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Heteroclinic travelling waves for the lattice sine-Gordon equation with linear pair interaction

Carl-Friedrich Kreiner* and Johannes Zimmer†

Abstract

The existence of travelling heteroclinic waves for the sine-Gordon lattice is proved for a linear interaction of neighbouring atoms. The asymptotic states are chosen such that the action functional is finite. The proof relies on a suitable concentration-compactness argument, which can be shown to hold even though the associated functional has no sub-additive structure.

1 Introduction

We consider the lattice sine-Gordon equation

$$\ddot{q}_k(t) = V'(q_{k+1}(t) - q_k(t)) - V'(q_k(t) - q_{k-1}(t)) - K \sin(q_k(t)), \quad k \in \mathbb{Z}, \quad (1)$$

with a constant $K > 0$. Equation (1) describes the evolution of an infinite chain of atoms with elastic nearest neighbour interaction and an on-site potential, according to Newton's law. The interaction potential $V: \mathbb{R} \rightarrow \mathbb{R}$ takes as argument the discrete strain, which is given by the difference of the positions of the atoms $q_{k+1}(t) - q_k(t)$.

In this article, we assume that V is a quadratic function $V(\varepsilon) := \frac{c_0^2}{2} \varepsilon^2$ with $c_0 > 0$ and seek a solution to (1) in the form of a travelling wave by setting $q_k(t) = u(k - ct)$ for $k \in \mathbb{Z}$. Then, a substitution into (1) yields immediately

$$c^2 u''(\tau) = c_0^2 (u(\tau + 1) - 2u(\tau) + u(\tau - 1)) - K \sin(u(\tau)). \quad (2)$$

In the setting introduced in Section 2, Equation (2) can be seen to be the Euler-Lagrange equation of the action functional

$$J(u) := \int_{\mathbb{R}} \left[\frac{c^2}{2} (u'(\tau))^2 - \frac{c_0^2}{2} (u(\tau + 1) - u(\tau))^2 + K(1 + \cos(u(\tau))) \right] d\tau. \quad (3)$$

The action functional is the kinetic energy $\int_{\mathbb{R}} \frac{c^2}{2} (u'(\tau))^2 d\tau$ minus the potential energy, consisting of interaction part, $\int_{\mathbb{R}} \frac{c_0^2}{2} (u(\tau + 1) - u(\tau))^2 d\tau$, and on-site part, $\int_{\mathbb{R}} -K(1 + \cos(u(\tau))) d\tau$. This specific choice of the on-site potential is made for simplicity of the presentation; however, all results in this paper can be generalised in a straightforward way to any non-negative, 2π -periodic C^1 -function with zero set $\{(2k + 1)\pi : k \in \mathbb{N}\}$ in place of $(1 + \cos(\cdot))$.

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We are interested in heteroclinic waves (that is, waves that connect two different stable asymptotic states at $\pm\infty$) for supersonic velocities $c > c_0$. Before stating the precise results, we give a brief overview of some related work.

Bates and Zhang [2] have shown that for a large class of similar models, homoclinic travelling waves exist for supersonic velocities. Their existence result also holds for long-range interaction, but the specialisation to nearest neighbour interaction covers the case

$$c^2 u''(\tau) = c_0^2 (u(\tau + 1) - 2u(\tau) + u(\tau - 1)) + K \sin(u(\tau)). \quad (4)$$

The on-site potential energy can here be taken to be $\int_{\mathbb{R}} [K \cos(u(\tau)) - 1] d\tau$. Bates and Zhang [2] consider homoclinic waves that have their asymptotic states in the maximum of the on-site potential. We study the analogous situation for heteroclinic waves. That is, we consider waves with asymptotic states in two different maxima of the on-site potential. For the choice $-K(1 + \cos(u(\tau)))$ made above for the on-site potential, this leads to the boundary conditions

$$\lim_{\tau \rightarrow -\infty} u(\tau) = -\pi \quad \text{and} \quad \lim_{\tau \rightarrow +\infty} u(\tau) = +\pi. \quad (5)$$

The existence proof will rely on minimisation and a novel type of concentration-compactness. The main difficulties are: Firstly, the action functional, which is to be minimised, is highly nonconvex due to the periodicity of the on-site potential. The second challenge is a lack of compactness due to the infinite domain \mathbb{R} . We show in Section 4 that these difficulties can be overcome with a suitable variant of concentration-compactness [8]. This is not obvious, since the functional (3) is not subadditive. We show that a concentration-compactness result holds nevertheless. This argument relies on the fact that the lattice action functional (3) can be related to the Mortola-Modica functional [9], so that a crucial L^∞ -*a-priori* bound can be inferred. This connection to the Mortola-Modica functional is made explicit in Section 3. Concentration-compactness arguments for lattice models were introduced by Friesecke and Wattis [5] (see also, e.g., [1]).

These two difficulties, namely a highly nonconvex functional and lack of compactness also persist for other boundary conditions, in particular

$$\lim_{\tau \rightarrow -\infty} u(\tau) = 0 \quad \text{and} \quad \lim_{\tau \rightarrow +\infty} u(\tau) = 2\pi, \quad (6)$$

that is, asymptotic states in the minima of the on-site potential (possibly to be understood in an averaged sense). These boundary conditions correspond to a moving dislocation in the Frenkel-Kontorova model [4]. There are results for heteroclinic waves with asymptotic states (6) for the special case of a piecewise quadratic on-site potential in the physics literature [7]. There, it is assumed that the solution satisfies the sign condition of the kind

$$u(\tau) < \pi \text{ for } \tau < 0 \text{ and } u(\tau) > \pi \text{ for } \tau > 0. \quad (7)$$

Under this assumption, the analogue of the Euler-Lagrange equation (2) for piecewise on-site potential simplifies to an equation with a nonlinearity that depends only on τ , rather than $u(\tau)$. This simplified system is then solved by Fourier methods, where the solution is represented as a sum of Fourier components. The difficulty is to show that the solution satisfies the sign condition (7). Kresse and Truskinovsky [7] observe that this condition probably does not hold for a specific interval

of subsonic velocities. A rigorous proof that the sign condition holds in some regime seems only to be available for the Fermi-Pasta-Ulam chain with piecewise quadratic pair interaction [10]. The extension of this result to more general potentials is an open problem.

2 Main result

We set $X := \{u \in H_{\text{loc}}^1(\mathbb{R}) : u' \in L^2(\mathbb{R})\}$ and remark that X is a Hilbert space when equipped with the inner product

$$\langle u, v \rangle_X := u(0)v(0) + \int_{\mathbb{R}} u'(\tau)v'(\tau) \, d\tau.$$

Further, let us define

$$\mathcal{M}_{-\pi, \pi} := \{u \in X : u(-\infty) = -\pi, u(\infty) = \pi\}. \quad (8)$$

We are now in a position to formalise the connection of Equation (2) and the action functional $J: X \rightarrow \mathbb{R} \cup \{\infty\}$ given in (3).

Let $v_0: \mathbb{R} \rightarrow [-\pi, \pi]$ be a monotone function in $C^\infty(\mathbb{R})$ such that $v_0(\tau) = -\pi$ for $\tau < -1$ and $v_0(\tau) = \pi$ for $\tau > 1$ and define $\Psi: H^1(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$\Psi(v) := J(v_0 + v).$$

It is not hard to see that for $c^2 > c_0^2$, $\Psi(v) < \infty$ for all $v \in H^1(\mathbb{R})$, and that, conversely, a minimiser u of J on $\mathcal{M}_{-\pi, \pi}$ can be written as $u = v_0 + v$ for some $v \in H^1(\mathbb{R})$ (for details see [6]). Furthermore, Ψ is continuously differentiable on $H^1(\mathbb{R})$.

