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Unsteady flow of viscous incompressible fluid with temperature-dependent viscosity due to a rotating disc in presence of transverse magnetic field and heat transfer

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Abstract — This paper studies the effect of a magnetic field and temperature-dependent viscosity on the unsteady flow and heat transfer for a viscous laminar incompressible and electrically conducting fluid due to an impulsively started rotating infinite disc. The unsteady axisymmetric boundary layer equations are solved using three methods, namely, (i) perturbation solution for small time, (ii) asymptotic analysis for large time and (iii) finite difference method together with Keller box elimination technique for intermediate times. The solutions are obtained in terms of local radial skin friction, local tangential skin friction, and local rate of heat transfer at the surface of the disc, for different values of the pertinent parameters: the Prandtl number Pr, the viscosity variation parameter ε and magnetic field parameter m. The computed dimensionless velocity and temperature profiles for $Pr = 0.72$ are shown graphically for different values of ε and m. © 2001 Éditions scientifiques et médicales Elsevier SAS

unsteady flow / incompressible flow / variable viscosity / rotating disc / magnetic field / heat transfer

Nomenclature

r	radial coordinate	m
z	normal coordinate	m
u	radial velocity component	$m \cdot s^{-1}$
v	tangential velocity component	$m \cdot s^{-1}$
w	axial velocity component	$m \cdot s^{-1}$
C_p	specific heat at constant pressure	$J \cdot kg^{-1} \cdot K^{-1}$
f	dimensionless radial velocity function	
g	dimensionless tangential velocity function	
h	dimensionless axial velocity function	
t	time	s
T	temperature in the flow region	K
T_w	surface temperature	K
T_∞	temperature of the ambient fluid	K
B_0	magnetic field	$kg \cdot s^{-2} \cdot A^{-1}$
m	magnetic field parameter	
σ	electrical conductivity	$m^{-2} \cdot kg^{-1} \cdot s^{3} \cdot A^{2}$
α	thermal diffusivity	$m^{2} \cdot s^{-1}$
ε	viscosity variation constant	
$\bar{q}$	rate of heat transfer	$J \cdot s^{-1} \cdot m^{-2}$
q	dimensionless rate of heat transfer	
Pr	Prandtl number	
ρ_∞	density of the fluid	$kg \cdot m^{-3}$
θ	dimensionless temperature function	
ϕ	angular coordinate	
μ	temperature-dependent viscosity	$kg \cdot m^{-1} \cdot s^{-1}$
μ_∞	viscosity in the ambient fluid	$kg \cdot m^{-1} \cdot s^{-1}$
κ	thermal conductivity of the fluid	$J \cdot s^{-1} \cdot m^{-1} \cdot K^{-1}$
η	dimensionless normal distance	
ν	kinematic coefficient of viscosity	$m^{2} \cdot s^{-1}$
Ω	angular velocity	
τ	dimensionless time	$kg \cdot m^{-1} \cdot s^{-2}$
$\bar{\tau}_r$	radial skin friction	$kg \cdot m^{-1} \cdot s^{-2}$
$\bar{\tau}_\phi$	tangential skin friction	
τ_r	dimensionless radial skin friction	
τ_ϕ	dimensionless tangential skin friction	

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1. INTRODUCTION

Rotating disc flow and heat transfer is one of the classical problems of fluid mechanics that has both theoretical and practical value. Heat transfer from a rotating body is of importance for the rotating components of various types of machinery, for example, computer disc drives (Herrero et al. [1]) and gas turbine rotors (Owen and Rogers [2]). In some applications where the rotating object is a candidate for overheating, and limitations exist on the allowable rotational speed, further heat removal is feasible by means of jet impingement. This is also a common cooling technique for some transmission gearing where the mechanism bearings are subject to impingement cooling by a liquid lubricant. The interaction of rotation and impingement creates a complex but powerful flow capable of increasing heat transfer considerably.

