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Abstract. We propose a phenomenological model for the evolutionary behaviour of shape memory alloys, where the possibility of plastic deformation is taken in to account. Two dissipative mechanisms are considered, namely the dissipation associated with solid-solid (martensitic) phase transformations, and plastic dissipation. The plastic contribution may lead to an irreversibility of the evolution. The existence of a so-called energetic solution is established for a suitable relaxation of the problem in the space of Young measures.

Key words. Elasto-plasticity, energetic solution, plastic strain gradients

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1 The mathematical model

Shape-memory alloys have been the focus of many investigations in the last decade. This interest can partially be attributed to the shape-memory effect itself (see Subsection 1.1), but even more the nonconvexity of the Helmholtz energy density due to the co-existence of several variants, which poses a significant mathematical challenge. It is remarkable that the inclusion of plastic effects in models for shape-memory alloys is a relatively new line of research, despite the significant influence on plastic effects of material properties. For example, cyclic plasticity may occur, which can negatively affect the performance of the material. This motivates the mathematical analysis of models of shape-memory materials accounting for plastic effects. Auricchio, Reali and Stefanelli have recently investigated a three-dimensional model for shape-memory alloys with inelastic effects [2]. We propose a different model, based on the theory of gradient plasticity advocated by, e.g., Dillon and Kratochvíl [5] and Gurtin [7].

The temporal evolution of the model presented here is rate-independent. The framework of energetic solutions is a suitable description of such an evolution; it is sketched in Subsection 1.3.

In the remainder of this section, we introduce the energetic approach to an elasto-plasticity problem of shape memory alloys. After a brief overview over shape memory alloys, microstructures and the associated non-(quasi-)convexity of the energy in Subsection 1.1, we describe in Subsection 1.2 the plasticity and the dissipation mechanisms employed in this paper. The mathematical formulation of the model is introduced in Section 2. The existence of a solution is proved via time-discretisation (Section 3).

1.1 Shape memory alloys

Shape-memory alloys (SMAs) are active materials, and have been the subject of intensive theoretical and experimental research during the past decades. Existing or potential applications can be found, for example, in medicine and mechanical or aerospace engineering. Shape-memory alloys are crystalline materials that exhibit specific a *hysteretic* stress / strain / temperature response; they have the ability to recover a trained shape after deformation and subsequent reheating. This is called the *shape-memory effect*. It is based on the ability of the shape-memory alloy to rearrange atoms in different crystallographic configurations (in particular, with different symmetry groups). The stability depends on the temperature. Normally, at higher temperatures a high-symmetry (for example, cubic) lattice is stable, which is referred to as the *austenite* phase. At lower temperatures, a lattice of lower symmetry (for example, tetragonal, orthorhombic, monoclinic, or triclinic) becomes stable, called the *martensite* phase. Due to the loss of symmetry, this phase may occur in different *variants*. The number of variants M, say, is the quotient of the order of the high-symmetry phase and the order of the low-symmetry group. So for a cubic high-symmetry phase, $M = 3$, 6, 12, or 4 for the tetragonal, orthorhombic, monoclinic, respectively triclinic martensites mentioned above. The variants can be combined coherently with each other, forming so-called *twins* of two variants. The resulting structure is then called a *laminate*.

The mathematical and computational modelling of SMAs represents a tool for the theoretical understanding of phase transition processes in solids. Such an analysis may complement experimental results, predict the response of new materials, or facilitate the usage of SMAs in applications. SMAs are genuine *multi-scale* materials and create a variety of challenges for mathematical modelling. We refer the reader to the literature [14] for a survey of a wide menagerie of SMA models ranging from nano- to macro-scales. In this article, we focus on a mesoscopic model in the framework of continuum mechanics. Beside the macroscopic deformation and its gradient, the model also involves the volume fractions of phases and variants and volume fraction gradients. This seems a fruitful compromise, since it allows for the modelling scales of large single crystals or polycrystals.

Let the specimen occupy a domain $\Omega \subset \mathbb{R}^n$. The stress-free parent austenite is a natural state of the material which makes it, in the context of continuum mechanics, a canonical choice for the reference configuration. As usual, $y\colon \Omega \to \mathbb{R}^n$ denotes the *deformation* and $u\colon \Omega \to \mathbb{R}^n$ the *displacement*, which are related to each other via the identity $y(x) = x + u(x)$, where $x \in \Omega$. Hence the *deformation gradient* is $F := \nabla y = \mathbb{I} + \nabla u$. Here, $\mathbb{I} \in \mathbb{R}^{n\times n}$ is the identity matrix and ∇ the gradient operator.

The total stored energy in the bulk occupying, in its reference configuration, the domain Ω is then

$$V(y) := \int_\Omega \varphi(\nabla y(x))\,\mathrm{d}x. \tag{1}$$

A common variational principle in continuum mechanics is the *minimisation of the stored energy*. Due to the coexistence of several variants at low temperature, φ has multiple minima and thus a multi-well character. We consider an isothermal situation with several variants coexisting. Since φ is a multi-well energy density, minimising sequences of V tend to develop, in general, finer and finer spatial oscillations of their gradients. In other words, the deformation gradient often tends to develop fine spatial oscillations due to lack of (quasi-)convexity of the stored energy density. These oscillations are difficult to model in full detail, although some studies in this direction exists [1]. The oscillations correspond to the development of finer and finer microstructures when the stored energy is to be minimised. The minimum of V, under specific boundary conditions for y, is usually not attained in a space of functions. Therefore one needs to extend the notion of a solution. *Young measures* are here an appropriate tool. They are capable of recording, on a mesoscopic level, the limit information of the finer and finer oscillating deformation gradient as we move towards the macroscopic scale. This can be described, for a current macroscopic point $x \in \Omega$, by a probability measure ν_x on the set of deformation gradients, that is, matrices in $\mathbb{R}^{n\times n}$. See Subsection 2.1 for a concise description of Young measures.

1.2 Plastic variables and dissipative mechanisms

In many situations, the austenite-martensite phase transformation is connected with plastic effects. In particular, cyclic plasticity may occur, which can negatively affect the performance of the material. Hence, mathematical models including both plasticity of shape-memory materials are needed. One such model is discussed here; see [2, 15] for related phenomenological models.

In order to include plasticity to the model, we assume that elastic properties of the material depend on plastic (internal) variables. In the setting described so far, the deformation y covers both the *elastic* and the *plastic* deformation. We employ the multiplicative split $F = F_\mathrm{e}F_\mathrm{p}$ of the deformation gradient into an elastic part F_e and an irreversible plastic part F_p. The latter belongs to $\mathrm{SL}(n) := \{A \in \mathbb{R}^{n\times n} \mid \det(A) = 1\}$. In addition to the so-called *plastic strain* F_p, we consider a vector $p \in \mathbb{R}^m$ of *hardening variables*. Both F_p and p are internal variables which influence the elasticity. It is to convenient to abbreviate $z = (F_\mathrm{p}, p)$. Furthermore, $\lambda\colon \Omega \to \mathbb{R}^{M+1}$ records the volume fraction of austenite and the M variants of martensite at a point $x \in \Omega$ (see [12]). As mentioned above, the stored energy density of shape memory materials is typically not quasiconvex, which explains why we consider a Young measure ν for the deformation gradient. For the moment, the intuitive interpretation of a Young measure $\nu = \{\nu_x\}_{x\in\Omega}$ as a recording device for the probability to find a phase or variant at the location $x \in \Omega$ suffices; Subsection 2.1 will make this intuition precise. In summary, we describe the state of the material by $q := (y, \nu, \lambda, F_\mathrm{p}, p) = (y, \nu, \lambda, z)$.

Following the approach of Dillon and Kratochvíl [5], Gurtin [7] and others, we work in the framework of so-called *gradient plasticity*, that is, ∇z enters the problem. As one can regard the volume fraction λ as an internal variable, too, it is natural to include a term involving $\nabla\lambda$ in the energetic contributions as well. This term also serves as a regularisation since it ensures compactness.

We consider two kinds of dissipation in our model, both of which are *rate-independent*. The first kind is related to the austenite–martensite transformation respectively the martensite–martensite transformation and will be characterised by the change of *volume fractions* in the composition of the material. The second kind is solely related to *plastic processes* in the material, e.g., to cyclic plasticity. To account for a possible irreversibility, the plastic dissipation may take the value $+\infty$, while the transformation dissipation is taken to be finite.