Lemma 2.1 (Euler-Lagrange equation and regularity) *Suppose $v \in H^1(\mathbb{R})$ is a critical point of Ψ ; set $u := v_0 + v \in \mathcal{M}_{-\pi, \pi} \subset X$. Then $u \in C^2(\mathbb{R})$ and u is a solution of (2) with boundary conditions (5).*

PROOF. Every critical point $v \in H^1(\mathbb{R})$ of Ψ satisfies by definition $\langle \Psi'(v), h \rangle = 0$ for all $h \in H^1(\mathbb{R})$, that is,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} [c^2 u'(\tau) h'(\tau) - c_0^2 (u(\tau+1) - u(\tau))(h(\tau+1) - h(\tau)) - K \sin(u(\tau)) h(\tau)] \, d\tau \\ &= \int_{\mathbb{R}} [c^2 u'(\tau) h'(\tau) + c_0^2 [u(\tau-1) - 2u(\tau) + u(\tau+1)] h(\tau) - K \sin(u(\tau)) h(\tau)] \, d\tau. \end{aligned} \quad (9)$$

This means that u is a weak solution of (2). Applying a classical bootstrap argument, we find that $u \in C^2(\mathbb{R})$ is a strong solution of (2). \square

Lemma 2.1 shows in particular that a minimiser of the variational problem

$$\text{minimise } J, \text{ as defined in (3), on } \mathcal{M}_{-\pi, \pi} \subset X \quad (10)$$

is a solution to (2) with boundary conditions (5).

We now formulate the existence result for (2), for sufficiently large supersonic wave speed and heteroclinic boundary conditions.

Theorem 2.2 *Let $c^2 > \frac{25}{24}c_0^2$. Then there exists a minimiser u_0 of J on $\mathcal{M}_{-\pi,\pi} \subset X$, that is, the variational problem (10) possesses a solution. This minimiser u_0 is a C^2 -function which satisfies (2) and the asymptotic boundary condition (5).*

The proof of this Theorem will follow easily from Lemma 2.1 and the statements in Sections 3 and 4; it is given in Section 5.

3 A-priori bound

For more a compact notation, we introduce on X a difference operator A as $Au(z) := u(z+1) - u(z)$. Observe that for fixed $T_1, T_2 \in \mathbb{R}$, $T_1 < T_2$,

$$\int_{T_1}^{T_2} [Au(\tau)]^2 d\tau \leq \int_0^1 \int_{T_1+s}^{T_2+s} [u'(\tau)]^2 d\tau ds; \quad (11)$$

this follows with Jensen's inequality and Fubini's theorem,

$$\begin{aligned} \int_{T_1}^{T_2} [Au(\tau)]^2 d\tau &= \int_{T_1}^{T_2} \left[\int_{\tau}^{\tau+1} u'(s) ds \right]^2 d\tau = \int_{T_1}^{T_2} \left[\int_0^1 u'(t+\tau) dt \right]^2 d\tau \\ &\leq \int_{T_1}^{T_2} \int_0^1 [u'(t+\tau)]^2 dt d\tau = \int_0^1 \int_{T_1+t}^{T_2+t} [u'(\zeta)]^2 d\zeta dt. \end{aligned}$$

By the same argument, $\int_{\mathbb{R}} [Au(\tau)]^2 d\tau \leq \int_{\mathbb{R}} [u'(\tau)]^2 d\tau$ (see also [11]). This implies

$$\begin{aligned} \int_{\mathbb{R}} \left[\frac{c^2 - c_0^2}{2} (u'(\tau))^2 + K(1 + \cos(u(\tau))) \right] d\tau \\ \leq J(u) \leq \int_{\mathbb{R}} \left[\frac{c^2}{2} (u'(\tau))^2 + K(1 + \cos(u(\tau))) \right] d\tau. \end{aligned}$$

for all $u \in X$. Modica and Mortola [9] have studied a very similar functional to those in this inequality. We quote a relevant result on the minimal values of such functionals from [3, Section 6.2].

Lemma 3.1 *For $\gamma > 0$, let $I_\gamma(u) := \int_{\mathbb{R}} [\gamma (u'(\tau))^2 + K(1 + \cos(u(\tau)))] d\tau$. Then the minimum of I_γ on $\mathcal{M}_{-\pi,\pi}$ is attained and*

$$\min_{u \in \mathcal{M}_{-\pi,\pi}} I_\gamma(u) = \vartheta := 2\sqrt{\gamma K} \int_{-\pi}^{\pi} \sqrt{1 + \cos(\xi)} d\xi. \quad (12)$$

Moreover, with the same ϑ ,

$$\inf_{T>0} \inf \left\{ \int_{-T}^T [\gamma (u')^2 + K(1 + \cos(u))] d\tau : \begin{array}{l} u \in H^1(-T, T), \\ u(-T) = -\pi, u(T) = \pi \end{array} \right\} = \vartheta. \quad (13)$$

As an immediate consequence we get, by evaluating the integral in (12),

$$8\sqrt{(c^2 - c_0^2)K} \leq \inf_{u \in \mathcal{M}_{-\pi,\pi}} J(u) \leq 8c\sqrt{K}. \quad (14)$$

This inequality and (13) will serve as basis for the L^∞ -a-priori bound in the next lemma.

Lemma 3.2 *Let $c^2 > c_0^2$. A global minimiser u_0 of J on $\mathcal{M}_{-\pi,\pi}$ satisfies*

$$\|u_0\|_{L^\infty(\mathbb{R})} < (2k + 3)\pi,$$

$$\text{where } k := \max \left\{ \kappa \in \mathbb{N}_0 : (2\kappa + 1) \leq \sqrt{\frac{c^2}{c^2 - c_0^2}} \right\}.$$

PROOF. The proof relies on the fact that (13) and (14) provide, loosely speaking, an estimate for the “cost” for u_0 to traverse a height of 2π from one minimum of $\cos(\cdot)$ to the next. More precisely, we have

$$J(u_0) \geq I_{\frac{1}{2}(c^2 - c_0^2)}(u_0) := \int_{\mathbb{R}} \left[\frac{c^2 - c_0^2}{2} [u_0'(\tau)]^2 + K [1 + \cos(u_0(\tau))] \right] d\tau. \quad (15)$$

Hence, (14) and $1 + \cos(u) \geq 0$ show

$$\frac{c^2 - c_0^2}{2} \|u_0'\|_{L^2(\mathbb{R})}^2 \leq J(u_0) \leq 8c\sqrt{K}. \quad (16)$$