The rotating disc problem was first formulated by von Kármán [3]. He showed that the Navier–Stokes equations for steady flow of a viscous incompressible fluid due to an infinite rotating disc can be reduced to a set of ordinary differential equations, and solved them by an approximate integral method. Later, Cochran [4] obtained more accurate results by patching two series expansions. It is found that the disc acts like a centrifugal fan, the fluid near the disc being thrown radially outwards. This in turn impulses an axial flow towards the disc to maintain continuity.

Benton [5] improved Cochran's solutions and extended the hydrodynamics problem to flow starting impulsively from rest. Bödewadt [6] studied the inverse problem of the disc at rest and fluid at infinity rotating with uniform angular velocity. Roger and Lance [7] studied numerically a similar problem with the disc rotating with different angular velocity to that of the surrounding fluid. Stuart [8], following a suggestion made by Batchelor [9], investigated the effect of uniform suction of fluid from the surface of the rotating disc. The effect of suction is essentially one of decreasing both radial and azimuthal components of the velocity and increasing the axial flow towards the disc at infinity. The boundary layer thinned, as a consequence. Ockendon [10] used the asymptotic method to determine the solutions of the problem for small values of suction parameter in the case of a rotating disc in a rotating fluid. Wagner [11] and Millsaps and Pohlhausen [12] determined the heat transfer from a disc with a uniform surface temperature different from that of isothermal surroundings. Later, Sparrow and Gregg [13] obtained the heat transfer from a rotating disc to a fluid for arbitrary Prandtl number. Ostrach and Thornton [14], considering the same isothermal rotating disc, extended their investigation to a fluid with Prandtl number of 0.72 and variable physical properties. Hartnett [15] examined the influence of variation in surface temperature on the heat transfer from a disc rotating in still air, allowing the temperature difference between the disc surface and the fluid at rest to vary as a power function of radius. Free convection flow above a heated rotating horizontal circular disc was investigated by Merkin [16], in the region from the outer edge of the disc to its center, which was previously studied by Zakerullah and Ackroyd [17] for a fluid with variable properties. The later investigation was restricted to the region near the circumference of the disc. The flow and heat transfer between torsionally oscillating disc has been determined by Hossain [18]. Later Hossain and Rahman [19] investigated the flow between two porous rotating discs in presence of a transverse magnetic field. Some interesting effects of magnetic field on the steady flow from a uniformly rotating disc of infinite or finite extent have also been examined by El-Mistikaway et al. [21, 22]. Recently, the transient problem posed by Benton [5] for flow due to an impulsively started infinite rotating disc has been investigated by Attia [20] numerically, with the effects of the transpiration velocity as well as a transverse magnetic field.

In all the above studies, the viscosity of the fluid was assumed to be constant. However, it is known that this physical property may change significantly with temperature, and to predict the flow behavior accurately it may be necessary to take into account viscosity variation for incompressible fluids. Gray et al. [23], and Mehta and Sood [24] showed that, when this effect is included, flow characteristics may be changed substantially compared to the constant viscosity assumption. With this understanding, Kafoussias and Williams [25] and Kafoussias and Rees [26] investigated the effect of temperature-dependent viscosity on mixed convection flow from a vertical plate in the region near the leading edge, using the local nonsimilarity method. Hossain and Munir [27] investigated the problem of mixed convection from a vertical plate, using the perturbation technique in the two extreme flow regimes, namely, the forced convection and the natural convection dominated regimes. Solutions were also obtained in the entire forced–free convection regime employing the finite difference method together with the Keller box elimination technique [28], showing the effect of the viscosity variation by considering viscosity to vary as a linear function of temperature. On the other hand, Hossain and Kabir [29] have investigated the natural convection flow from a vertical wavy surface, with variable viscosity proportional to an inverse linear function of temperature which are appropriate for liquid metals.

The present investigation is concerned with the effect of temperature-dependent viscosity on the flow and heat transfer along a uniformly heated impulsively rotating disc subjected to a transverse magnetic field. Here, although purely laminar flow is considered, the results may have some relevance to turbulent flow, as heat transfer and surface friction will be affected by the flow in the viscous sub-layer very close to the disc surface.