1.2.1 Dissipation originating in phase transitions

In order to describe dissipation due to transformations we adopt the (to some extent rather simplified) standpoint that the amount of dissipated energy associated with a particular phase transition between austenite and a martensitic variant or between two martensitic variants can be described by a specific energy (of the dimension $\mathrm{J/m^3} = \mathrm{Pa}$). This viewpoint has been independently adopted in physics, see [8, 17, 18]. For an explicit definition of the transformation dissipation, we need to identify the particular phases or phase variants. To this behalf, we define a continuous mapping $\mathcal{L}\colon \mathbb{R}^{n\times n} \to \triangle$, where

$$\triangle := \left\{ \zeta \in \mathbb{R}^{1+M} \;\middle|\; \zeta_\ell \geq 0 \text{ for } \ell = 0, \dots, M, \text{ and } \sum_{\ell=0}^{M} \zeta_\ell = 1 \right\}$$

is a simplex with $M+1$ vertices, with M being the number of martensitic variants. Here $\mathcal{L}$ is related with the material itself and thus has to be frame indifferent. We assume, beside $\zeta_\ell \geq 0$ and $\sum_{\ell=0}^{M} \zeta_\ell = 1$, that the coordinate ζ_ℓ of $\mathcal{L}(F)$ takes the value 1 if F is in the ℓ-th (phase) variant, that is, F is in a vicinity of ℓth well $\mathrm{SO}(n)U_\ell$ of φ, which can be identified by the stretch tensor $F^\top F$ being close to $U_\ell^\top U_\ell$. If $\mathcal{L}(F)$ is not in any vertex of $\triangle$, then it means that F in the spinodal region where no definite phase or variant is specified. We assume, however, that the wells are sufficiently deep and the phases and variants are geometrically sufficiently far from each other that the tendency for minimisation of the stored energy will essentially prevent F to range into the spinodal region. Thus, the concrete form of $\mathcal{L}$ is not important as long as $\mathcal{L}$ enjoys the properties listed above. We remark that $\mathcal{L}$ plays the rôle of what is often called vector of *order parameters* or a vector-valued *internal variable*.

For two states q_1 and q_2, with $q_j = (y_i, \nu_i, \lambda_i, z_i)$ for $j = 1, 2$, we now define the dissipation due to martensitic transformation which "measures" changes in the volume fraction $\lambda \in L^\infty\left(\Omega; \mathbb{R}^{M+1}\right)$. This dissipation is given by

$$\mathcal{D}_{\mathrm{tr}}\left(q_1, q_2\right) := \int_\Omega \left|\lambda_1(x) - \lambda_2(x)\right|_{\mathbb{R}^{M+1}} \, \mathrm{d}x \ ,$$

where

$$\lambda_j(x) := \int_{\mathbb{R}^{n\times n}} \mathcal{L}(s)\nu_{j,x}(\mathrm{d}s)$$

(see Subsection 2.1 for the definition of this expression involving a Young measure).

1.2.2 Plastic dissipation

The second source of dissipation is related to temporal changes in the plastic (hardening) variables gathered in $z = (F_{\mathrm{p}}, p)$. We write $Z := \mathrm{SL}(n) \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \times \mathbb{R}^m$ for the Cartesian product of the set of plastic variables with the set of their time derivatives. We write $(z, \dot{z})$ for elements of Z. Let us consider a nonnegative function $\delta\colon \Omega \times Z \to \mathbb{R}$ which is positively one-homogeneous in the last variable $\dot{z}$, i.e., $\delta(x, z, \eta\dot{z}) = \eta\delta(x, z, \dot{z})$ for all $\eta \geq 0$. Then the *infinitesimal dissipation distance* is [11]

$$D_{\mathrm{p}}\left(z_1, z_2\right) = \inf_z \left\{ \int_0^1 \delta\left(x, z(s), \dot{z}(s)\right) \, \mathrm{d}s \;\middle|\; z\colon C^1[0,1] \to Z \text{ with } z(0) = z_1 \text{ and } z(1) = z_2 \right\} .$$

The *plastic dissipation* is then

$$\mathcal{D}_{\mathrm{p}}\left(q_1, q_2\right) := \int_\Omega D_{\mathrm{p}}\left(x, z_1(x), z_2(x)\right) \, \mathrm{d}x \ . \tag{2}$$

Consequently, the overall dissipation $\mathcal{D}\colon \mathbb{Q} \times \mathbb{Q} \to [0, +\infty]$ is given by

$$\mathcal{D}\left(q_1, q_2\right) := \mathcal{D}_{\mathrm{p}}\left(q_1, q_2\right) + \mathcal{D}_{\mathrm{tr}}\left(q_1, q_2\right) \ . \tag{3}$$

Technical assumptions on the dissipation are stated at the end of Subsection 2.2.

1.2.3 Loading and boundary conditions

In experiments, a specimen occupying the region Ω will be subjected to external loads. In order to simplify our exposition, we consider only dead body forces and surface forces. We assume that we are given two disjoint sets $\Gamma_0, \Gamma_1 \subset \partial\Omega$, where the $(n-1)$-dimensional Hausdorff measure of Γ_0 is positive. We consider Dirichlet boundary conditions $y = y_0$ on Γ_0 for some prescribed (time-independent) mapping y_0. As for the surface forces acting on Γ_1, we define a linear functional

$$L(y) := \int_\Omega f(x) \cdot y(x)\,\mathrm{d}x + \int_{\Gamma_1} g(x) \cdot y(x)\,\mathrm{d}S \ , \tag{4}$$

where $f \colon \Omega \to \mathbb{R}^n$ and $g \colon \Gamma_1 \to \mathbb{R}^n$ are the densities of volume and surface forces acting on the material, respectively. Below, we write $L = L(t, y)$ to indicate the possibility of temporally changing forces.

1.3 Energetic solution

Combining the previous considerations, we arrive at the energy functional $\mathcal{I}$ of the form

$$\mathcal{I}(t,q) := \int_\Omega \int_{\mathbb{R}^{d\times d}} W\left(x, sF_\mathrm{p}^{-1}(x), F_\mathrm{p}(x), \nabla F_\mathrm{p}(x), p(x), \nabla p(x)\right) \nu_x(\mathrm{d}s)\,\mathrm{d}x \\ + \varepsilon \left\|\nabla\lambda\right\|_{L^2\left(\Omega;\mathbb{R}^{(1+M)\times n}\right)} - L(t, y(t)) \ . \tag{5}$$

In this paper, we restrict our attention to so-called *separable* materials, that is,

$$W\left(x, F_\mathrm{e}, F_\mathrm{p}, \nabla F_\mathrm{p}, p, \nabla p\right) := \hat{\varphi}\left(x, F_\mathrm{e}\right) + W_\mathrm{p}\left(F_\mathrm{p}, \nabla F_\mathrm{p}, p, \nabla p\right) \ .$$

It is often convenient to write

$$V(q) := \int_\Omega \int_{\mathbb{R}^{n\times n}} W\left(x, sF_\mathrm{p}^{-1}(x), F_\mathrm{p}(x), \nabla F_\mathrm{p}, p(x), \nabla p(x)\right) \nu_x(\mathrm{d}s)\,\mathrm{d}x \\ + \varepsilon \left\|\nabla\lambda\right\|_{L^2\left(\Omega;\mathbb{R}^{(1+M)\times n}\right)} . \tag{6}$$

We seek to analyse the time evolution of a process $q(t) \in \mathbb{Q}$ during the time interval $[0, T]$; $\mathbb{Q}$ is here the configuration space, whose mathematical definition is given in (18) below. The following two properties are key ingredients of the so-called energetic solution introduced by Mielke and Theil [13].

(i) *Stability inequality*: for every $t \in [0, T]$ and every $\tilde{q} \in \mathbb{Q}$, it holds that

$$\mathcal{I}(t, q(t)) \le \mathcal{I}\left(t, \tilde{q}\right) + \mathcal{D}\left(q(t), \tilde{q}\right). \tag{7}$$

(ii) *Energy balance*: For every $0 \leq t \leq T$,

$$\mathcal{I}(t, y(t), z(t)) + \mathrm{Var}(\mathcal{D}, q; [0, t]) = \mathcal{I}(0, q(0)) + \int_0^t \dot{L}(\xi, q(\xi))\, \mathrm{d}\xi \ , \tag{8}$$

where

$$\mathrm{Var}(\mathcal{D}, z; [s, t]) := \sup \left\{ \sum_{j=1}^{N} \mathcal{D}\left(q\left(t_{j-1}\right), q\left(t_j\right)\right) \;\middle|\; \{t_j\}_{j=0}^{N} \text{ is a partition of } [s, t] \right\}$$

is the *variation* of the dissipation.

Definition 1.1 *The mapping $q\colon [0, T] \to \mathbb{Q}$ is an* energetic solution *to the problem $(\mathcal{I}, \mathcal{D}, L)$ with the energy functional $\mathcal{I}$ as in* (5)*, the dissipation $\mathcal{D}$ of* (3) *and the load L as in* (4) *if the stability inequality* (7) *and energy balance* (8) *are satisfied for every $t \in [0, T]$.*

2 Mathematical background and assumptions

2.1 Young measures

We briefly recall the concept of Young measures [19] and follow the presentation in [9]. Young measures describe the limit of a sequence $\{y_k\}_{k\in\mathbb{N}}$ of functions $y_k\colon \Omega \to \mathbb{R}^d$ which converges weakly in $L^q\left(\Omega; \mathbb{R}^d\right)$ for $1 \leq q < \infty$ or weakly$*$ if $q = \infty$. The precise concept is as follows. A *Young measure* on a bounded domain $\Omega \subset \mathbb{R}^n$ is a weakly* measurable mapping

$$\Omega \to \mathrm{Prob}\left(\mathbb{R}^d\right), \qquad x \mapsto \nu_x,$$

with values in the probability measures. We recall that a mapping with values in the Radon measures is *weakly* measurable* if for any $f \in C_0\left(\mathbb{R}^d\right)$, the mapping

$$\Omega \to \mathbb{R}, \qquad x \mapsto \langle f, \nu_x \rangle := \int_{\mathbb{R}^d} f(s) \nu_x(\mathrm{d}s)$$

is measurable in the usual sense. We denote the set of all Young measures by $\mathcal{Y}\left(\Omega; \mathbb{R}^d\right)$.

