{\partial\Omega_2} \hat{w} n_i a_2 u_{0,i} dS + \tilde{A}_2(\varepsilon) = \int_{\Omega_2} \hat{w} f^\varepsilon dx - \int_{\partial\Omega_2} \hat{w} n_i a_2 u_{0,i} dS + \tilde{A}_2(\varepsilon),$$

having used (3.14) and (B.5). The above expression can be rewritten as

$$A_2(\varepsilon) = \int_{\Omega_2} w f^\varepsilon dx - \int_{\partial\Omega_2} \hat{w} n_i a_2 u_{0,i} dS + \hat{A}_2(\varepsilon),$$

where

$$\hat{A}_2(\varepsilon) := \int_{\Omega_2 \cap \tilde{Q}_0^\varepsilon} a_2 \nabla(w - \hat{w}) \cdot \nabla U_1^\varepsilon dx + \int_{\Omega_2 \cap \tilde{Q}_0^\varepsilon} (w - \hat{w}) U_1^\varepsilon dx + \int_{\Omega_2 \cap \tilde{Q}_0^\varepsilon} (\hat{w} - w) f^\varepsilon dx.$$

Arguing further as in Proposition B.2, we obtain (B.62). □

We can now complete the proof of the Lemma B.1 with the aid of the established Propositions B.2–B.5 as follows.

Combine (B.61) with (B.21), (B.22), (B.31) which are all substituted into (B.17), and then employ (3.7). As a result, $I_1(\varepsilon)$, see(B.15), is evaluated as follows:

$$I_1(\varepsilon) = b_\varepsilon(w, U_1^\varepsilon) = \int_{\Omega_2 \bigcup \Omega_0^\varepsilon} w f^\varepsilon dx + (\lambda_0 + 1)|Q_1| \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} u_0 dx + \hat{I}_1(\varepsilon), \quad \text{(B.65)}$$

where

$$|\hat{I}_1(\varepsilon)| \leq C\varepsilon^{1/2} b_\varepsilon(w, w)^{1/2}. \quad \text{(B.66)}$$

Let us now consider $I_2(\varepsilon)$, see (B.16). Noticing that the last term in the right hand side of (B.16) is a constant times (B.56), we can employ again (B.58)-(B.59) which results in:

$$\int_{\Omega_1^\varepsilon} w u_0 dx = |Q_1| \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} u_0 dx + R^\varepsilon, \quad \text{(B.67)}$$

where

$$|R^\varepsilon| \leq C\varepsilon b_\varepsilon(w, w)^{1/2}. \quad \text{(B.68)}$$

Notice next that the integral over $\tilde{\Omega}_0^\varepsilon$ in (B.16) is small due to the fact that the measure of $\tilde{\Omega}_0^\varepsilon$ is small:

$$\left| \int_{\tilde{\Omega}_0^\varepsilon} w u_0 dx \right| \le C\varepsilon^{1/2} b_\varepsilon(w,w)^{1/2}. \tag{B.69}$$

As a result of employing (B.67)–(B.69) in (B.16) it can be rewritten in the following form:

$$I_2(\varepsilon) = -(\lambda_0+1)\left(\int_{\Omega_0^\varepsilon} w u_0 (1+V) dx + \int_{\Omega_2} w u_0 dx + |Q_1| \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} u_0 dx \right) + \hat{R}^\varepsilon, \tag{B.70}$$

where

$$|\hat{R}^\varepsilon| \le C\varepsilon^{1/2} b_\varepsilon(w,w)^{1/2}. \tag{B.71}$$

Adding finally (B.70) and (B.65) and using (B.5), we conclude that (B.14) can be bounded as follows:

$$|b_\varepsilon(w, U_1^\varepsilon - \tilde{u}^\varepsilon)| \le C\varepsilon^{1/2} b_\varepsilon(w,w)^{1/2}, \tag{B.72}$$

which is identical to (B.6) and hence proves Lemma B.1. □

Acknowledgements

The work was supported by Bath Institute for Complex Systems (EPSRC grant GR/S86525/01). The authors are grateful to Vladimir Kamotski, R.R. Gadyl'shin, V.V. Zhikov for discussions and useful comments. Thanks are also due to D.M. Bird for stimulating discussions on physical origins.

References

[1] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1965.