2. MATHEMATICAL FORMALISM

Let the disc lie in the plane $z = 0$ and the space $z \geq 0$ be occupied by homogeneous incompressible, electrically conducting fluid, where z is the vertical axis in the cylindrical coordinate system with r and ϕ as the radial and tangential axes, respectively. The geometry of the problem is shown in *figure 1*.

The disc rotates with uniform angular velocity Ω, B_0 is the externally applied magnetic field in the z direction, T_w is the uniform temperature at the disc surface and T_∞ is the temperature of the ambient fluid. The basic equations governing the flow of the fluid in presence of magnetic field and electromagnetic equations of Maxwell are as follows:

$$\nabla \cdot \mathbf{V} = 0 \tag{1}$$

$$\rho_\infty\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right) = -\nabla p + \nabla\cdot(\mu\nabla\mathbf{V}) + \mathbf{J}\times\mathbf{B} \tag{2}$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V}\times\mathbf{B}) \tag{3}$$

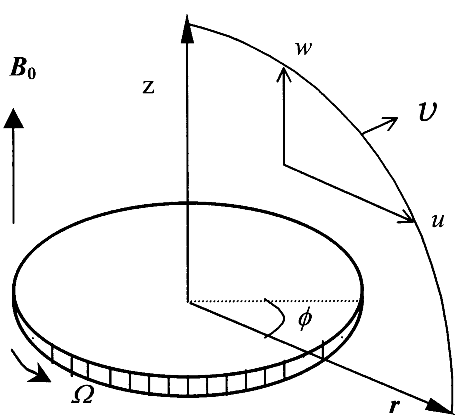


Figure 1. The flow configuration and coordinate system.

$$\nabla\cdot\mathbf{J} = 0, \qquad \nabla\times\mathbf{E} = 0, \qquad \nabla\cdot\mathbf{B} = 0 \tag{4}$$

$$\rho_\infty C_p\left(\frac{\partial T}{\partial t} + (V\cdot\nabla)T\right) = k\nabla^2 T \tag{5}$$

where $\mathbf{V}$ is the velocity field having the radial, tangential and vertical components as u, v, and w, $\mathbf{B}$ is the magnetic induction vector having the components (B_r, B_ϕ, B_z), $\mathbf{E}$ is the electric field vector having the components (E_r, E_ϕ, E_z), $\mathbf{J}$ is the current density vector having the components (J_r, J_ϕ, J_z), C_p is specific heat at constant pressure, ρ_∞ is the fluid density, κ is the thermal conductivity of the fluid, p is the pressure in the flow region, σ is the electric conductivity and μ is the viscosity of the fluid. If the magnetic Reynolds number is small the induced magnetic field can be neglected in comparison with the applied field B_0, which is assumed to be constant in space and time, transversely to the disc. In addition, we assume that the viscosity depends on temperature, i.e. $\mu = \mu_\infty/\{1 + \varepsilon(T - T_\infty)/(T_w - T_\infty)\}$ (see Ling and Libby [30]). All other material functions, such as the fluid density ρ_∞, the thermal conductivity of the fluid κ, the pressure in the flow region p and the electric conductivity σ, are treated as constant.

Further, assume that the Joule and viscous dissipation effects are neglected from the energy equation and flow is unsteady as well as symmetric about the vertical axis z. Finally, we assume a short-circuit problem for which applied electric field $\mathbf{E} = 0$, and, hence, current density $\mathbf{J} = (J_r, J_\phi, 0)$. Thus, equation (3) implies $j_r = \sigma B_0 v$, $j_\phi = -\sigma B_0 u$. Hence, by the usual boundary layer approximation, the basic equations transform into the following form:

$$u_r + \frac{u}{r} + w_z = 0 \tag{6}$$

$$\rho_\infty\left(u_t + uu_r + wu_z - \frac{v^2}{r} + \sigma B_0^2 u\right) = -p_r + \frac{\partial}{\partial r}(\mu u_r) + \frac{\partial}{\partial r}\left(\mu\frac{u}{r}\right) + \frac{\partial}{\partial z}(\mu u_z) \tag{7}$$