It is known that $\mathcal{Y}\left(\Omega; \mathbb{R}^d\right)$ is a convex subset of $L^\infty_{\mathrm{w}}\left(\Omega; M\left(\mathbb{R}^d\right)\right) \cong L^1\left(\Omega; C_0\left(\mathbb{R}^d\right)\right)^*$, where $L^\infty_{\mathrm{w}}\left(\Omega; M\left(\mathbb{R}^d\right)\right)$ is the space of weakly* measurable bounded functions. The *parametrised Young measure theorem* [16] states that for every sequence $\{y_k\}_{k\in\mathbb{N}}$ which is bounded in $L^\infty\left(\Omega; \mathbb{R}^d\right)$, there exists a subsequence (denoted by the same indices for notational simplicity) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}\left(\Omega; \mathbb{R}^d\right)$ such that for every continuous function $f\colon \mathbb{R}^d \to \mathbb{R}$,

$$f(y_k) \stackrel{\star}{\rightharpoonup} x \mapsto \langle f, \nu_x \rangle \text{ weakly* in } L^\infty(\Omega), \tag{9}$$

where

$$\langle f, \nu_x \rangle := \int_{\mathbb{R}^d} f(s)\nu_x(\mathrm{d}s) \tag{10}$$

is the *expectation* of f. Let $\mathcal{Y}^\infty\left(\Omega;\mathbb{R}^d\right)$ denote set of all Young measures which are generated by taking all bounded sequences $\{y_k\}_{k\in\mathbb{N}}$ in $L^\infty\left(\Omega;\mathbb{R}^d\right)$.

The above concept is applicable if $\{y_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty\left(\Omega;\mathbb{R}^d\right)$. If in addition to the uniform bound in $L^\infty\left(\Omega;\mathbb{R}^d\right)$, $y_k \rightharpoonup y$ in $L^q\left(\Omega;\mathbb{R}^d\right)$ with $1 \leq q < \infty$, then $y_k \to y$ if and only if the corresponding Young measure is a Dirac mass, $\nu_x = \delta_{y(x)}$. Non-Dirac Young measures thus record possible oscillations in the limit process.

The assumption that $\{y_k\}_{k\in\mathbb{N}}$ is bounded in $L^\infty\left(\Omega;\mathbb{R}^d\right)$ can be relaxed to the assumption of such a bound in $L^q\left(\Omega;\mathbb{R}^d\right)$ with $1 < q < \infty$. The parametrised Young measure theorem is then valid under stronger growth conditions on the nonlinearity f. The precise formulation has been given by Schonbek [16, Theorem 2.2] (see also [3] for a general formulation of the parametrised Young measure theorem). Namely, for every sequence $\{y_k\}_{k\in\mathbb{N}}$ which is uniformly bounded in $L^q\left(\Omega;\mathbb{R}^d\right)$ for some $q > 1$, there exists a subsequence, still indexed by k for notational convenience, and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}\left(\Omega;\mathbb{R}^d\right)$ such that for every $f \in C\left(\mathbb{R}^d\right)$ with

$$f(x) = o\left(|x|^q\right) \text{ for } |x| \to \infty, \tag{11}$$

the following holds in $L^1\left(\Omega;\mathbb{R}^d\right)$:

$$f(y_k) \rightharpoonup \langle f, \nu_x \rangle. \tag{12}$$

We say that $\{y_k\}_{k\in\mathbb{N}}$ generates ν if (12) holds; we denote the set of all Young measures obtained as limits of bounded sequences in $L^q\left(\Omega;\mathbb{R}^d\right)$ by $\mathcal{Y}^q\left(\Omega;\mathbb{R}^d\right)$. If $\{y_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,q}\left(\Omega;\mathbb{R}^d\right)$, then the set of Young measures generated by subsequences of $\{\nabla y_k\}_{k\in\mathbb{N}}$ will be denoted $\mathbb{G}^q\left(\Omega;\mathbb{R}^{d\times d}\right)$. In the spirit of (9), we extend the energy V to $\bar{V}(\nu) := \int_\Omega \int_{\mathbb{R}^{d\times d}} \varphi(s)\nu_x(\mathrm{d}s)\,\mathrm{d}x$ for $\nu \in \mathbb{G}^q\left(\Omega;\mathbb{R}^{d\times d}\right)$.

2.2 Mathematical framework, assumptions and main result

We recall $W = W\left(x, sF_\mathrm{p}^{-1}(x), F_\mathrm{p}(x), \nabla F_\mathrm{p}(x), p(x), \nabla p(x)\right)$; to abbreviate the notation, let us write $A := sF_\mathrm{p}^{-1}$, $G := \nabla F_\mathrm{p}$ and $\pi := \nabla p$. We assume that W satisfies the following requirements:

$$W(x,\cdot) \text{ is continuous for a.e. } x \in \Omega\ , \tag{13}$$

$$W\left(\cdot, A, F_\mathrm{p}, G, p, \pi\right) \text{ is measurable for all } A, F_\mathrm{p}, G, p, \pi\ . \tag{14}$$

Next, growth conditions: we assume that there are constants $C, c > 0$ and $\alpha, \beta, \omega > 1$ such that

$$C\left(1+|A|^\alpha+|F_p|^\beta+|G|^\beta+|p|^\omega+|\pi|^\omega\right) \geq W\left(x, A, F_\mathrm{p}, G, p, \pi\right) \geq c\left(-1+|A|^\alpha+|F_p|^\beta+|G|^\beta+|p|^\omega+|\pi|^\omega\right) . \quad (15)$$

It is reasonable to require convexity in the gradient terms $G = \nabla F_\mathrm{p}$ and $\pi = \nabla p$,

$$W(x, A, F_\mathrm{p}, \cdot, p, \cdot) \text{ is convex for a.e. } x \in \Omega \text{ and every } A, F_\mathrm{p}, p . \quad (16)$$

In order to simplify the notation, we omit the dependence of W on x. However, we point out that the entire theory developed in this paper applies to spatially inhomogeneous W as well.

In what follows, we suppose that

$$y \in \mathbb{Y}\left(\Omega; \mathbb{R}^n\right) := \left\{y \in W^{1,d}\left(\Omega; \mathbb{R}^n\right) \,\middle|\, y = y_0 \text{ on } \Gamma_0\right\} , \quad (17)$$

where $\Gamma_0 \subset \partial\Omega$ with a positive surface measure, as described in Subsection 1.2.3. We recall from that subsection that $\Gamma_0 \cap \Gamma_1 = \emptyset$ by assumption. Further,

$$\mathbb{P}\left(\Omega; \mathbb{R}^{n\times n}, \mathbb{R}^m\right) := \left\{(F_p, p) \in W^{1,\beta}\left(\Omega; \mathbb{R}^{n\times n}\right) \times W^{1,\omega}\left(\Omega; \mathbb{R}^m\right) \,\middle|\, F_p(x) \in \mathrm{SL}(n) \text{ for a.e. } x \in \Omega\right\} .$$

Then we look for $q \in \mathcal{Q} := \mathbb{Y}\left(\Omega; \mathbb{R}^n\right) \times \mathbb{G}\left(\Omega; \mathbb{R}^n\right) \times L^\infty\left(\Omega; \mathbb{R}^{M+1}\right) \times \mathbb{P}\left(\Omega; \mathbb{R}^{n\times n}, \mathbb{R}^m\right)$ and restrict the space further by imposing the *admissibility condition*

$$\mathbb{Q} := \left\{q \in \mathcal{Q} \,\middle|\, \lambda = \mathcal{L} \bullet \nu \text{ and } \nabla y = \mathbb{I} \bullet \nu\right\} , \quad (18)$$

where, for almost all $x \in \Omega$, $[\lambda \bullet \nu](x) := \int_{\mathbb{R}^{n\times n}} \mathcal{L}(s)\nu_x(\mathrm{d}s)$; $\mathbb{I} \bullet \nu$ is defined analogously. We need to define the notion of convergence in this space, and do so as follows.

Definition 2.1 *Suppose that* $\{q_k\}_{k\in\mathbb{N}} \subset \mathbb{Q}$, *where* $q_k = (y_k, \nu_k, \lambda_k, z_k)$. *We say that* $q_k \rightharpoonup q := (y, \nu, \lambda, z) \in \mathbb{Q}$ *as* $k \to \infty$ *if* $y_k \rightharpoonup y$ *in* $W^{1,d}\left(\Omega; \mathbb{R}^n\right)$, $\nu^k \rightharpoonup^* \nu$ *in* $L^\infty_\mathrm{w}\left(\Omega; \mathcal{M}(\mathbb{R}^{n\times n})\right)$, $\lambda_k \rightharpoonup \lambda$ *in* $W^{1,2}\left(\Omega; \mathbb{R}^{M+1}\right)$ *and* $z_k \rightharpoonup z$ *in* $W^{1,\beta}\left(\Omega; \mathbb{R}^{n\times n}\right) \times W^{1,\omega}\left(\Omega; \mathbb{R}^m\right)$.

In line with related work [6, 10], we impose the following conditions on $\mathcal{D}$:

(i) Lower semicontinuity:

$$\mathcal{D}(q, \tilde{q}) \leq \liminf_{k\to\infty} \mathcal{D}(q_k, \tilde{q}_k) , \quad (19)$$

whenever $q_k \rightharpoonup q$ and $\tilde{q}_k \rightharpoonup \tilde{q}$ as $k \to \infty$.