Let $T_1, T_2 \in \mathbb{R} \cup \{\pm\infty\}$ with $T_1 < T_2$ be such that $u_0(T_1) = \nu\pi$ and $u_0(T_2) = (\nu + 2)\pi$ for some odd integer ν . Then the contribution of the interval $[T_1, T_2]$ to the value $I_{\frac{1}{2}(c^2 - c_0^2)}(u_0)$ is, from (13),

$$\begin{aligned} & \int_{T_1}^{T_2} \left[\frac{c^2 - c_0^2}{2} [u_0'(\tau)]^2 + K [1 + \cos(u_0(\tau))] \right] d\tau \\ & \geq \inf_{\substack{u \in H_{\text{loc}}^1(T_1, T_2) \cap C^0[T_1, T_2], \\ u(T_1) = \nu\pi, u(T_2) = (\nu + 2)\pi}} \int_{T_1}^{T_2} \left[\frac{c^2 - c_0^2}{2} [u'(\tau)]^2 + K [1 + \cos(u(\tau))] \right] d\tau \\ & \geq 8\sqrt{(c^2 - c_0^2)K}. \end{aligned} \quad (17)$$

The boundary conditions (5) imply that the height 2π needs to be covered; any further increase in height of 2π has to be compensated by a decrease in height of 2π and vice versa. Hence there is an odd number of such increases or decreases. We write this odd number as $2\kappa + 1$ with $\kappa \in \mathbb{N}$, so that $(\kappa + 1) \cdot 2\pi \leq |\{u_0(z) : z \in \mathbb{R}\}| < (\kappa + 2) \cdot 2\pi$; thus κ can be understood as the number of times that u_0 grows by full 2π in excess to the one time required by $u_0 \in \mathcal{M}_{-\pi,\pi}$. Then (15) and (17) show that

$$J(u_0) \geq I_{\frac{1}{2}(c^2 - c_0^2)}(u_0) \geq 8(2\kappa + 1)\sqrt{(c^2 - c_0^2)K}.$$

On the other hand, $J(u_0)$ is bounded by the Modica-Mortola bound (14). Therefore, using (16),

$$8(2\kappa + 1)\sqrt{(c^2 - c_0^2)K} \leq J(u_0) \leq 8c\sqrt{K} \quad (18)$$

so that

$$\kappa \leq k := \max \left\{ \kappa \in \mathbb{N}_0 : (2\kappa + 1) \leq \sqrt{\frac{c^2}{c^2 - c_0^2}} \right\}.$$

Hence $(k + 1) \cdot 2\pi \leq |\{u_0(z) : z \in \mathbb{R}\}| < (k + 2) \cdot 2\pi$. Due to (5), $(-\pi, \pi) \subseteq \{u_0(z) : z \in \mathbb{R}\}$, so

$$\sup_{\tau \in \mathbb{R}} |u_0(\tau)| < (k + 2) \cdot 2\pi - \pi = (2k + 3)\pi;$$

note in particular that the inequality is strict. □

4 Concentration-compactness

The next step is to prove a variant of the concentration-compactness lemma of P.-L. Lions [8, Lemma I.1] that is adapted to our situation.

The setting in this classical paper [8] (see also [5]) is as follows. The general problem is to minimise a functional $E: U \rightarrow \mathbb{R}$ on a Banach space U subject to a constraint $K(u) = \lambda > 0$. It is shown that, for fixed λ , that any minimising sequence is, up to a subsequence, either relatively compact, or vanishes, or splits into two or more parts which drift away infinitely from each other. Vanishing can usually be excluded quite easily. Setting

$$I_\lambda := \inf\{E(u) : u \in U, K(u) = \lambda\},$$

splitting cannot occur, heuristically speaking, if and only if

$$I_\lambda < I_\alpha + I_{\lambda-\alpha} \text{ for all } \alpha \in (0, \lambda).$$

In comparison to the classical setting, a major difference in the present paper is that the constraint $u(\pm\infty) = \pm\pi$ cannot be varied continuously. Hence it is impossible to consider the minimum value of the functional on level sets of the constraint as a function of a continuous parameter in the constraint. As a consequence of this, no meaningful analog to the above subadditivity inequality can be formulated. Instead, we will exclude splitting by means of the *a priori* bound from Lemma 3.2.

The most important difference in contrast to other variants of the concentration-compactness lemma is therefore in the alternative of splitting. The value of the functional J is split up between sequences $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}}$ (whose sum is essentially the original sequence $(u_n)_{n \in \mathbb{N}}$)—not the value of the constraint, as usual. On the other hand, the present lemma holds not just for minimising sequences $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{-\pi, \pi}$, but for all sequences for which the values of the functional converge.

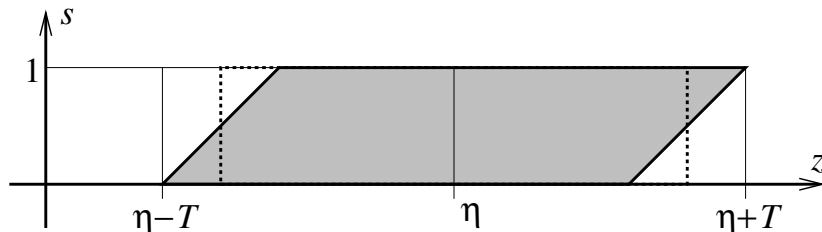


Figure 1: Domain of integration of $\int_0^1 \int_{\eta-T+s}^{\eta+T-1+s} [u'(\tau)]^2 d\tau ds$, which is up to a multiplicative factor the first term in the definition of $J_T(\cdot; \eta)$ in (19).

We introduce a truncated version of J . For parameters $T > 1$ and $\eta \in \mathbb{R}$, set

$$J_T(u; \eta) := \int_0^1 \int_{\eta-T+s}^{\eta+T-1+s} \frac{c^2}{2} [u'(\tau)]^2 \, d\tau \, ds - \int_{\eta-T}^{\eta+T-1} \frac{c_0^2}{2} [Au(\tau)]^2 \, d\tau + \int_{\eta-T+\frac{1}{2}}^{\eta+T-\frac{1}{2}} K [1 + \cos(u(\tau))] \, d\tau \quad (19)$$

We point out that all integrals are taken over symmetric intervals around η which simplifies some estimates later in this proof. The fact that the last integral is over $[\eta - T + \frac{1}{2}, \eta + T - \frac{1}{2}]$ rather than $[\eta - T, \eta + T]$ is not very important, but will slightly ease technicalities in the proof of Lemma 4.1.

The domain of integration for the first term in the definition of J_T can be thought of as the shaded parallelogram shown in Figure 1, with the integrand being constant on vertical lines. This choice is motivated by the second term because

$$Au_n(\tau) = \int_{\tau}^{\tau+1} u'(s) \, ds$$

shows that, roughly speaking, the second term could be interpreted as an integration over the same domain. This idea has already been suggested by (11).

The following proof will be formulated using symmetrised differences $u(\tau + \frac{1}{2}) - u(\tau - \frac{1}{2})$, rather than $u(\tau + 1) - u(\tau)$, in order to exploit the symmetry of the integration domains. It is immediate that the minimisation problem remains unchanged by this shift in the integration variable.