[2] A. Alama, M. Avellaneda, P.A. Deift, and R. Hempel, On the existence of eigenvalues of a divergence-form operator $A + \lambda B$ in a gap of $\sigma(A)$. Asymptotic Anal. 8 (1994), no. 4, 311–344.

[3] G. Allaire. Homogenization and two-scale convergence, SIAM J. Math. Anal 23 (1992), 1482–1518.

[4] G. Allaire, Y. Capdeboscq and A. Piatnitski, Homogenization and localization with an interface, Indiana Univ. Math. J. 52 (2003), no. 6, 1413–1446.

[5] G. Allaire, and C. Conca, Bloch wave homogenization and spectral asymptotic analysis. J. Math. Pures Appl. (9) 77 (1998), no. 2, 153–208.

[6] G. Allaire and A. Piatnitski, Homogenization of the Schrödinger equation and effective mass theorems, Comm. Math. Phys. 258 (2005), no. 1, 1–22.

[7] N.S. Bakhvalov and G.P. Panasenko, Homogenization: Averaging Processes in Periodic Media, Nauka, Moscow. 1984 (in Russian). English translation in: Mathematics and Its Applications (Soviet Series) 36, Kluwer Academic Publishers, Dordrecht-Boston-London, 1989.

[8] A.Yu. Belyaev, Compression waves in a fluid with gas bubbles, J. Appl. Math. Mech. 52 (1988), no. 3, 344–348.

[9] A. Bensoussan, J.-L. Lions and G.C. Papanicolaou, Asymptotic analysis for periodic structures, North Holland, Amsterdam, 1978.

[10] M.Sh. Birman, On the spectrum of singular boundary-value problems, (Russian) Mat. Sb. 55 (1961), no. 97 , 125–174. English transl. in Eleven Papers on Analysis, AMS Transl. 2, 53, AMS, Providence, RI, 1966, pp. 23–60.

[11] M.Sh. Birman and T.A. Suslina, Periodic second-order differential operators. Threshold properties and averaging, (Russian) Algebra i Analiz 15 (2003), no. 5, 1–108; translation in St. Petersburg Math. J. 15 (2004), no. 5, 639–714

[12] G. Bouchitté and D. Felbacq, Homogenization near resonances and artificial magnetism from dielectrics, C. R. Math. Acad. Sci. Paris 339 (2004), no. 5, 377–382.

[13] K.D. Cherednichenko, V.P. Smyshlyaev and V.V. Zhikov , Non-local homogenised limits for composite media with highly anisotropic periodic fibres, Proceedings of the Royal Society of Edinburgh: Section A , 136A (2006).

[14] A. Figotin and A. Klein, Localized classical waves created by defects, J. Statist. Phys. 86 (1997), no. 1-2, 165–177.

[15] A. Figotin and A. Klein, Localization of light in lossless inhomogeneous dielectrics, J. Opt. Soc. Amer. 15 (1998), no. 5, 1423–1435.

[16] A. Figotin and P. Kuchment, Band-gap structure of spectra of periodic dielectric and acoustic media. I. Scalar model. SIAM J. Appl. Math. 56 (1996), no. 1, 68–88.

[17] A. Figotin and P. Kuchment, Band-gap structure of spectra of periodic dielectric and acoustic media. II. Two-dimensional photonic crystals, SIAM J. Appl. Math. 56 (1996), no. 6, 1561–1620.

[18] S.E. Golowich and M.I. Weinstein, Scattering resonances of microstructures and homogenization theory, SIAM J. Multiscale Model. Simul. 3 (2005), no. 3, 477–521.

[19] S. Guenneau, A. Nicolet, C. Geuzaine, F. Zolla and A.B. Movchan, Comparisons of finite element and Rayleigh methods for the study of conical Bloch waves in arrays of metallic cylinders, COMPEL 23 (2004), no. 4, 932–949.

[20] R. Hempel and K. Lienau, Spectral properties of periodic media in the large coupling limit, Comm. Partial Differential Equations 25 (2000), no. 7-8, 1445–1470.

[21] V.V. Jikov, S.M. Kozlov and O.A. Oleinik, Homogenization of Differential Operators and Integral Functionals, Springer-Verlag, Berlin, 1994.