(ii) Positivity: If $\{q_k\}_{k\in\mathbb{N}} \subset \mathbb{Q}$ is bounded and

$$\min\{\mathcal{D}\left(q_k, q\right), \mathcal{D}\left(q, q_k\right)\} \to 0 \text{ as } k \to \infty \text{ then } q_k \rightharpoonup q . \quad (20)$$

Some more assumptions on the dissipation are required to deal with the possibility of the plastic dissipation becoming infinite. We state suitable restrictions for D_{p} (see [10] for similar conditions).

Assumption 2.2 *The plastic dissipation D_{p} satisfies the following conditions.*

1. $D_{\mathrm{p}}(x,\cdot,\cdot)\colon D(x) \to [0,+\infty)$ *is continuous, where*

$$D(x) := \left\{ (z_0, z_1) \;\middle|\; D_{\mathrm{p}}\left(x, z_0, z_1\right) < +\infty \right\} . \tag{21}$$

2. *For every $R > 0$ there is $K > 0$ such that, for almost all $x \in \Omega$, $D_{\mathrm{p}}\left(x, z_0, z_1\right) < K$ if $(z_0, z_1) \in D(x)$ and $|z_0|, |z_0| < R$.*

3. *There is $v^* \in \mathbb{R}^m$ such that for all $\eta, R > 0$, there is $\rho > 0$ such that for almost every $x \in \Omega$ and every $z, z_0, z_1 \in \mathbb{R}^{n\times n} \times \mathbb{R}^m$:*

$$|z - z_0| < \rho \text{ and } (z_0, z_1) \in D(x) \text{ implies } (z, z_1 + (0, \eta v^*)) \in D(x) \ ,$$

where $\eta \to 0$ when $\rho \to 0$.

As for the load, we impose the following qualifications:

$$f \in C^1\left([0,T]; L^{\tilde{d}}\left(\Omega; \mathbb{R}^d\right)\right) \text{ with } \tilde{d} \geq \begin{cases} \frac{dn}{n-d} & \text{if } 1 \leq d < n \\ 1 & \text{else} \end{cases} \text{ and} \tag{22}$$

$$g \in C^1\left([0,T]; L^{\hat{d}}\left(\Gamma_1; \mathbb{R}^d\right)\right) \text{ with } \hat{d} \geq \begin{cases} \frac{nd-d}{nd-n} & \text{if } 1 \leq d < n \\ 1 & \text{else.} \end{cases} \tag{23}$$

Our main result is the following theorem regarding the existence of an energetic solution.

Theorem 2.3 *Let $\frac{1}{\alpha} + \frac{1}{\beta} \leq \frac{1}{d} < \frac{1}{n}$, and let Assumption 2.2,* (13)–(16)*,* (19)–(23) *hold. Then there is a process $q\colon [0,T] \to \mathbb{Q}$ with $q(t) = (y(t), \nu(t), z(t), \lambda(t))$ such that q is an energetic solution according to Definition 1.1. for a given initial condition $q_0 \in \mathbb{Q}$.*

The proof of this result relies on approximations by time-discrete (incremental) problems constructed for a given time step. These are minimisation problems over spatial variables. Each minimisation problem takes into account the solution obtained for the previous time step while the initial condition serves as input for the first minimisation problem. Hence, the proof is rather constructive. In addition to Theorem 2.3, we prove various convergence results for the deformation, the martensitic volume fractions, and the plastic variables, see Theorem 3.7 at the end of the paper.

3 Existence of a solution process

3.1 Incremental problems

We start the mathematical analysis by defining the set of stable states,

$$\mathcal{S}(t) := \left\{ q \in \mathbb{Q} \;\middle|\; \mathcal{I}(t,q) \leq \mathcal{I}(t,\tilde{q}) + \mathcal{D}(q,\tilde{q}) \text{ for every } \tilde{q} \in \mathcal{Q} \right\} \; ; \tag{24}$$

let us also define

$$\mathcal{S}_{[0,T]} := \cup_{t\in[0,T]} \{t\} \times \mathcal{S}(t) \; . \tag{25}$$

We say that a sequence $\{(t_k, q_k)\}_{k\in\mathbb{N}}$ is *stable* if $q_k \in \mathcal{S}(t_k)$.

The proof of existence of a rate-independent evolution commonly proceeds via time-discretisation. Thus, in a first step, a sequence of incremental problems is defined. Let us remind ourselves of the notation $z := (F_\mathrm{p}, p)$. We define a time discretisation $0 = t_0 < \ldots < t_n = T$ via a time step $\tau > 0$, chosen in such a way that $N = T/\tau \in \mathbb{N}$. Let an initial state $\mathcal{S}(0) \ni q^0 =: q^0_\tau \in \mathbb{Q}$ be given. For $1 \leq k \leq N$ we find $q^k_\tau \in \mathbb{Q}$ by solving

$$\text{minimise } \mathcal{I}(t_k, q) + \mathcal{D}\left(q^{k-1}_\tau, q\right), \text{subject to } q \in \mathbb{Q} \; . \tag{26}$$

The existence of a solution to the time step problem (26) is ensured by the following lemma.

Lemma 3.1 (Existence) *Let* (13)–(16), (19), (22) *and* (23) *hold. Suppose further* $\frac{1}{\alpha} + \frac{1}{\beta} \leq \frac{1}{d}$. *Then the problem* (26) *has a solution for all* $k = 1, \ldots, N = T/\tau$.

Proof: Suppose that $q^{k-1} \in \mathbb{Q}$ is known; let $\{q_j\}_{j\in\mathbb{N}} \subset \mathbb{Q}$ be a minimising sequence for $q \mapsto \mathcal{I}(t_k, q) + \mathcal{D}(q^{k-1}_\tau, q)$. The assumption (15) implies that $\{z_j\}_{j\in\mathbb{N}}$ is uniformly bounded in $W^{1,\beta}(\Omega;\mathbb{R}^{n\times n}) \times W^{1,\omega}(\Omega;\mathbb{R}^m)$. Hence, as $\beta > 1$ and $\omega > 1$, we can extract a weakly converging subsequence (not relabelled) $z_j \rightharpoonup z$ in $W^{1,\beta}(\Omega;\mathbb{R}^{n\times n}) \times W^{1,\omega}(\Omega;\mathbb{R}^m)$, with $z_j = \left(F_{\mathrm{p}j}, p_j\right)$. The strong convergence of $z_j \to z := (F_\mathrm{p}, p)$ in $L^\beta(\Omega;\mathbb{R}^{n\times n}) \times L^\omega(\Omega;\mathbb{R}^m)$ as $j \to \infty$ ensures that $F_p(x) \in \mathrm{SL}(n)$ almost everywhere. By weak-$\star$ compactness $\nu_j \overset{*}{\rightharpoonup} \nu \in L^\infty_\mathrm{w}(\Omega;\mathcal{M}(\mathbb{R}^{n\times n}))$ as $j \to \infty$; $L^\infty_\mathrm{w}(\Omega;\mathcal{M}(\mathbb{R}^{n\times n}))$ is defined in Subsection 2.1. However, we must show that $\nu \in \mathbb{G}$. Each Young-measure component ν_j, say, of $q_j = \left(y_j, \nu_j, \lambda_j, F_{\mathrm{p}j}, p_j\right)$ is generated by a sequence of gradients $\{\nabla y^l_j\}_{l\in\mathbb{N}}$ of maps $y^l_j \in W^{1,d}(\Omega;\mathbb{R}^n)$. As $\{F_{\mathrm{p}j}\}_{j\in\mathbb{N}}$ is bounded in $L^\beta(\Omega;\mathbb{R}^{n\times n})$, we obtain from Hölder's inequality the estimate

$$\left\|\nabla y^l_j\right\|_{L^d(\Omega;\mathbb{R}^{n\times n})} \leq \left\|\nabla y^l_j F_{\mathrm{p}j}^{-1}\right\|_{L^\alpha(\Omega;\mathbb{R}^{n\times n})} \left\|F_{\mathrm{p}j}\right\|_{L^\beta(\Omega;\mathbb{R}^{n\times n})} \leq C.$$

The first norm on the right hand side is bounded by assumption (15) so we have that $\{\nabla y_j^l\}_{j,l\in\mathbb{N}}$ is bounded independently of $j,l\in\mathbb{N}$. A diagonalisation argument then shows that there is a generating sequence of gradients for η, so that $\eta\in\mathbb{Q}$. Finally, due to the convexity of the energy in $\nabla\lambda$, we obtain a subsequence of volume fractions $\lambda_j\rightharpoonup\lambda$ in $W^{1,2}\left(\Omega;\mathbb{R}^L\right)$. The assumptions on the joint convexity of W in the gradient arguments G and π and (19) ensure that $\mathcal{I}$ is sequentially weakly lower semicontinuous on $\mathbb{Q}$. The existence of a minimum then follows by the direct method of the calculus of variations. □

3.2 Interpolation in time

We now introduce a piecewise constant interpolation q_τ of $q_\tau^k := \left(y_\tau^k,\nu_\tau^k,\lambda_\tau^k,z_\tau^k\right)$. Namely, $q_\tau(t) := q_\tau^k$ if $t\in[k\tau,(k+1)\tau)$ and $k=0,\dots,N-1=T/\tau-1$. Finally, $q_\tau(T) := q^N|_\tau$. Likewise, $L_\tau(t,q)=L(k\tau,q)$ is a piecewise constant interpolation of the load L for suitable piecewise constant q. Analogously, $\mathcal{I}_\tau(t,q)=\mathcal{I}(k\tau,q)$ is a piecewise constant interpolation of $\mathcal{I}$ defined in the same way as L_τ.