Lemma 4.1 (Concentration-compactness) *Let $c^2 > c_0^2$ and $\theta \geq \inf J(u)|_{\mathcal{M}_{-\pi, \pi}}$, $\theta < \infty$. Then every sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{-\pi, \pi}$ with the property*

$$\lim_{n \rightarrow \infty} J(u_n) = \theta \quad (20)$$

possesses a subsequence, not relabelled and still denoted by $(u_n)_{n \in \mathbb{N}}$, which satisfies one of the following three alternatives:

- (i) **Tightness:** *There is a sequence $(\eta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that, for every $\varepsilon > 0$, there exists $T_0 > 0$ such that for all $T > T_0$*

$$J(u_n) - J_T(u_n; \eta_n) < \varepsilon \text{ for every } n \in \mathbb{N}.$$

- (ii) **Vanishing:** *For all $T > 0$,*

$$\lim_{n \rightarrow \infty} \sup_{\eta \in \mathbb{R}} J_T(u_n; \eta) = 0. \quad (21)$$

- (iii) **Splitting:** *There exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$, there are $f_n, g_n \in X$ such that*

$$\begin{aligned} |u_n - (f_n + g_n - \pi)| &\leq \varepsilon \\ |J(u_n) - (J(f_n) + J(g_n))| &\leq \varepsilon, \quad \lim_{n \rightarrow \infty} \text{dist}(\text{supp}(f'_n), \text{supp}(g'_n)) = \infty, \\ \lim_{n \rightarrow \infty} J(f_n) &= \alpha, \quad \lim_{n \rightarrow \infty} J(g_n) = \beta, \end{aligned}$$

for some $0 < \alpha, \beta < \theta$. (π is needed in the first inequality to ensure $J(f_n) < \infty$ and $J(g_n) < \infty$.)

The condition (20) is in particular satisfied if $(u_n)_{n \in \mathbb{N}}$ is a minimising sequence for J . We actually need Prop. 4.1 only for that case, in which $\theta = \inf J(u)|_{\mathcal{M}_{-\pi, \pi}}$.
PROOF. The proof is given in four steps. First we introduce a concentration functional, discuss its properties (Step 1). The rest is concerned with the proof that the only alternative to cases (i) and (ii) is case (iii). Step 2 identifies the intervals which will become the support of f_n and g_n , respectively. Further estimates show the statements about the sequences f_n (Step 3) and g_n (Step 4).

Step 1. As in Lions' proof [8], a concentration function is introduced. Namely, given a sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{-\pi, \pi}$ with (20) and a parameter $\eta \in \mathbb{R}$, define a sequence of functions $P_n(\cdot; \eta): (0, \infty) \rightarrow \mathbb{R}$,

$$\begin{aligned} P_n(T; \eta) &:= J_{T+\frac{1}{2}}(u_n; \eta) \quad (\text{compare Figure 1}) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\eta-T+s}^{\eta+T+s} \frac{c^2}{2} [u'_n(\tau)]^2 \, d\tau \, ds \\ &\quad - \int_{\eta-T}^{\eta+T} \frac{c_0^2}{2} \left[u_n\left(\tau + \frac{1}{2}\right) - u_n\left(\tau - \frac{1}{2}\right) \right]^2 \, d\tau + \int_{\eta-T}^{\eta+T} K [1 + \cos(u_n(\tau))] \, d\tau; \end{aligned} \quad (22)$$

the shift by $\frac{1}{2}$ from the definition of J_T saves some summands $\frac{1}{2}$ in subsequent estimates of this proof while the form of J_T is more useful in Section 5 below.

Note that, for every fixed $n \in \mathbb{N}$ and $\eta \in \mathbb{R}$, P_n is nondecreasing in T . Namely, for all $\varepsilon > 0$ and all $\eta \in \mathbb{R}$, (11) and $c > c_0$ show that the increment of the second integral in (22) is bounded by the increment of the first one,

$$\begin{aligned} \frac{c_0^2}{2} \int_{\eta+T}^{\eta+T+\varepsilon} \left[u_n\left(\tau + \frac{1}{2}\right) - u_n\left(\tau - \frac{1}{2}\right) \right]^2 \, d\tau &= \frac{c_0^2}{2} \|Au_n\|_{(\eta+T-\frac{1}{2}, \eta+T-\frac{1}{2}+\varepsilon)}^2 \\ &\leq \frac{c^2}{2} \int_0^1 \int_{\eta+T-\frac{1}{2}+s}^{\eta+T-\frac{1}{2}+\varepsilon+s} [u'_n(\tau)]^2 \, d\tau \, ds = \frac{c^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\eta+T+s}^{\eta+T+\varepsilon+s} [u'_n(\tau)]^2 \, d\tau \, ds, \end{aligned} \quad (23)$$

and the very same estimate holds on $(\eta - T - \varepsilon, \eta - T)$. This implies $P_n(T + \varepsilon; \eta) \geq P_n(T; \eta)$ for all $T, \varepsilon > 0$ because $1 + \cos(u_n)$ is always non-negative.

Now we can define for each $n \in \mathbb{N}$ the concentration function

$$Q_n(T) := \sup_{\eta \in \mathbb{R}} P_n(T; \eta). \quad (24)$$

As supremum of monotone and nondecreasing functions, Q_n enjoys the same properties. It is clear that Q_n is bounded on $(0, \infty)$ because, for each $n \in \mathbb{N}$,

$$\lim_{T \rightarrow \infty} Q_n(T) = J(u_n).$$

By assumption (20), $(J(u_n))_{n \in \mathbb{N}}$ converges to θ and is therefore *a fortiori* bounded in \mathbb{R} ; thus the sequence $(Q_n)_{n \in \mathbb{N}}$ is bounded from above in $L^\infty(0, \infty)$. Hence,

by Helly's selection theorem (see, e.g., [12, Section 17.4]), a subsequence, not relabelled, converges pointwise almost everywhere to a monotone nondecreasing function $Q: (0, \infty) \rightarrow \mathbb{R}$ and

$$l := \lim_{T \rightarrow \infty} Q(T) \in [0, \theta]. \quad (25)$$

Obviously, alternative (i) in the statement occurs for $l = \theta$, and alternative (ii), vanishing, occurs when $l = 0$. What remains is to show that $0 < l < \theta$ corresponds to alternative (iii), splitting.

Step 2. Let $\varepsilon > 0$. By definition of l in (25), there exists $T_0 \in \mathbb{R}$ such that $Q(T_0) \geq l - \frac{1}{3}\varepsilon$. Since $Q_n(T) \rightarrow Q(T)$ as $n \rightarrow \infty$ for almost every T , we may assume, possibly after increasing T_0 , that $Q_n(T_0) \rightarrow Q(T_0)$. Thus, $Q_n(T_0) \geq l - \frac{2}{3}\varepsilon$, if we consider only large enough n . The definition (24) of Q_n implies that we can find $\eta_n \in \mathbb{R}$ such that for all large enough n

$$P_n(T_0; \eta_n) \geq l - \varepsilon.$$

We can also find a sequence $(T_n)_{n \in \mathbb{N}}$ with $T_n \rightarrow \infty$ as $n \rightarrow \infty$ (and in particular $T_n \gg T_0$ for all $n \in \mathbb{N}$) such that $Q_n(T_n) \leq l + \varepsilon$; this follows from the facts that $Q_n(T) \rightarrow Q(T)$ as $n \rightarrow \infty$ for almost every T , and that $Q(T) \rightarrow l$ as $T \rightarrow \infty$, see (25). Since Q_n has been defined as supremum over P_n in (24), the sequence $(T_n)_{n \in \mathbb{N}}$ satisfies $P_n(T_n; \eta_n) \leq l + \varepsilon$. As P_n is monotone and nondecreasing in T for each $n \in \mathbb{N}$,

$$|P_n(T; \eta_n) - l| \leq \varepsilon \text{ for all } T \in [T_0, T_n]. \quad (26)$$

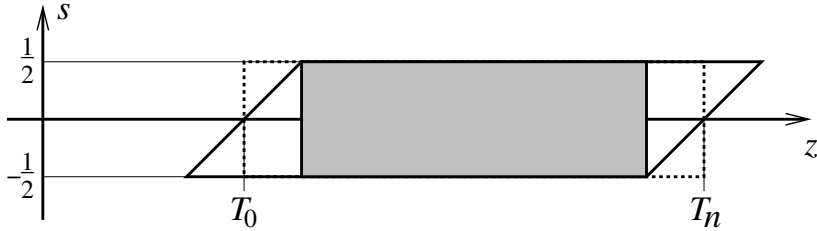


Figure 2: Domains of integration of the first term (parallelogram) and the second term (dashed rectangle) in (28) and of the integral in the following line (shaded rectangle). To prevent a complicated labelling of axes, the situation has been sketched for $\eta_n = 0$.