$$\rho_\infty\left(v_t + uv_r + \frac{uv}{r} + wv_z + \sigma B_0^2 v\right) = \frac{\partial}{\partial r}(\mu v_r) + \frac{\partial}{\partial r}\left(\mu\frac{v}{r}\right) + \frac{\partial}{\partial z}(\mu v_z) \tag{8}$$

$$\rho_\infty C_p(T_t + uT_r + wT_z) = \kappa\left(T_{rr} + \frac{1}{r}T_r + T_{zz}\right) \tag{9}$$

The boundary conditions for the present problem are:

$$\begin{array}{llll} u, w = 0, & v = r\Omega, & T = T_w & \text{at } z = 0 \\ u, v \to 0, & T = T_\infty & & \text{as } z \to \infty \end{array} \tag{10}$$

To obtain the solutions of the governing equations, these are first converted into a convenient form using appropriate transformations. Considering this, we can introduce the following transformations:

$$\begin{aligned} &u = r\Omega\left(\frac{\tau}{1+\tau}\right) f(\eta,\tau), \qquad v = r\Omega g(\eta,\tau) \\ &w = -4(\nu\Omega)^{1/2}\left(\frac{\tau}{1+\tau}\right)^{3/2} h(\eta,\tau) \\ &\frac{T-T_\infty}{T_w-T_\infty} = \theta(\eta,\tau) \\ &\eta = \frac{1}{2}\sqrt{\frac{\Omega}{\nu}}\left(\frac{\tau}{1+\tau}\right)^{-1/2} z, \qquad \tau = \Omega t \end{aligned} \tag{11}$$

Now substituting the above transformations into equations (6)–(9), the following nonsimilarity equations are obtained:

$$\begin{aligned} &(1+\varepsilon\theta)h''' - \varepsilon\theta' h'' - (1+\varepsilon\theta)^2 \left\{ 4\left(\frac{1}{1+\tau}\right)^2 h' \right. \\ &-2\eta\left(\frac{1}{1+\tau}\right)^2 h'' + 4\left(\frac{\tau}{1+\tau}\right)\frac{\partial h'}{\partial \tau} + 4\left(\frac{\tau}{1+\tau}\right)^2 h'^2 \\ &\left. -8\left(\frac{\tau}{1+\tau}\right)^2 hh'' - 4g^2 + 4\left(\frac{\tau}{1+\tau}\right) mh' \right\} = 0 \end{aligned} \tag{12}$$

$$\begin{aligned} &(1+\varepsilon\theta)g'' - \varepsilon\theta' g' - (1+\varepsilon\theta)^2 \left\{ 4\left(\frac{\tau}{1+\tau}\right)\frac{\partial g}{\partial \tau} \right. \\ &-2\eta\left(\frac{1}{1+\tau}\right)^2 g' + 8\left(\frac{\tau}{1+\tau}\right)^2 h'g \\ &\left. -8\left(\frac{\tau}{1+\tau}\right)^2 hg' + 4\left(\frac{\tau}{1+\tau}\right) mg \right\} = 0 \end{aligned} \tag{13}$$

$$\begin{aligned} &\frac{1}{Pr}\theta'' + 2\eta\left(\frac{1}{1+\tau}\right)^2 \theta' + 8\left(\frac{\tau}{1+\tau}\right)^2 \theta' h \\ &= 4\left(\frac{\tau}{1+\tau}\right)\frac{\partial \theta}{\partial \tau} \end{aligned} \tag{14}$$

The above equations should satisfy the following boundary conditions:

$$\begin{aligned} &h(0,\tau) = h'(0,\tau) = 0, \\ &g(0,\tau) = 1, \qquad \theta(0,\tau) = 1 \\ &h'(\infty,\tau) = g(\infty,\tau) = \theta(\infty,\tau) = 0 \end{aligned} \tag{15}$$

where $Pr\ (= \mu_\infty C_p/\kappa)$ is the Prandtl number, ε is termed the viscosity variation parameter and $m\ (= \sigma B_0^2/\rho_\infty \Omega)$ is the magnetic field parameter. Throughout, prime denotes the differentiation with respect to η.