Proposition 3.2 (Stability) *We make the same assumptions as in Lemma 3.1: let* (13)–(16), (19), (22) *and* (23) *be satisfied, and suppose that* $\frac{1}{\alpha}+\frac{1}{\beta}\le\frac{1}{d}$. *Then the problem* (26) *has a solution* $q_\tau(t)$ *which is stable, i.e., for all* $t\in[0,T]$ *and for every* $\tilde{q}\in\mathbb{Q}$,

$$\mathcal{I}_\tau(t,q_\tau(t))\le\mathcal{I}_\tau(t,\tilde{q})+\mathcal{D}\left(q_\tau(t),\tilde{q}\right). \tag{27}$$

Moreover, for all $t_1\le t_2$ *from the set* $\{k\tau\}_{k=0}^N$, *the following discrete energy inequalities hold if one extends the definition of* $q_\tau(t)$ *by setting* $q_\tau(t):=q_0$ *if* $t<0$.

$$-\int_{t_1}^{t_2}\dot{L}\left(t,q_\tau(t-\tau)\right)\,\mathrm{d}t\le\mathcal{I}\left(t_2,q_\tau(t_2)\right)+\mathrm{Var}\left(q_\tau,[t_1,t_2]\right)-\mathcal{I}\left(t_1,q_\tau(t_1)\right)$$
$$\le-\int_{t_1}^{t_2}\dot{L}\left(t,q_\tau(t)\right)\,\mathrm{d}t\ . \tag{28}$$

Proof: The existence of a solution to (26) was proved in Lemma 3.1. The stability estimate (27) follows from the minimising property of q_τ^k and the properties of $\mathcal{D}$. Indeed, by minimality of q_τ^k,

$$\mathcal{I}\left(k\tau,q_\tau^k\right)+\mathcal{D}\left(q_\tau^{k-1},q_\tau^k\right)\le\mathcal{I}\left(k\tau,\tilde{q}\right)+\mathcal{D}\left(q_\tau^{k-1},\tilde{q}\right), \tag{29}$$

which immediately implies that

$$\mathcal{I}\left(k\tau,q_\tau^k\right)\le\mathcal{I}\left(k\tau,\tilde{q}\right)+\mathcal{D}\left(q_\tau^{k-1},\tilde{q}\right)-\mathcal{D}\left(q_\tau^{k-1},q_\tau^k\right). \tag{30}$$

We remark that both dissipative terms satisfy the triangle inequality, $\mathcal{D}_{\mathrm{p}}\left(q_1, q_2\right)+\mathcal{D}_{\mathrm{p}}\left(q_2, q_3\right) \leq \mathcal{D}_{\mathrm{p}}\left(q_1, q_3\right)$ and analogously for $\mathcal{D}_{\mathrm{tr}}$. Thus

$$\mathcal{D}\left(q_\tau^{k-1}, \tilde{q}\right) - \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k\right) \leq \mathcal{D}\left(q_\tau^k, \tilde{q}\right),$$

so that (27) follows from (30).

Next, we demonstrate the validity of the energy inequality (28), using arguments of [13]. For this part, let us test the stability of q_τ^{k-1} with $\tilde{q} := q_\tau^k$. This gives

$$\begin{aligned}\mathcal{I}\left((k-1)\tau, q_\tau^{k-1}\right) &\leq \mathcal{I}\left((k-1)\tau, q_\tau^k\right) + \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k,\right) \\ &= \mathcal{I}\left(k\tau, q_\tau^k\right) + L\left(k\tau, q_\tau^k\right) - L\left((k-1)\tau, q_\tau^k\right) + \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k\right). \qquad (31)\end{aligned}$$

Suppose that $0 \leq k_1 \leq k_2 \leq N$ and that $t_1 = k_1\tau$ and $t_2 = k_2\tau$. A summation of (31) over $k = k_1+1, \ldots, k_2$ yields

$$\sum_{k=k_1+1}^{k_2} \left[L\left((k-1)\tau, q_\tau^k\right) - L\left(k\tau, q_\tau^k\right)\right] \leq \mathcal{I}\left(k_2\tau, q_\tau^{k_2}\right) - \mathcal{I}\left(k_1\tau, q_\tau^{k_1}\right) + \sum_{k=k_1+1}^{k_2} \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k\right). \qquad (32)$$

We rewrite this inequality in terms of q_τ to see that it is the first inequality in (28),

$$\begin{aligned}-\int_{t_1}^{t_2} \dot{L}\left(t, q_\tau(t-\tau)\right)\,\mathrm{d}t &\leq \mathcal{I}\left(k_2\tau, q_\tau^{k_2}\right) - \mathcal{I}\left(k_1\tau, q_\tau^{k_1}\right) + \sum_{k=k_1+1}^{k_2} \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k\right) \\ &= \mathcal{I}\left(k_2\tau, q_\tau^{k_2}\right) - \mathcal{I}\left(k_1\tau, q_\tau^{k_1}\right) + \mathrm{Var}\left(q_\tau, [t_1, t_2]\right)\end{aligned}$$

(the explicit form of $\mathrm{Var}\left(q_\tau, [t_1, t_2]\right)$ holds since we consider a step function). To prove the validity of the second inequality in (28), we rely on the minimality of q_τ^k, when compared with q_τ^{k-1} in (29). That is,

$$\begin{aligned}\mathcal{I}\left(k\tau, q_\tau^k\right) + \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k\right) &\leq \mathcal{I}\left(k\tau, q_\tau^{k-1}\right) \\ &= \mathcal{I}\left((k-1)\tau, q_\tau^{k-1}\right) + L\left((k-1)\tau, q_\tau^{k-1}\right) - L\left(k\tau, q_\tau^{k-1}\right).\end{aligned}$$

Similarly as in the previous argument, a summation over $k = k_1+1, \ldots, k_2$ is employed to find that

$$\begin{aligned}&\mathcal{I}\left(k_2\tau, q_\tau^{k_2}\right) - \mathcal{I}\left(k_1\tau, q_\tau^{k_1}\right) + \sum_{k=k_1+1}^{k_2} \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k\right) \\ &\qquad\qquad \leq \sum_{k=k_1+1}^{k_2} \left[L\left((k-1)\tau, q_\tau^{k-1}\right) - L\left(k\tau, q_\tau^{k-1}\right)\right], \qquad (33)\end{aligned}$$

so that

$$\mathcal{I}\left(k_2\tau, q_\tau^{k_2}\right) - \mathcal{I}\left(k_1\tau, q_\tau^{k_1}\right) + \mathrm{Var}\left(q_\tau, [t_1, t_2]\right) \leq -\int_{t_1}^{t_2} \dot{L}\left(t, q_\tau(t-\tau)\right)\,\mathrm{d}t\,,$$

which is the second inequality in (28). □

3.3 Limit passage for vanishing time step

The next proposition gives the *a priori* bounds needed to pass to the limit as the step size goes to zero.

Proposition 3.3 (A priori bounds) *Assume that* W *satisfies the conditions* (13)–(16), *and that* (19), (22), (23) *are satisfied. Let* $\frac{1}{\alpha}+\frac{1}{\beta}\le\frac{1}{d}$. *Further, suppose that* $W^{1,r}\left(\Omega;\mathbb{R}^d\right)$ *embeds continuously to* $L^{r'}\left(\Omega;\mathbb{R}^d\right)$ *and to* $L^{\hat r}\left(\Gamma_1;\mathbb{R}^d\right)$. *Then there exists a constant* $\kappa\in\mathbb{R}$ *such that*

$$\|y_\tau\|_{L^\infty\left(0,T;W^{1,r}\left(\Omega;\mathbb{R}^d\right)\right)}<\kappa, \tag{34}$$

$$\left\|\left(1\otimes|s|^d\right)\bullet\nu_\tau\right\|_{L^\infty(0,T;L^1(\Omega))}<\kappa\ , \tag{35}$$

$$\|\nu_\tau\|_{L^\infty(0,T;L^\infty_{\mathrm{w}}(\Omega;\mathcal{M}(\mathbb{R}^{n\times n})))}<\kappa\ , \tag{36}$$

$$\|\lambda_\tau\|_{L^\infty(0,T;W^{1,2}(\Omega;\mathbb{R}^{M+1}))\cap BV(0,T;L^1(\Omega;\mathbb{R}^{M+1}))}<\kappa\ , \tag{37}$$

$$\mathrm{Var}\left(\mathcal{D},q_\tau;[0,T]\right)<\kappa, \tag{38}$$

and for $\hat{\mathcal{I}}_\tau(t):=\mathcal{I}_\tau(t,q_\tau(t))$,

$$\left\|\hat{\mathcal{I}}_\tau\right\|_{BV(0,T)}<\kappa. \tag{39}$$

Proof: We recall from (6) that

$$V(q)=\int_\Omega\int_{\mathbb{R}^{n\times n}}W\left(x,sF_{\mathrm{p}}^{-1}(x),F_{\mathrm{p}}(x),\nabla F_{\mathrm{p}},p(x),\nabla p(x)\right)\nu_x(\mathrm{d}s)\,\mathrm{d}x \\ +\varepsilon\,\|\nabla\lambda\|_{L^2\left(\Omega;\mathbb{R}^{(1+M)\times n}\right)}\,.$$