Now we are going to analyse the behaviour of $u_n(z)$ for $|z - \eta_n| \in [T_0, T_n]$; the goal is to show that there exist $k_n^\pm \in \mathbb{Z}$ such that

$$|u_n(z) - (2k_n^\pm + 1)\pi| \leq \delta(\varepsilon) \text{ whenever } (z \mp \eta_n) \in [T_0 + \frac{1}{2}, T_n - \frac{1}{2}]. \quad (27)$$

Starting with (26) and considering first only the interval $[\eta_n + T_0, \eta_n + T_n]$, we find, first using (23) and then shrinking the domain of integration (compare Figure 2),

$$\begin{aligned} 2\varepsilon &\geq P_n(T_n; \eta_n) - P_n(T_0; \eta_n) \\ &\geq \frac{c^2 - c_0^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\eta_n + T_0 + s}^{\eta_n + T_n + s} [u'_n(\tau)]^2 d\tau ds + K \int_{\eta_n + T_0}^{\eta_n + T_n} [1 + \cos u_n(\tau)] d\tau \end{aligned} \quad (28)$$

and, for any $T', T'' \in \mathbb{R}$ with $T_0 + \frac{1}{2} \leq T' < T'' \leq T_n - \frac{1}{2}$, by shrinking the domain of integration again,

$$\geq \int_{\eta_n + T'}^{\eta_n + T''} \left[\frac{c^2 - c_0^2}{2} [u'_n(\tau)]^2 + K [1 + \cos(u_n(\tau))] \right] d\tau,$$

and using the trivial estimate $x^2 + y^2 \geq 2xy$ and the change of variables $\xi = u_n(\tau)$,

$$\begin{aligned} &\geq 2 \int_{\eta_n + T'}^{\eta_n + T''} \left[\sqrt{\frac{c^2 - c_0^2}{2}} |u'_n(\tau)| \cdot \sqrt{K(1 + \cos(u_n(\tau)))} \right] d\tau \\ &= \sqrt{2K(c^2 - c_0^2)} \left| \int_{u_n(\eta_n + T')}^{u_n(\eta_n + T'')} \sqrt{1 + \cos(\xi)} d\xi \right|. \end{aligned}$$

This shows that $|u_n(\eta_n + T') - u_n(\eta_n + T'')| \leq \delta_1$, where $\delta_1 = \delta_1(\varepsilon)$ is given by the relation

$$\int_{\pi - \frac{\delta_1}{2}}^{\pi + \frac{\delta_1}{2}} \sqrt{1 + \cos(x)} dx = \frac{2\varepsilon}{\sqrt{2K(c^2 - c_0^2)}}.$$

To see that this interval $\{u_n(\tau) : \tau \in [\eta_n + T_0 + \frac{1}{2}, \eta_n + T_n - \frac{1}{2}]\}$ of length $\leq \delta_1$ is near an element of $\{(2k+1)\pi : k \in \mathbb{Z}\}$, observe that (28) implies also

$$2\varepsilon \geq K(T_n - T_0 - 1) \cdot \min_{\tau \in [\eta_n + T_0 + \frac{1}{2}, \eta_n + T_n - \frac{1}{2}]} [1 + \cos(u_n(\tau))].$$

Suppose this minimum is attained at $\tau_{0,n}$. Then $\frac{2\varepsilon}{K(T_n - T_0 - 1)} \geq 1 + \cos(u_n(\tau_{0,n}))$. As we may assume $\varepsilon \ll 1$ and $T_n \gg T_0$, there exists a k_n^+ such that

$$|u_n(\tau_{0,n}) - (2k_n^+ + 1)\pi| \leq \pi - \arccos\left(\frac{2\varepsilon}{K(T_n - T_0 - 1)} - 1\right) =: \delta_2$$

Hence, with $\delta = \delta(\varepsilon) = \delta_1 + \delta_2$,

$$|u_n(\tau) - (2k_n^+ + 1)\pi| \leq \delta \quad \text{for all } \tau \in [\eta_n + T_0 + \frac{1}{2}, \eta_n + T_n - \frac{1}{2}],$$

and $\delta \rightarrow 0$ for $\varepsilon \rightarrow 0$. To establish (27), it suffices to observe that the same argument yields k_n^- with the corresponding property

$$|u_n(\tau) - (2k_n^- + 1)\pi| \leq \delta \quad \text{for all } \tau \in [\eta_n - T_n + \frac{1}{2}, \eta_n - T_0 - \frac{1}{2}].$$

Step 3. Define

$$f_n(\tau) := \begin{cases} (2k_n^- + 1)\pi & \text{for } \tau < -T_0 - 2, \\ u_n(\eta_n + \tau) & \text{for } \tau \in [-T_0 - 1, T_0 + 1], \\ (2k_n^+ + 1)\pi & \text{for } \tau > T_0 + 2 \end{cases}$$

and interpolate linearly on $[-T_0 - 2, -T_0 - 1]$ and $[T_0 + 1, T_0 + 2]$. In analogy to P_n , we introduce, replacing u_n by f_n in (22), $\tilde{P}_n: (0, \infty) \rightarrow \mathbb{R}$ by

$$\begin{aligned} \tilde{P}_n(T) := & \frac{c^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-T+s}^{T+s} [f'_n(\tau)]^2 d\tau ds - \frac{c_0^2}{2} \int_{-T}^T [f_n(\tau + \frac{1}{2}) - f_n(\tau - \frac{1}{2})]^2 d\tau \\ & + K \int_{-T}^T [1 + \cos(f_n(\tau))] d\tau. \end{aligned} \quad (29)$$

For $|\tau| > T_0 + \frac{5}{2}$, each of the integrands vanishes because f_n is on $\{\tau \in \mathbb{R} : |\tau| > T_0 + 2\}$ by definition constant and equal to an odd multiple of π . Therefore

$$\tilde{P}_n(T_0 + \frac{5}{2}) = \tilde{P}_n(T) = \tilde{P}_n(\infty) = J(f_n) \text{ for all } T > T_0 + \frac{5}{2}. \quad (30)$$