The present problem, in absence of magnetic field, has recently been investigated by Hossain and Hossain [31].

3. ALL TIME SOLUTION

Since the system of equations (12)–(14) are locally nonsimilar by nature, we may obtain the solution by both the local nonsimilarity method introduced by Sparrow and Minkowycz [32] and the implicit finite difference method together with the Keller box elimination technique [28]. Here we propose to simulate equations (12)–(14) by the finite difference method, since it is found to be efficient and accurate as well-documented and widely used by Cebici and Bradshaw [33], and recently applied by Hossain et al. [27, 29]. According to the aforementioned method, the system of partial differential equations are first converted to a system of seven first-order differential equations by introducing new functions of the η derivatives. This system is then put into a finite difference scheme in which the resulting nonlinear difference equations are linearized by the use of Newton's quasi-linearization method. The resulting linear difference equations, along with the boundary conditions, are finally solved by an efficient block-tridiagonal factorization method introduced by Keller [28].

The action of the viscosity in the fluid adjacent to the disc sets up a tangential shear stress, which opposes the rotation of the disc. As a consequence, it is necessary to provide a torque at the shaft to maintain a steady rotation. To find the tangential shear stress, $\bar{\tau}_\phi$, we apply the Newtonian formula:

$$\bar{\tau}_\phi = \left[\mu\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \phi}\right)\right]_{z=0} \tag{16}$$

There is also a surface shear stress $\bar{\tau}_r$ in the radial direction, which can be obtained by applying the Newtonian formula:

$$\bar{\tau}_r = \left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right]_{z=0} \tag{17}$$

The rate of heat transfer from the disc surface to the fluid is computed by the application of Fourier's law as given below:

$$\bar{q} = -k\left(\frac{\partial T}{\partial z}\right)_{z=0} \tag{18}$$

When the values of the functions h, g and θ are known, using the transformations given in (11), we can calculate the values of dimensionless radial skin friction, tangential skin friction and heat transfer rate, from the following relations:

$$\begin{aligned}\tau_r(1+\varepsilon) &= \left(\frac{\tau}{1+\tau}\right)^{1/2} h''(0,\tau)\\ \tau_\phi(1+\varepsilon) &= \left(\frac{\tau}{1+\tau}\right)^{-1/2} g'(0,\tau)\\ q &= -\left(\frac{\tau}{1+\tau}\right)^{-1/2}\theta'(0,\tau)\end{aligned} \tag{19}$$

4. SMALL TIME SOLUTION

For small time, i.e. when $\tau \ll 1$, then the transformations given in (11) take the following form:

$$\begin{aligned}& u = r\Omega\tau f(\eta,\tau), \qquad v = r\Omega g(\eta,\tau)\\ & w = -4(\nu\Omega)^{1/2}\tau^{3/2}h(\eta,\tau)\\ & \frac{T-T_\infty}{T_w-T_\infty} = \theta(\eta,\tau)\\ & \eta = \tfrac{1}{2}\sqrt{\frac{\Omega}{\nu\tau}}z, \qquad \tau = \Omega t\end{aligned} \tag{20}$$