The growth conditions (15) imply that

$$\int_\Omega\int_{\mathbb{R}^{n\times n}}\left|sF_{\mathrm{p}}^{-1}(x)\right|^\alpha\nu_x(\mathrm{d}s)\,\mathrm{d}x+\|F_p\|^\beta_{W^{1,\beta}\left(\Omega;\mathbb{R}^{d\times d}\right)}+\|p\|^\omega_{W^{1,\beta}(\Omega;\mathbb{R}^m)}\le V(q)\ . \tag{40}$$

We use that for some $C>0$, since $\frac{1}{\alpha}+\frac{1}{\beta}\le\frac{1}{d}$,

$$\int_\Omega\left|\int_{\mathbb{R}^{m\times n}}s\nu_x(\mathrm{d}s)\right|^d\mathrm{d}x\le C\int_\Omega\int_{\mathbb{R}^{m\times n}}|s|^d\,\nu_x(\mathrm{d}s)\,\mathrm{d}x \\ \le C\,\|F_p\|^\beta_{W^{1,\beta}\left(\Omega;\mathbb{R}^{d\times d}\right)}\int_\Omega\int_{\mathbb{R}^{m\times n}}\left|sF_{\mathrm{p}}^{-1}(x)\right|^\alpha\nu_x(\mathrm{d}s)\,\mathrm{d}x\ ;$$

this, the fact that $\nabla y(x)=\int_{\mathbb{R}^{n\times n}}s\nu_x(\mathrm{d}s)$ for almost all $x\in\Omega$, and the Poincaré inequality yield together with (40)

$$C\,\|y\|^d_{W^{1,d}(\Omega;\mathbb{R}^n)}+\|F_p\|^\beta_{W^{1,\beta}\left(\Omega;\mathbb{R}^{d\times d}\right)}+\|p\|^\omega_{W^{1,\beta}(\Omega;\mathbb{R}^m)}\le V(q)\ . \tag{41}$$

Since $\mathcal{I} = V - L$ by (5), we find from (33) for $k_1 = 0$ that

$$V\left(q_\tau^{k_2}\right) - L\left(k_2\tau, q_\tau^{k_2}\right) - V\left(q_\tau^0\right) + L\left(0, q_\tau^0\right) \le \sum_{k=1}^{k_2} \left[L\left((k-1)\tau, q_\tau^{k-1}\right) - L\left(k\tau, q_\tau^{k-1}\right)\right] \ ,$$

which we rewrite as

$$V\left(q_\tau^{k_2}\right) \le \sum_{k=1}^{k_2} \left[L\left((k-1)\tau, q_\tau^{k-1}\right) - L\left(k\tau, q_\tau^{k-1}\right)\right] + L\left(k_2\tau, q_\tau^{k_2}\right) + C \ . \tag{42}$$

Combining this with the estimate (41) for $q := q_\tau^{k_2}$, we find for $Y_\tau := \max_{1\le\ell\le N} \left\|y_\tau^\ell\right\|_{W^{1,d}\left(\Omega;\mathbb{R}^d\right)}^d$ that

$$Y_\tau \le \sum_{k=1}^{k_2} \left[L\left((k-1)\tau, q_\tau^{k-1}\right) - L\left(k\tau, q_\tau^{k-1}\right)\right] + C \ . \tag{43}$$

This gives the bound (34) since y_τ appears in the load on the right-hand-side, but only linearly (see (4)); since the power $d > 1$ on the left-hand side is larger, (34) follows. With (34) at our disposal we immediately obtain (35), and (36)–(39) are easy (e.g., $\nabla\lambda$ is bounded in $L^2\left(\Omega, \mathbb{R}^{(M+1)\times n}\right)$, and λ is bounded as a volume fraction; the BV bound comes from its contribution in the dissipation). □

Proposition 3.4 *Let $\mathcal{I}$ be weakly sequentially lower semicontinuous. Suppose that for all $(t_*, q_*) \in [0,T] \times \mathbb{Q}$, for all stable sequences $\{(t_k, q_k)\}_{k\in\mathbb{N}}$ with $t_k \to t_*$ and $q_k \rightharpoonup q_*$ in the sense of Definition 2.1, there is a sequence $\{\tilde{q}_k\}_{k\in\mathbb{N}} \subset \mathbb{Q}$ such that for all $\tilde{q} \in \mathbb{Q}$*

$$\limsup_{k\to\infty} \left[\mathcal{I}\left(t_k, \tilde{q}_k\right) + \mathcal{D}\left(q_k, \tilde{q}_k\right)\right] \le \mathcal{I}\left(t_*, \tilde{q}\right) + \mathcal{D}\left(q_*, \tilde{q}\right) \ . \tag{44}$$

Then $\mathcal{I}$ is weakly continuous along stable sequences and $q_* \in \mathcal{S}(t_*)$.

Proof: We follow the proof of [10, Proposition 4.2]. Take $\tilde{q} = q_*$ in (44) and notice that (44) holds with $\tilde{q}_k := q_k$, for $k \in \mathbb{N}$. By stability and then (44), we obtain for this choice of $\tilde{q}_k$

$$\limsup_{k\to\infty} \mathcal{I}\left(t_k, q_k\right) \le \limsup_{k\to\infty} \left[\left(\mathcal{I}\left(t_k, \tilde{q}_k\right) + \mathcal{D}\left(q_k, \tilde{q}_k\right)\right] \le \mathcal{I}\left(t_*, \tilde{q}\right) + \mathcal{D}\left(q_*, \tilde{q}\right) = \mathcal{I}\left(t_*, q_*\right) \ .$$

We have further

$$\lim_{k\to\infty} \left|\mathcal{I}\left(t_k, q_k\right) - \mathcal{I}\left(t_*, q_k\right)\right| = \lim_{k\to\infty} \left|L\left(t_k, q_k\right) - L\left(t_*, q_k\right)\right| = 0 \ ,$$

due to the assumptions (22) and (23) on f and g, respectively.

Since $\mathcal{I}$ is weakly lower semicontinuous it follows that

$$\liminf_{k\to\infty} \mathcal{I}\left(t_k, q_k\right) = \liminf_{k\to\infty} \left[\mathcal{I}\left(t_k, q_k\right) - \mathcal{I}\left(t_*, q_k\right)\right] + \liminf_{k\to\infty} \mathcal{I}\left(t_*, q_k\right) \geq \mathcal{I}\left(t_*, q_*\right) \ .$$

This together with (44) gives weak continuity of $\mathcal{I}\left(t_k, q_k\right) \to \mathcal{I}\left(t_*, q_*\right)$. Finally, we have for every $\tilde{q} \in \mathbb{Q}$

$$\mathcal{I}\left(t_*, q_*\right) = \lim_{k\to\infty} \mathcal{I}\left(t_k, q_k\right) \leq \liminf_{k\to\infty} \left[\mathcal{I}\left(t_k, \tilde{q}_k\right) + \mathcal{D}\left(q_k, \tilde{q}_k\right)\right] \leq \mathcal{I}\left(t_*, \tilde{q}\right) + \mathcal{D}\left(q_*, \tilde{q}\right) \ .$$

The arbitrariness of $\tilde{q} \in \mathbb{Q}$ shows the stability of q_*. □

The key point in the existence proof for a rate-independent process is to show the validity of (44). Let us suppose for the moment that irreversibility for the plastic process is excluded, that is, ∞ is not contained in the range of $\mathcal{D}$ defined in (2), that is, $\mathcal{D}_{\mathrm{p}} \colon \mathbb{Q}\times\mathbb{Q} \to [0, +\infty)$. Then it is sufficient to assume that for $\epsilon > 0$ small enough

$$D_{\mathrm{p}}\left(x, z_1, z_2\right) \leq c(x) + C\left(\left|F_{\mathrm{p}1}\right|^{\beta^*-\epsilon} + \left|F_{\mathrm{p}2}\right|^{\beta^*-\epsilon} + \left|p_1\right|^{\omega^*-\epsilon} + \left|p_2\right|^{\omega^*-\epsilon}\right)$$

holds, with

$$\beta^* := \begin{cases} \frac{n\beta}{n-\beta} & \text{if } n > \beta \\ 1+\xi & \text{else, for some } \xi > 0 \end{cases} \quad \text{and } \omega^* := \begin{cases} \frac{n\omega}{n-\omega} & \text{if } n > \omega \\ > 1+\xi & \text{else, for some } \xi > 0. \end{cases}$$

Then the compact embedding ensures continuity of $\mathcal{D}_{\mathrm{p}}$. Similar, $\mathcal{D}_{\mathrm{tr}}$ is continuous by compactness. Thus, the dissipation $\mathcal{D}$ defined in (3) is continuous.

However, we allow irreversibility by including ∞ in the range of $\mathcal{D}_{\mathrm{p}}$, so that $\mathcal{D}_{\mathrm{p}} \colon \mathbb{Q}\times\mathbb{Q} \to [0, +\infty]$, and we thus must be more careful. Assumption 2.2 will play a central rôle in the next argument. We recall the notation $z_j := (F_j, p_j) \in \mathbb{R}^{n\times n} \times \mathbb{R}^m$ for $j = 1, 2$ used in that assumption.