The goal is now to show that, up to a subsequence, $J(f_n) \rightarrow \alpha \in (0, \theta)$ for $n \rightarrow \infty$. To do so, we are going to estimate, with l from (25),

$$\left| \tilde{P}_n(T_n) - l \right| = \left| \left[\tilde{P}_n(T_n) - P_n(T_n; \eta_n) \right] - [P_n(T_n; \eta_n) - l] \right|$$

in terms of ε . We know already $|P_n(T_n; \eta_n) - l| \leq \varepsilon$ from (26), so we are left with $\left| \tilde{P}_n(T_n) - P_n(T_n; \eta_n) \right|$. Note that $P_n(T_0 + \frac{1}{2}; \eta_n) = \tilde{P}_n(T_0 + \frac{1}{2})$ because, by definition, $f_n(\tau) = u_n(\eta_n + \tau)$ for all $|\tau| \leq T_0 + 1$. Thus the triangle inequality yields

$$\left| P_n(T_n; \eta_n) - \tilde{P}_n(T_n) \right| \leq \left| P_n(T_n; \eta_n) - P_n(T_0 + \frac{1}{2}; \eta_n) \right| + \left| \tilde{P}_n(T_0 + \frac{1}{2}) - \tilde{P}_n(T_n) \right|. \quad (31)$$

It follows from (26) that the first term on the right-hand side of (31) can be estimated as

$$\left| P_n(T_n; \eta_n) - P_n(T_0 + \frac{1}{2}; \eta_n) \right| \leq 2\varepsilon. \quad (32)$$

As for the second term on the right-hand side of (31), observe that the domains of integration in $\tilde{P}_n(T_0 + \frac{1}{2})$ and $\tilde{P}_n(T_n)$ overlap so that, with $\mathcal{I}_n := (-T_n, -T_0 - \frac{1}{2}) \cup (T_0 + \frac{1}{2}, T_n)$,

$$\begin{aligned} \left| \tilde{P}_n(T_0 + \frac{1}{2}) - \tilde{P}_n(T_n) \right| &= \frac{c^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\substack{(-T_n+s, -T_0-\frac{1}{2}+s) \\ \cup (T_0+\frac{1}{2}+s, T_n+s)}} [f'_n(\tau)]^2 d\tau ds \\ &\quad - \frac{c_0^2}{2} \int_{\mathcal{I}_n} [f_n(\tau + \frac{1}{2}) - f_n(\tau - \frac{1}{2})]^2 d\tau + K \int_{\mathcal{I}_n} [1 + \cos(f_n(\tau))] d\tau \\ &\leq \frac{c^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\substack{(-T_n+s, -T_0-\frac{1}{2}+s) \\ \cup (T_0+\frac{1}{2}+s, T_n+s)}} [f'_n(\tau)]^2 d\tau ds + K \int_{\mathcal{I}_n} [1 + \cos(f_n(\tau))] d\tau \end{aligned} \quad (33)$$

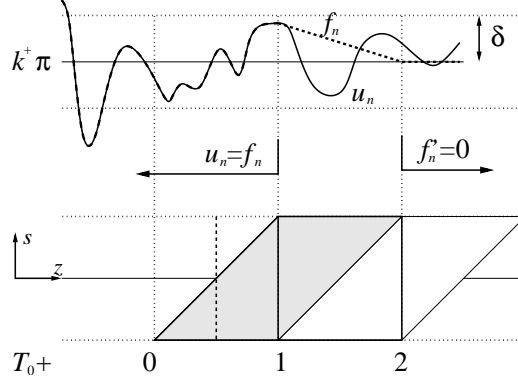


Figure 3: Illustration of the deduction of (34). The equality $u_n = f_n$ is to be understood modulo a translation in the argument of the two functions as described in the text.

because $[f_n(\tau + \frac{1}{2}) - f_n(\tau - \frac{1}{2})]^2 \geq 0$. We estimate the terms first on the intervals which lie in \mathbb{R}^+ , and start with the first summand of (33).

Using $f'(\tau) = 0$ for $\tau > T_0 + 2$, we obtain

$$\frac{c^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{T_0 + \frac{1}{2} + s}^{T_n + s} [f'_n(\tau)]^2 d\tau ds = \frac{c^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{T_0 + \frac{1}{2} + s}^{T_0 + \frac{5}{2} + s} [f'_n(\tau)]^2 d\tau ds$$

(the domain of integration is here the largest parallelogram shown in Figure 3) and, splitting the integral at $z = T_0 + 1$ and employing $f'_n = u'_n(\eta_n + \cdot)$ on $(T_0, T_0 + 1)$ (this is used on the left half of the shaded parallelogram in Figure 3) and $f'_n = 0$ on $(T_0 + 2, T_0 + 3)$ (thus no contribution comes from the rightmost triangle in Figure 3)

$$= \frac{c^2}{2} \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{T_0 + \frac{1}{2} + s}^{T_0 + 1} [u'_n(\eta_n + \tau)]^2 d\tau ds + \int_{T_0 + 1}^{T_0 + 2} [f'_n(\tau)]^2 d\tau \right]$$

and, extending the first integral to the whole shaded parallelogram of Figure 3, while evaluating the second integral, in which $|f'| \leq \delta$ a.e. due to (27),

$$\leq \frac{c^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\eta_n + T_0 + \frac{1}{2} + s}^{\eta_n + T_0 + \frac{3}{2} + s} [u'_n(\tau)]^2 d\tau ds + \frac{c^2}{2} \cdot 1 \cdot \delta^2$$

and, bounding the remaining integral by $\frac{2}{c^2 - c_0^2} [P_n(T_0 + \frac{3}{2}; \eta_n) - P_n(T_0 + \frac{1}{2}; \eta_n)]$ by (22) and (23),

$$\leq \frac{c^2}{c^2 - c_0^2} [P_n(T_0 + \frac{3}{2}; \eta_n) - P_n(T_0 + \frac{1}{2}; \eta_n)] + \frac{1}{2} \cdot c^2 \cdot \delta^2$$

and finally, using the monotonicity of P_n and (26),

$$\leq \frac{c^2}{c^2 - c_0^2} \cdot 2\varepsilon + \frac{1}{2} \cdot c^2 \cdot \delta^2 = c^2 \left(\frac{2\varepsilon}{c^2 - c_0^2} + \frac{\delta^2}{2} \right). \quad (34)$$

We continue estimating the right-hand side of (33) on $(T_0 + \frac{1}{2}, T_n)$ and find for the remaining third integral, from (27),

$$\left| K \int_{T_0 + \frac{1}{2}}^{T_n} [1 + \cos(f_n(\tau))] \, d\tau \right| = K \int_{T_0 + \frac{1}{2}}^{T_0 + 2} [1 + \cos(f_n(\tau))] \, d\tau \leq \frac{3}{2} K (1 - \cos(\delta)). \quad (35)$$

The very same estimates hold for the interval $(-T_n, -T_0 - \frac{1}{2})$. Thus when combining (33), (34) and (35), we obtain

$$\left| \tilde{P}_n(T_n) - \tilde{P}_n(T_0 + \frac{1}{2}) \right| \leq c^2 \left(\frac{4\varepsilon}{c^2 - c_0^2} + \delta^2 \right) + 3K(1 - \cos \delta). \quad (36)$$

Therefore, using $\tilde{P}_n(T_n) = J(f_n)$ (from (30)) and inserting (32) and (36) into (31), we obtain

$$\begin{aligned} |P_n(T_n; \eta_n) - J(f_n)| &= \left| P_n(T_n; \eta_n) - \tilde{P}_n(T_n) \right| \\ &\leq 2\varepsilon + c^2 \left(\frac{4\varepsilon}{c^2 - c_0^2} + \delta^2 \right) + 3K(1 - \cos \delta) =: \tilde{\varepsilon}, \end{aligned} \quad (37)$$

hence, with (26),

$$|J(f_n) - l| \leq |J(f_n) - P_n(T_n; \eta_n)| + |P_n(T_n; \eta_n) - l| \leq \tilde{\varepsilon} + \varepsilon.$$

Now choose $\varepsilon_0 > 0$ such that

$$l + 2(\tilde{\varepsilon} + \varepsilon) < \theta \quad \text{and} \quad l - 2(\tilde{\varepsilon} + \varepsilon) > 0 \quad \text{for all} \quad \varepsilon < \varepsilon_0; \quad (38)$$

this is possible because $\delta = \delta(\varepsilon) \rightarrow 0$ for $\varepsilon \rightarrow 0$ while c , c_0 and K are constants, and $0 < l < \theta$ by assumption. We will from now on assume that $\varepsilon < \varepsilon_0$.