Introducing the above transformation, the flow governing equations (6)–(9) are reduced to the following non-similarity equations that are valid for small time:

$$\begin{aligned}&(1+\varepsilon\theta)h''' - \varepsilon h''\theta'\\ &\quad = (1+\varepsilon\theta)^2\bigg(4h' - 2\eta h'' + 4\tau^2 h'^2 - 8\tau^2 h'' h\\ &\qquad -4g^2 + 4m\tau h' + 4\tau\frac{\partial h'}{\partial\tau}\bigg) = 0\end{aligned} \tag{21}$$

$$\begin{aligned}&(1+\varepsilon\theta)g'' - \varepsilon g'\theta'\\ &\quad = (1+\varepsilon\theta)^2\bigg(8\tau^2 h' g - 2\eta g'\\ &\qquad -8\tau^2 h g' + 4\tau\frac{\partial g}{\partial\tau} + 4m\tau g\bigg) = 0\end{aligned} \tag{22}$$

$$\frac{1}{Pr}\theta'' + 2\eta\theta' + 8\tau^2\theta' h = 4\tau\frac{\partial\theta}{\partial\tau} \tag{23}$$

The boundary conditions are as follows:

$$\begin{aligned}& h(0,\tau) = h'(0,\tau) = 0,\\ & g(0,\tau) = 1, \qquad \theta(0,\tau) = 1\\ & h'(\infty,\tau) = g(\infty,\tau) = \theta(\infty,\tau) = 0\end{aligned} \tag{24}$$

It may be noticed that equations (21)–(23) are non-similar partial differential equations by nature and of the parabolic type. Since $\tau \ll 1$, we can approximate the perturbation solutions of equations (21)–(23) by treating τ as the perturbation parameter. Hence, the functions h, g and θ can be assumed to be of the following form:

$$\begin{aligned}& h(\eta,\tau) = \sum_{i=0}^{\infty}\tau^i h_i(\eta)\\ & g(\eta,\tau) = \sum_{i=0}^{\infty}\tau^i g_i(\eta) \quad \text{and}\\ & \theta(\eta,\tau) = \sum_{i=0}^{\infty}\tau^i \theta_i(\eta)\end{aligned} \tag{25}$$

where $h_i(\eta)$, $g_i(\eta)$ and $\theta_i(\eta)$ are the functions depending on η.

Now, substituting the expression (25) into the equations (21)–(23) and taking the terms only up to $O(\tau^2)$, gives:

$$\begin{aligned}&(1+\varepsilon\theta_0)h_0''' - \varepsilon h_0'\theta_0'\\ &\quad = (1+\varepsilon\theta_0)^2\big(4h_0' - 2\eta h_0'' - 4g_0^2\big)\end{aligned} \tag{26}$$

$$(1+\varepsilon\theta_0)g_0'' = \varepsilon g_0'\theta_0' - 2\eta(1+\varepsilon\theta_0)^2 g_0' \tag{27}$$

$$\frac{1}{Pr}\theta_0'' + 2\eta\theta_0' = 0 \tag{28}$$

$$\begin{aligned}& h_0(0) = h_0'(0) = 0, \qquad g_0(0) = \theta_0(0) = 1\\ & h_0'(\infty) = 0, \qquad g_0(\infty) = \theta_0(\infty) = 0\end{aligned} \tag{29}$$

$$\begin{aligned}&(1+\varepsilon\theta_0)h_1''' + \varepsilon\big(\theta_1 h_0''' - h_0''\theta_1' - h_1''\theta_0'\big)\\ &\quad -(1+\varepsilon\theta_0)^2\big(8h_1' - 2\eta h'' - 8g_0 g_1 + 4mh_0'\big)\\ &\quad -\big(2\varepsilon\theta_1 + 2\varepsilon^2\theta_0\theta_1\big)\big(4h_0' - 2\eta h_0'' - 4g_0^2\big) = 0\end{aligned} \tag{30}$$

$$\begin{aligned}&(1+\varepsilon\theta_0)g_1'' + \varepsilon\big(\theta_1 g_0'' - g_0'\theta_1' - g_1'\theta_0'\big)\\ &\quad -(1+\varepsilon\theta_0)^2\big(4g_1 - 2\eta g' + 4mg_0\big)\\ &\quad +2\eta\big(2\varepsilon\theta_1 + 2\varepsilon^2\theta_0\theta_1\big)g_0' = 0\end{aligned} \tag{31}$$