Proposition 3.5 *Let $\beta, \omega > n$. Let Assumption 2.2 hold. Then* (44) *holds.*

Proof: If $\mathcal{D}\left(q_*, \tilde{q}\right) = +\infty$ in (44), then nothing is to show. So we can assume that $\mathcal{D}_{\mathrm{p}}\left(q_*, \tilde{q}\right) \in \mathbb{R}$; then $(z_*, \tilde{z}) \in D(x)$, with $D(x)$ defined in (21). If $q_k \rightharpoonup q_*$ as $k \to \infty$, then due to the compact embedding

$$\rho_k := \left\|F_{\mathrm{p}k} - F_{\mathrm{p}*}\right\|_{C\left(\bar{\Omega};\mathbb{R}^{n\times n}\right)} + \left\|p_k - p_*\right\|_{C\left(\bar{\Omega};\mathbb{R}^m\right)} \to 0 \ .$$

Thus, there is $R > 0$ such that $|z_k| + |z_*| + |\tilde{z}| < R$ if k is large enough. We define $\tilde{z}_k := \left(\tilde{F}_{\mathrm{p}}, \tilde{p} + \eta_k v^*\right)$ where η_k relates to ρ_k as in Assumption 2.2 (3). Then $(z_k, \tilde{z}_k) \in D(x)$ by Assumption 2.2 (3) with $z_0 := z_*$ and $z_1 := \tilde{z}$. The continuity of D_{p} (Assumption 2.2 (1)) gives the pointwise convergence $D_{\mathrm{p}}\left(x, z_k, \tilde{z}_k\right) \to D_{\mathrm{p}}\left(z, z_*, \tilde{z}\right)$. Furthermore, we have $|z_k| <$

R and $|\tilde{z}_k| < R$ in addition to the property $(z_k, \tilde{z}_k) \in D(x)$ established above. Condition 2 of Assumption 2.2 together with the Lebesgue dominated convergence theorem implies that $\mathcal{D}_{\rm p}(q_k, \tilde{q}_k) \to \mathcal{D}_{\rm p}(q_*, \tilde{q})$. Further, $\mathcal{D}_{\rm tr}$ is continuous by compactness. Thus, the dissipation $\mathcal{D}$ defined in (3) is continuous. As for $\mathcal{I}$, assumptions (22) and (23) imply that (44) is fulfilled with equality. □

The following lemma is taken from [11, Theorem 5.21]. Let us first denote $\mathbb{X} := L^\beta(\Omega; \mathbb{R}^{n\times n}) \times L^\omega(\Omega; \mathbb{R}^m)$. Notice that if (19) and (20) hold for $\mathcal{D}_{\rm p}$ in $\mathbb{P}$ then they hold in $\mathbb{X}$ with the strong convergence in $\mathbb{X}$.

Lemma 3.6 (Helly for plastic dissipation) *Let $\mathcal{D}_{\rm p}\colon \mathbb{X} \times \mathbb{X} \to [0, +\infty]$. Let $\mathcal{K}$ be a compact subset of $\mathbb{X}$. Then for every sequence $\{z_k\}_{k\in\mathbb{N}}$, $z_k\colon [0,T] \to \mathcal{K}$ for which $\sup_{k\in\mathbb{N}} \mathrm{Var}(\mathcal{D}_{\rm p}, z_k; [0,T]) < C$ with some $C > 0$, there exists a subsequence (not relabelled), a function $z\colon [0,T] \to \mathcal{K}$, and a function $\delta\colon [0,T] \to [0,C]$ such that:*

1. $\mathrm{Var}(\mathcal{D}_{\rm p}, z_k; [0,t]) \to \delta(t)$ *for all* $t \in [0,T]$,
2. $z_k \to z$ *for all* $t \in [0,T]$, *and*
3. $\mathrm{Var}(\mathcal{D}_{\rm p}, z; [t_0,t_1]) \le \lim_{t\searrow t_1}\delta(t) - \lim_{t\nearrow t_0}\delta(t)$ *for all* $0 \le t_0 < t_1 \le T$, *with the limits evaluated as* $\delta(0) = 0$ *respectively* $\delta(T)$ *in the cases* $t_0 = 0$ *respectively* $t_1 = T$.

The assertion of the following theorem includes also the assertion of Theorem 2.3.

Theorem 3.7 (Existence of a rat-independent process) *Let $\frac{1}{\alpha} + \frac{1}{\beta} \le \frac{1}{d} < \frac{1}{n}$. Suppose further that (13)–(16), (19), (20) (22) and (23) hold. Then there exists a process $q\colon [0,T] \to \mathbb{Q}$ with $q(t) = (y(t), \nu(t), z(t), \lambda(t))$ such that q is an energetic solution in the sense of Definition 1.1. The following limit passages are also valid:*
(i) for a t-dependent (not relabelled) subsequence w^*-$\lim \nu_\tau(t) = \nu(t)$ in $L^\infty_{\rm w}(\Omega; \mathcal{M}(\mathbb{R}^{n\times n}))$ for all $t \in [0,T]$,
for a t-dependent (not relabelled) subsequence w-$\lim_{\tau\to0} y_\tau(t) = y(t)$ in $W^{1,d}(\Omega; \mathbb{R}^n)$ for all $t \in [0,T]$
(ii) for a (not relabelled) subsequence w^*-$\lim_{\tau\to0} \lambda_\tau(t) = \lambda(t)$ in $L^\infty(\Omega; \mathbb{R}^{M+1}) \cap W^{1,2}(\Omega; \mathbb{R}^{M+1})$ for all $t \in [0,T]$ and $\lambda \in \mathrm{BV}([0,T]; L^1(\Omega; \mathbb{R}^L))$,
(iii) for a (not relabelled) subsequence $\lim_{\tau\to0} z_\tau(t) = z(t)$ in $\mathbb{X}$ for all $t \in [0,T]$,
(iv) for a (not relabelled) subsequence $\lim_{\tau\to0} \mathcal{I}_\tau(t, q_\tau) = \mathcal{I}(t, q(t))$ for all $t \in [0,T]$, and
(v) for a (not relabelled) subsequence $\lim_{\tau\to0} \mathrm{Var}(\mathcal{D}, q_\tau; [0,t]) = \mathrm{Var}(\mathcal{D}, q; [0,t])$ for all $t \in [0,T]$.

Proof: The proof is divided into several steps and combines ideas from [6] and [11].

Step 1: The points (i), (ii), and (iii) follow from the *a-priori* estimates in Proposition 3.3 and Lemma 3.6 (recall that λ is a volume fraction and thus trivially bounded in L^∞).

Altogether, Step 1 implies the existence of the limit $q(t) = (y(t), \nu(t), \lambda(t), z(t))$. Moreover, (35) implies that $\int_\Omega \int_{\mathbb{R}^{n\times n}} |s|^d \nu(t)(\mathrm{d}s)\,\mathrm{d}x < +\infty$ for all $t \in [0,T]$. As $\nabla y_\tau(t,x) = \int_{\mathbb{R}^{n\times n}} s\nu_{\tau,x}(t)(\mathrm{d}s)$ and $\lambda_\tau(t,x) = \int_{\mathbb{R}^{n\times n}} \mathfrak{L}(s)\nu_{\tau,x}(t)(\mathrm{d}s)$ depend linearly on ν_τ, we immediately get that $q \in \mathbb{Q}$.

We set $S(t,\tau) := \min_{k\in\mathbb{N}\cup\{0\}}\{k\tau \mid k\tau \geq t\}$. Then $\lim_{\tau\to 0} S(t,\tau) = t$; then $q_\tau(t) := q_\tau(S(t,\tau)) \in \mathcal{S}(S(t,\tau))$. Moreover, by our assumptions on $\mathcal{D}$ (Assumption 2.2 and Proposition 3.5), we know that (44) holds. Therefore $q(t) \in S(t)$, i.e., the limit is stable by Proposition 3.4. Proposition 3.4 also implies (iv) .

Step 2: We have $q_\tau(t) = q_\tau(k\tau)$ if $0 \leq t - k\tau \leq \tau$. Hence, using (28) in the first and in the second line, we find that for some $C, C_1 > 0$

$$\begin{aligned}\mathcal{I}\left(t, q_\tau(t)\right) + \mathrm{Var}\left(\mathcal{D}, q_\tau; [0,t]\right) &\leq \mathcal{I}\left(k\tau, q_\tau(k\tau)\right) + \mathrm{Var}\left(\mathcal{D}, q_\tau; [0,k\tau]\right) + C\tau \\ &\leq \mathcal{I}\left(0, q_\tau(0)\right) - \int_0^{k\tau} \dot{L}\left(s, q_\tau(s)\right)\,\mathrm{d}s + C\tau \\ &\leq \mathcal{I}\left(0, q_\tau(0)\right) - \int_0^{t} \dot{L}\left(s, q_\tau(s)\right)\,\mathrm{d}s + C_1\tau\ .\end{aligned}$$

Notice also, $\theta_\tau(t) := \dot{L}\left(t, q_\tau\right)$ is bounded in $L^\infty(0,T)$ by (22) and (23), so that there is a weak* limit of a subsequence (not relabelled), which we denote θ. We set $\theta_{\mathrm{i}}(t) := \liminf_{\tau\to 0}\theta_\tau(t)$. Further, using Lemma 3.6 (i) and the weak lower semicontinuity of the variation we get in the limit $\tau \to 0$

$$\mathcal{I}(t, q(t)) + \delta(t) + \mathrm{Var}\left(\mathcal{D}_{\mathrm{tr}}, q; [0,t]\right) \leq \mathcal{I}(0, q(0)) - \int_0^t \theta(s)\,\mathrm{d}s\ . \tag{45}$$