With this choice, we find that there exists a subsequence (not relabelled) of $(f_n)_{n \in \mathbb{N}}$ for which the limit of $J(f_n)$ for $n \rightarrow \infty$ exists, and

$$\lim_{n \rightarrow \infty} J(f_n) = \alpha,$$

for some $\alpha \in \mathbb{R}$ with $|l - \alpha| \leq \tilde{\varepsilon}$ and, by choice of ε_0 , $0 < \alpha < \theta$.

Step 4. Introduce now, with k^\pm as in (27),

$$g_n(\tau) := \begin{cases} u_n(\eta_n + \tau) - 2k_n^- \pi & \text{for } \tau < -T_n + 1, \\ \pi & \text{for } \tau \in [-T_n + 2, T_n - 2], \\ u_n(\eta_n + \tau) - 2k_n^+ \pi & \text{for } \tau > T_n - 1, \end{cases}$$

and interpolate linearly on $[-T_n + 1, -T_n + 2]$ and $[T_n - 2, T_n - 1]$. Clearly

$$\text{dist}(\text{supp}(f'_n), \text{supp}(g'_n)) \geq (T_n - T_0 - 4) \rightarrow \infty$$

for $n \rightarrow \infty$. By definition of f_n and g_n ,

$$u_n(\eta_n + \tau) = f_n(\tau) + g_n(\tau) - \pi \quad \text{for } |\tau| \in [0, T_0 + 1] \cup [T_n - 1, \infty),$$

and for $\pm\tau \in [T_0 + 1, T_n - 1]$ we have $f_n(\tau) + g_n(\tau) - \pi = (2k^\pm + 1)\pi$, while (27) shows $|u_n - (2k^\pm + 1)\pi| \leq 2\varepsilon$.

We are now going to estimate $|J(u_n) - J(f_n) - J(g_n)|$. The last statement to be shown, $J(g_n) \rightarrow \beta \in (0, \theta)$ for $n \rightarrow \infty$, will then be an easy consequence of it. By the triangle inequality, (26) and (37),

$$\begin{aligned} |J(u_n) - J(f_n) - J(g_n)| &\leq |J(u_n) - J(g_n) - P_n(T_n - 1; \eta_n)| \\ &\quad + |P_n(T_n - 1; \eta_n) - P_n(T_n; \eta_n)| \\ &\quad + |P_n(T_n; \eta_n) - J(f_n)| \\ &\leq |J(u_n) - J(g_n) - P_n(T_n - 1; \eta_n)| + 2\varepsilon + \tilde{\varepsilon}. \end{aligned} \quad (39)$$

In a very similar manner to (33)–(36) (see also [6]), it is possible to show in analogy to (37) that

$$|J(u_n) - P(T_0; \eta_n) - J(g_n)| \leq c^2 \left(\frac{4\varepsilon}{c^2 - c_0^2} + \delta^2 \right) + 3K(1 - \cos \delta) = \tilde{\varepsilon} - 2\varepsilon, \quad (40)$$

with $\tilde{\varepsilon}$ as in (37). This inequality implies together with (39)

$$|J(u_n) - J(f_n) - J(g_n)| \leq \tilde{\varepsilon} - 2\varepsilon + 2\varepsilon + \tilde{\varepsilon} = 2\tilde{\varepsilon},$$

thus for sufficiently large n

$$|J(g_n) - (\theta - \alpha)| \leq 2\tilde{\varepsilon} + \varepsilon,$$

and switching to a subsequence if necessary, we find that the limit

$$\lim J(g_n) =: \beta$$

exists, and that $0 < \beta < \theta$, by choice of ε_0 in (38). This finishes the proof. \square

5 Proof of the main result

We prove the existence of a heteroclinic wave by ruling out two of the three cases in the concentration-compactness Lemma 4.1, namely vanishing (possibility (ii)) and splitting (possibility (iii)).

Lemma 5.1 (No vanishing) *Let $c^2 > c_0^2$ and $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{-\pi, \pi}$ be a minimising sequence for J . Then $(u_n)_{n \in \mathbb{N}}$ does not vanish.*

PROOF. The argument is by contradiction. Suppose that (21) holds. Fix $T \gg 1$, $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that for all u_n with $n > N$

$$\sup_{\eta \in \mathbb{R}} \int_{\eta-T}^{\eta+T} \frac{c^2}{2} (u'_n(\tau))^2 \, d\tau - \int_{\eta-T}^{\eta+T-1} \frac{c_0^2}{2} (Au_n(\tau))^2 \, d\tau + \int_{\eta-T}^{\eta+T} K(1 + \cos(u_n(\tau))) \, d\tau \leq \varepsilon.$$

Observe that for supersonic waves the quantity on the left-hand side is always non-negative since the first term dominates the second one by (11), and the on-site potential $1 + \cos(\xi) \geq 0$ non-negative for every $\xi \in \mathbb{R}$. In particular,

$$\frac{c^2 - c_0^2}{2} \|u'\|_{L^2(-T, T)}^2 \leq \frac{c^2}{2} \|u'\|_{L^2(-T, T)}^2 - \frac{c_0^2}{2} \|Au\|_{L^2(-T, T-1)}^2 \leq \varepsilon$$

holds. We claim that this implies

$$|u(T) - u(-T)| \leq \sqrt{\frac{4\varepsilon}{c^2 - c_0^2}} =: \varepsilon_1. \quad (41)$$

To see this, consider the auxiliary variational problem

$$\text{Maximise } u(1), \text{ subject to } u(0) = 0, \|u'\|_{L^2(0,1)} = \alpha \quad (42)$$

on $H^1(0, 1)$. Since

$$|u(1)| = \left| \int_0^1 u'(s) \, ds \right| \leq \|u'\|_{L^1(0,1)} \leq \|u'\|_{L^2(0,1)},$$

where equality holds only for $u' = \text{const.}$, we conclude that the maximum must be attained by the linear functions $u(s) = \pm \alpha s$, which yields $u(1) = \alpha$ as maximum. The claim follows by choosing the appropriate scaling.