$$\begin{aligned}&\frac{1}{Pr}\theta_1'' + 2\eta\theta_1' = 4\theta_1\\ & h_1(0) = h_1'(0) = 0, \qquad g_1(0) = \theta_1(0) = 0\\ & h_1'(\infty) = 0, \qquad g_1(\infty) = \theta_1(\infty) = 0\end{aligned} \tag{32}$$

$$\begin{aligned}&(1+\varepsilon\theta_0)h_2''' + \varepsilon\big(\theta_2 h_0''' + \theta_1 h_1''' - h_0''\theta_2' - h_1''\theta_1' - h_2''\theta_0'\big)\\ &\quad -(1+\varepsilon\theta_0)^2\big(12h_2' - 2\eta h_2'' + 4h_0'^2\\ &\quad -8h_0''h_0 - 8g_0 g_2 - 4g_1^2 + 4mh_1'\big)\\ &\quad -\big(2\varepsilon\theta_1 + 2\varepsilon^2\theta_0\theta_1\big)\big(8h_1' - 2\eta h_1'' - 8g_0 g_1 + 4mh_0'\big)\end{aligned}$$

$$-\{2\varepsilon\theta_2+\varepsilon^2(2\theta_0\theta_2+\theta_1^2)\}(4h_0'-2\eta h_0''-4g_0^2)=0 \tag{33}$$

$$\begin{aligned}&(1+\varepsilon\theta_0)g_2''+\varepsilon(\theta_2 g_0''+\theta_1 g_1''-g_0'\theta_2'-g_1'\theta_1'-g_2'\theta_0')\\&\quad-(1+\varepsilon\theta_0)^2(8g_2-2\eta g_2'+8h_0'g_0-8g_0'h_1+4mg_1)\\&\quad-(2\varepsilon\theta_1+2\varepsilon^2\theta_0\theta_1)(4g_1-2\eta g'+4mg_0)\\&\quad+2\eta\{2\varepsilon\theta_2+\varepsilon^2(2\theta_0\theta_2+\theta_1^2)\}g_0'=0\end{aligned} \tag{34}$$

$$\frac{1}{Pr}\theta_2''+2\eta\theta_2'-8\theta_0'h_0=8\theta_2 \tag{35}$$

$$\begin{aligned}&h_2(0)=h_2'(0)=0, && g_2(0)=\theta_2(0)=0\\&h_2'(\infty)=0, && g_2(\infty)=\theta_2(\infty)=0\end{aligned} \tag{36}$$

It can be seen that equations (26) and (27) are coupled and nonlinear by nature for the case of a fluid possessing variable viscosity ($\varepsilon \neq 0.0$) and the solution of which is not possible analytically. A similar situation prevails for the subsequent sets. Numerical solutions of equations (26)–(29) are obtained using the Nachtshiem–Swigert iteration [34] together with the sixth-order implicit Runge–Kutta–Butcher [35] initial value solver. Solutions of the subsequent sets of equations (30)–(36) are also obtained by the above method for different values of the pertinent parameters.

Once the values of the functions h_n, g_n and θ_n for $n = 0, 1, 2, \ldots$ and their derivatives are known, the values of dimensionless radial skin friction τ_r, tangential skin friction τ_ϕ and heat transfer rate q can easily be obtained, from the expressions given below:

$$\begin{aligned}&\tau_r(1+\varepsilon)=\tau^{1/2}h''(0,\tau)\\&\tau_\phi(1+\varepsilon)=\tau^{-1/2}g'(0,\tau)\\&q=-\tau^{-1/2}\theta'(0,\tau)\end{aligned} \tag{37}$$

5. LARGE TIME SOLUTION

When $\tau \gg 1$ then the transformations given in (11) reduce to the following form:

$$\begin{aligned}&u=r\Omega f(\eta,\tau), \qquad v=r\Omega g(\eta,\tau)\\&w=-4(\nu\Omega)^{1/2}h(\eta,\tau)\\&\frac{T-T_\infty}{T_w-T_\infty}=\theta(\eta,\tau)\\&\eta=\frac{1}{2}\sqrt{\frac{\Omega}{\nu}}z, \qquad \tau=\Omega t\end{aligned} \tag{38}$$