[22] P. Kuchment, The mathematics of photonic crystals, in: Mathematical modeling in optical science, Frontiers Appl. Math. 22, SIAM, Philadelphia, PA, 2001, pp. 207–272.

[23] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization, SIAM J. Math. Anal. 20 (1989), 608–623.

[24] G.J. Pearce, T.D. Hedley and D.M. Bird 2005. Adaptive curvilinear coordinates in a plane-wave solution of Maxwell's equations in photonic crystals. Physical Review B 71 (19): Art. No. 195108 May (2005).

[25] G. Pearce, J. Pottage, D. Bird, P. Roberts, J. Knight, and P. Russell, Hollow-core PCF for guidance in the mid to far infra-red. Optics Express 13 (2005), no. 18, 6937–6946.

[26] J. Pottage, D. Bird, T. Hedley, J. Knight, T. Birks, P. Russell and P. Roberts, Robust photonic band gaps for hollow core guidance in PCF made from high index glass. Optics Express 11 (2003), no. 2, 2854–2861.

[27] M. Neuss-Radu, A result on the decay of the boundary layers in the homogenization theory. Asymptotic Analysis 23 (2000), 313–328.

[28] M. Neuss-Radu, The failure of uniform exponential decay for boundary layers. In: N. Antonic, C.J. van Duijn, W. Jäger, A. Mikelic (Eds.): Multiscale Problems in Science and Technology, Challenges to Mathematical Analysis and Perspectives. Proceedings of the Conference on Multiscale

Problems in Science and Technology, Dubrovnik, Croatia, 3-9 September 2000. Springer, 2002, 243-251.

[29] C.G. Poulton, L.C. Botten, R.C. McPhedran, N.A. Nicorovici and A.B. Movchan, Noncommuting limits in electromagnetic scattering: asymptotic analysis for an array of highly conducting inclusions, SIAM J. Appl. Math. 61 (2001), no. 5, 1706–1730.

[30] P. Russell, Photonic crystal fibers, Science 299 (2003), 358–362.

[31] E. Sanchez-Palencia, Nonhomogeneous media and vibration theory, Lecture Notes in Physics, 127, Springer-Verlag, Berlin-New York, 1980.

[32] G.V. Sandrakov, Averaging of nonstationary problems in the theory of strongly nonhomogeneous elastic media, (Russian) Dokl. Akad. Nauk 358 (1998), no. 3, 308–311.

[33] T.J. Shepherd, P.J. Roberts and R. Loudon. Soluble two-dimensional photonic crystal models, Phys. Rev. E 5 (1997), no. 5, 6024–6038.

[34] V.P. Smyshlyaev and K.D. Cherednichenko. On derivation of strain gradient effects in the overall behaviour of periodic heterogeneous media, J. Mech. Phys. Solids 48 (2000), 1325–1357.

[35] E. Stein, Singular integrals and differentiability properties of functions. Princeton University Press, Princeton, New Jersey, 1970.

[36] M.I. Vishik and A.A. Lyusternik, Regular degeneration and boundary layer for linear differential equations with small parameter, (Russian) Uspehi Mat. Nauk (N.S.) 12 (1957), no. 5(77), 3–122.

[37] J.D. Joannopoulos, R.D. Meade and J.N. Winn, Photonic crystals, molding the flow of light. Princeton University Press, Princeton, New Jersey, 1995.

[38] V.V. Zhikov, Spectral approach to asymptotic diffusion problems, (Russian) Differentsialnye Uravneniya 25 (1989), no. 1, 44–50 ; translation in Differential Equations 25 (1989), no. 1, 33–39

[39] V.V. Zhikov, On an extension of the method of two-scale convergence and its applications, (Russian) Mat. Sb. 191 (2000), no. 7, 31–72; translation in Sb. Math. 191 (2000), no. 7-8, 973–1014

[40] V.V. Zhikov, 2004. Gaps in the spectrum of some elliptic operators in divergent form with periodic coefficients, (Russian) Algebra i Analiz 16 (2004), no. 5, 34–58; English version to appear in St. Petersburg Mathematical Journal.

[41] V.V. Zhikov, On operator estimates in homogenization theory. (Russian) Dokl. Akad. Nauk 403 (2005), no. 3, 305–308.

[42] V.V. Zhikov, Spectral method in homogenization theory, Proc. Steklov Inst. Mathematics 250 (2005).