Since J is invariant under the transformation $\tau \mapsto \tau + \tau_0$ for arbitrary $\tau_0 \in \mathbb{R}$, we may assume without loss of generality that $u_n(0) = 0$ for all $n \in \mathbb{N}$. It follows

$$K \int_{-T}^T [1 + \cos(u(\tau))] \, d\tau \geq K (1 + \cos(\varepsilon_1)) \cdot 2T.$$

This, however, is a contradiction to

$$\sup_{\eta \in \mathbb{R}} \int_{\eta-T}^{\eta+T} K(1 + \cos(u(\tau))) \, d\tau < \varepsilon$$

for ε small enough. □

Lemma 5.2 (No splitting) *Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{-\pi, \pi}$ be a minimising sequence for J . Then, for every c with $c^2 > \frac{25}{24}c_0^2$, $(u_n)_{n \in \mathbb{N}}$ does not split in the sense of Lemma 4.1 (iii).*

PROOF. The condition $c^2 > \frac{25}{24}c_0^2$ ensures that $\sqrt{\frac{c^2}{c^2 - c_0^2}} < 5$, thus $k \in \{0, 1\}$ in the statement of Lemma 3.2, so any minimiser u_0 of $J|_{\mathcal{M}_{-\pi, \pi}}$ satisfies $\|u_0\|_{L^\infty(\mathbb{R})} < 5\pi$. By the same arguments as in the proof of Lemma 3.2, the same $L^\infty(\mathbb{R})$ -bound applies to the members of a minimising sequence (possibly after dropping finitely many of them); indeed, if $(J(u_n) - \inf J|_{\mathcal{M}_{-\pi, \pi}}) < \delta$ then, in analogy to (18),

$$8(2\kappa + 1)\sqrt{(c^2 - c_0^2)K} \leq J(u_0) \leq 8c\sqrt{K} + \delta,$$

so if $\sqrt{\frac{c^2}{c^2 - c_0^2}} < 5$, as guaranteed by assumption, then $\frac{8\sqrt{c^2 K} + \delta}{8\sqrt{(c^2 - c_0^2)K}} < 5$ for δ small enough.

Suppose now, for a contradiction, that $(u_n)_{n \in \mathbb{N}}$ splits. Let $\varepsilon > 0$ and choose, as in the proof of Lemma 4.1 (iii), f_n, g_n such that $|u_n - (f_n + g_n - \pi)| < \varepsilon$. Due to the L^∞ -bound and the boundary conditions $u_n(\infty) - u_n(-\infty) = 2\pi$, we have either

$$\begin{aligned} & \{f_n(\infty) - f_n(-\infty), g_n(\infty) - g_n(-\infty)\} = \{0, 2\pi\} \\ \text{or } & \{f_n(\infty) - f_n(-\infty), g_n(\infty) - g_n(-\infty)\} = \{4\pi, -2\pi\}. \end{aligned}$$

Thus, let \tilde{f}_n, \tilde{g}_n be defined by $\tilde{f}_n(\tau) := f_n(-\tau)$ and $\tilde{g}_n(\tau) := g_n(-\tau)$. Then for each $n \in \mathbb{N}$ one of the four functions f_n, g_n, \tilde{f}_n and \tilde{g}_n belongs to $\mathcal{M}_{-\pi, \pi}$. Let this function be denoted by \tilde{u}_n . As both

$$\lim_{n \rightarrow \infty} J(f_n) < \inf_{u \in \mathcal{M}_{-\pi, \pi}} J(u) \quad \text{and} \quad \lim_{n \rightarrow \infty} J(g_n) < \inf_{u \in \mathcal{M}_{-\pi, \pi}} J(u),$$

by Lemma 4.1 (iii), there exists a subsequence of $(\tilde{u}_n)_{n \in \mathbb{N}}$, not relabelled, such that

$$\lim_{n \rightarrow \infty} J(\tilde{u}_n) < \inf_{u \in \mathcal{M}_{-\pi, \pi}} J(u) = \lim_{n \rightarrow \infty} J(u_n),$$

a contradiction to the fact that $(u_n)_{n \in \mathbb{N}}$ is a minimising sequence in $\mathcal{M}_{-\pi, \pi}$. \square

Now we are in a position to prove the main theorem 2.2.

PROOF. The Modica-Mortola bound (14) implies that J is bounded from below on X . Let $(u_n)_{n \in \mathbb{N}}$ be a minimising sequence in $M_{-\pi, \pi}$. We know from Lemma 4.1 that $(u_n)_{n \in \mathbb{N}}$ has to concentrate, or vanish, or split. Lemma 5.1 rules out vanishing, and Lemma 5.2 rules out splitting for $c > \frac{25}{24}c_0^2$. Thus $(u_n)_{n \in \mathbb{N}}$ must concentrate. Hence, for fixed $\varepsilon > 0$, it is possible to choose a sequence $(\eta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ and $T_0 > 0$ such that

$$|J(u_n) - J_{T_0}(u_n; \eta_n)| < \varepsilon.$$

We write $w_n(\tau) := u_n(\eta_n + \tau)$. The sequence $(w_n)_{n \in \mathbb{N}}$ is bounded in X , since $\|w'_n\|_{L^2(\mathbb{R})} = \|u'_n\|_{L^2(\mathbb{R})} \leq \frac{2}{c^2 - c_0^2} J(u_n)$ and $|w(0)| < 5\pi$ by (16) and Lemma 3.2. As X is a Hilbert space, there exists a subsequence, not relabelled, which converges weakly. On the interval $[-T_0, T_0]$, the weak convergence of $(w_n)_{n \in \mathbb{N}}$ implies strong convergence in $L^2(-T_0, T_0)$ and $C^0[-T_0, T_0]$ to some limit u . Hence for all $n > N$, N sufficiently large,

$$\left| \int_{-T_0}^{T_0-1} [(Aw_n(\tau))^2 - (Au(\tau))^2] \, d\tau \right| < \varepsilon$$

and

$$\left| \int_{-T_0}^{T_0} [\cos(w_n(\tau)) - \cos u(\tau)] \, d\tau \right| < \varepsilon.$$

As the weak convergence implies $\|u'\|_{L^2(-T_0, T_0)}^2 \leq \liminf_{n \in \mathbb{N}} \|w'_n\|_{L^2(-T_0, T_0)}^2$, we may conclude $J_{T_0}(u) \leq \liminf_{n \in \mathbb{N}} J_{T_0}(w_n; 0)$.

We now extend the domain of u inductively to \mathbb{R} . Take any monotone sequence $T_k \rightarrow \infty$ with $k \in \mathbb{N}_0$ and assume that u has already been defined as uniform limit of $(w_n)_{n \in \mathbb{N}}$ on the interval $[-T_k, T_k]$. As $(w_n)_{n \in \mathbb{N}}$ is still bounded in X we can again choose a subsequence, not relabelled, which converges uniformly in $C^0[-T_{k+1}, T_{k+1}]$ to some limit \tilde{u} , which, by construction, coincides with u on $[-T_k, T_k]$.

It follows that this function u on \mathbb{R} satisfies the boundary conditions (5) and, with a constant $C = C(c, c_0, K)$,

$$\begin{aligned} J(u) &= \lim_{T \rightarrow \infty} J_T(u; 0) \leq \lim_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} J_T(w_n; 0) \\ &\leq \lim_{T \rightarrow \infty} \lim_{n \rightarrow \infty} J(w_n) + C\varepsilon = \lim_{n \rightarrow \infty} J(u_n) + C\varepsilon; \end{aligned}$$

in particular $u' \in L^2(\mathbb{R})$, thus $u \in M_{-\pi, \pi}$. As ε was arbitrary, the previous inequality shows

$$J(u) \leq \liminf_{n \rightarrow \infty} J(u_n).$$

This means that u is a minimiser of J on $M_{-\pi, \pi}$, and the theorem follows from the fact that (2) is the Euler-Lagrange equation of J (in the sense of Lemma 2.1). \square

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