As $\delta(t) \geq \mathrm{Var}\left(\mathcal{D}_{\mathrm{p}}, q; [0,t]\right)$ by Lemma 3.6 and by Fatou's lemma $\int_0^t \theta(s)\,\mathrm{d}s \geq \int_0^t \theta_{\mathrm{i}}(s)\,\mathrm{d}s$ for a.e. $t \in [0,T]$, we obtain

$$\mathcal{I}(t, q(t)) + \mathrm{Var}\left(\mathcal{D}, q; [0,t]\right) \leq \mathcal{I}(0, q(0)) - \int_0^t \theta_{\mathrm{i}}(s)\,\mathrm{d}s\ .$$

We observe that $\theta_{\mathrm{i}}(s) = \dot{L}(s, q(s))$. Altogether we get the upper energy estimate

$$\mathcal{I}(t, q(t)) + \mathrm{Var}\left(\mathcal{D}, q; [0,t]\right) \leq \mathcal{I}(0, q(0)) - \int_0^t \dot{L}(s, q(s)\,\mathrm{d}s\ . \tag{46}$$

In order to get the lower estimate we exploit the fact that $q(t)$ is stable for all $t \in [0,T]$. Take a (possibly non-uniform) partition of a time interval $[t_1, t_2] \subset [0,T]$ such that $t_1 =$

$\vartheta_0 < \vartheta_1 < \vartheta_2 < \vartheta_K = t_2$ such that $\max_i (\vartheta_i - \vartheta_{i-1}) =: \vartheta \to 0$ as $K \to \infty$. We test the stability of $q(\vartheta_{k-1})$ with $q(\vartheta_k)$, for $k = k_1 + 1, \ldots, k_2$. Analogously to (32) this yields

$$\sum_{k=1}^{K} \left[L(\vartheta_{k-1}, q(\vartheta_k)) - L(\vartheta_k, q(\vartheta_k)) \right] \\ \leq \mathcal{I}(t_2, q(t_2)) - \mathcal{I}(t_1, q(t_1)) + \sum_{k=1}^{K} \mathcal{D}(q(\vartheta_{k-1}), q(\vartheta_k)) \ . \quad (47)$$

Hence,

$$\sum_{k=1}^{K} - \int_{\vartheta_{k-1}}^{\vartheta_k} \dot{L}(s, q(\vartheta_k)) \, \mathrm{d}s \leq \mathcal{I}(t_2, q(t_2)) - \mathcal{I}(t_1, q(t_1)) + \mathrm{Var}(\mathcal{D}, q; [t_1, t_2]) \ . \quad (48)$$

Finally, we observe that

$$\sum_{k=1}^{K} \int_{\vartheta_{k-1}}^{\vartheta_k} \dot{L}(s, q(\vartheta_k)) \, \mathrm{d}s \\ = \sum_{k=1}^{K} \dot{L}(\vartheta_k, q(\vartheta_k)) (\vartheta_k - \vartheta_{k-1}) + \sum_{k=1}^{K} \int_{\vartheta_{k-1}}^{\vartheta_k} \Big(\dot{L}(s, q(\vartheta_k)) - \dot{L}(\vartheta_k, q(\vartheta_k)) \Big) \, \mathrm{d}s \ . \quad (49)$$

The second term on the right-hand side of (49) tends to zero as $\vartheta \to 0$ because the time derivative of external forces is uniformly continuous in time by (22) and (23). The first term on the right-hand side converges to $\int_{t_1}^{t_2} \dot{L}(s, q(s)) \, \mathrm{d}s$ by [4, Lemma 4.12]. Thus, (48) and (49) together yield the lower energy bound

$$- \int_{t_1}^{t_2} \dot{L}(s, q(s)) \, \mathrm{d}s \leq \mathcal{I}(t_2, q(t_2)) - \mathcal{I}(t_1, q(t_1)) + \mathrm{Var}(\mathcal{D}, q; [t_1, t_2]) \ . \quad (50)$$

The upper and lower estimates (46) and (50) combined yield the energy balance

$$\mathcal{I}(t, q(t)) + \mathrm{Var}(\mathcal{D}, q; [0, t]) = \mathcal{I}(0, q(0)) - \int_0^t \dot{L}(s, q(s) \, \mathrm{d}s \ . \quad (51)$$

Step 3: We obtain (the first inequality relies on (50) and the observation $\theta_{\mathrm{i}}(s) = \dot{L}(s, q(s))$ made in Step 2, the second inequality is Lemma 3.6 (3), while the third estimate is (45))

$$\mathcal{I}(0, q(0)) - \int_0^t \theta_{\mathrm{i}}(s) \, \mathrm{d}s \leq \mathcal{I}(t, q(t)) + \mathrm{Var}(\mathcal{D}, q; [0, t]) \\ \leq \mathcal{I}(t, q(t)) + \delta(t) + \mathrm{Var}(\mathcal{D}_{\mathrm{tr}}, q; [0, t]) \\ \leq \mathcal{I}(0, q(0)) - \int_0^t \theta(s) \, \mathrm{d}s \leq \mathcal{I}(0, q(0)) - \int_0^t \theta_{\mathrm{i}}(s) \, \mathrm{d}s \ . \quad (52)$$

Thus, all inequalities in (52) are equalities and consequently we have shown that (v) holds. □

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References

[1] Marcel Arndt. *Upscaling from Atomistic Models to Higher Order Gradient Continuum Models for Crystalline Solids.* Dissertation, Institute for Numerical Simulation, University of Bonn, 2004.

[2] F. Auricchio, A. Reali, and U. Stefanelli. A phenomenological 3D model describing stress-induced solid phase transformations with permanent inelasticity. In *Topics on mathematics for smart systems*, pages 1–14. World Sci. Publ., Hackensack, NJ, 2007.

[3] J. M. Ball. A version of the fundamental theorem for Young measures. In M. Rascle, D. Serre, and M. Slemrod, editors, *PDEs and continuum models of phase transitions (Nice, 1988)*, pages 207–215. Springer, Berlin, 1989.

[4] Gianni Dal Maso, Gilles A. Francfort, and Rodica Toader. Quasistatic crack growth in nonlinear elasticity. *Arch. Ration. Mech. Anal.*, 176(2):165–225, 2005.

[5] O. W. Dillon, Jr. and J. Kratochvíl. A strain gradient theory of plasticity. *Int. J. Solids Struct.*, 6:1513–1533, 1970.

[6] Gilles Francfort and Alexander Mielke. Existence results for a class of rate-independent material models with nonconvex elastic energies. *J. Reine Angew. Math.*, 595:55–91, 2006.

[7] Morton E. Gurtin. On the plasticity of single crystals: free energy, microforces, plastic-strain gradients. *J. Mech. Phys. Solids*, 48(5):989–1036, 2000.

[8] Y. Huo and I. Müller. Nonequilibrium thermodynamics of pseudoelasticity. *Contin. Mech. Thermodyn.*, 5(3):163–204, 1993.

[9] Martin Kružík and Johannes Zimmer. Evolutionary problems in nonreflexive spaces. Accepted for publication, *ESAIM Control Optim. Calc. Var.*

[10] Andreas Mainik and Alexander Mielke. Global existence for rate-independent gradient plasticity at finite strain. WIAS Preprint 1299, 2008.

[11] Alexander Mielke. Evolution of rate-independent systems. In *Evolutionary equations. Vol. II*, Handb. Differ. Equ., pages 461–559. Elsevier/North-Holland, Amsterdam, 2005.

[12] Alexander Mielke and Tomáš Roubíček. A rate-independent model for inelastic behavior of shape-memory alloys. *Multiscale Model. Simul.*, 1(4):571–597 (electronic), 2003.

[13] Alexander Mielke, Florian Theil, and Valery I. Levitas. A variational formulation of rate-independent phase transformations using an extremum principle. *Arch. Ration. Mech. Anal.*, 162(2):137–177, 2002.

[14] T. Roubíček. Models of microstructure evolution in shape memory alloys. In *Nonlinear homogenization and its applications to composites, polycrystals and smart materials*, volume 170 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 269–304. Kluwer Acad. Publ., Dordrecht, 2004.

[15] Amir Sadjadpour and Kaushik Bhattacharya. A micromechanics inspired constitutive model for shape-memory alloys: the one-dimensional case. *Smart Mater. Struct.*, 16(1):S51–S62, 2007.

[16] Maria Elena Schonbek. Convergence of solutions to nonlinear dispersive equations. *Comm. Partial Differential Equations*, 7(8):959–1000, 1982.

[17] P. Thamburaja and L. Anand. Thermo-mechanically coupled superelastic response of initially-textured Ti-Ni sheet. *Acta Mater.*, 51(2):325–338, 2003.

[18] A. Vivet and Ch. Lexcellent. Micromechanical modelling for tension–compression pseudoelastic behavior of AuCd single crystals. *The European Physical Journal Applied Physics*, 4(2):125–132, 1998.

[19] L. C. Young. Generalized curves and the existence of an attained absolute minimum in the calculus of variations. *Comptes Rendus de la Société des Sciences et des Lettres de Varsovie, Classe III*, (30):212–234, 1937.