Using the above transformation, the flow governing equations (6)–(9) take the following form:

$$\begin{aligned}&(1+\varepsilon\theta)h'''-\varepsilon\theta'h''-(1+\varepsilon\theta)^2\\&\quad\cdot\left(4h'^2-4g^2-8h''h+4\frac{\partial h'}{\partial\tau}+4mh'\right)=0\end{aligned} \tag{39}$$

$$\begin{aligned}&(1+\varepsilon\theta)g''-\varepsilon\theta'g'\\&\quad-(1+\varepsilon\theta)^2\left(8h'g-8hg'+4\frac{\partial g}{\partial\tau}+4mg\right)=0\end{aligned} \tag{40}$$

$$\frac{1}{Pr}\theta''+8\theta'h=4\frac{\partial\theta}{\partial\tau} \tag{41}$$

The boundary conditions to be satisfied by the above equations are:

$$\begin{aligned}&h(0,\tau)=h'(0,\tau)=0,\\&g(0,\tau)=1, \qquad \theta(0,\tau)=1\\&h'(\infty,\tau)=g(\infty,\tau)=\theta(\infty,\tau)=0\end{aligned} \tag{42}$$

At the steady state situation the τ-derivative in the equations (39)–(41) can be neglected. Hence,

$$\begin{aligned}&(1+\varepsilon\theta)h'''-\varepsilon\theta'h''\\&\quad-(1+\varepsilon\theta)^2(4h'^2-4g^2-8h''h+4mh')=0\end{aligned} \tag{43}$$

$$\begin{aligned}&(1+\varepsilon\theta)g''-\varepsilon\theta'g'\\&\quad-(1+\varepsilon\theta)^2(8h'g-8hg'+4mg)=0\end{aligned} \tag{44}$$

$$\frac{1}{Pr}\theta''+8\theta'h=0 \tag{45}$$

and the boundary conditions become:

$$\begin{aligned}&h(0)=h'(0)=0,\\&g(0)=1, \qquad \theta(0)=1\\&h'(\infty)=g(\infty)=\theta(\infty)=0\end{aligned} \tag{46}$$

The solutions of the above sets of equations are obtained using the methods adopted in the preceding section. As before, once the values of the functions h, g and θ are known, we can calculate the values of dimensionless radial skin friction, tangential skin friction and heat transfer rate from the following relations:

$$\begin{aligned}&\tau_r(1+\varepsilon)=h''(0)\\&\tau_\phi(1+\varepsilon)=g'(0)\\&q=-\theta'(0)\end{aligned} \tag{47}$$

6. RESULTS AND DISCUSSION

Numerical simulations were carried out for the motion of a fluid having Prandtl number Pr equal to 0.72 (suitable for air), while the viscosity variation parameter $\varepsilon = 0.0$, 2.0 and 4.0 and magnetic field parameter $m =$ 0.0, 1.0, 2.0 and 3.0. The results are presented in terms of nondimensional local radial skin friction and tangential skin friction, as well as the rate of heat transfer at the disc surface, against the time-dependent parameter τ.

The growth of the axial velocity component against time is depicted in *figure 2*: (a) for $\varepsilon = 0.0$ and (b) for $\varepsilon = 2.0$, for different values of magnetic field parameter $m = 0.0$, 1.0, 2.0 and 3.0 and for $Pr = 0.72$. In *figure 2(a)*, for constant viscosity ($\varepsilon = 0.0$), it can be seen that increasing the value of the magnetic field parameter m leads to a reduction in the (negative) axial velocity towards the disc, from -0.88 to -0.061 in the steady state. In *figure 2(b)* axial velocity reduces from -0.59 to -0.036 with increase in m for $\varepsilon = 2.0$. For the uniform viscosity case, the magnetic field has more apparent effect on the axial velocity than in the case of variable viscosity (i.e. $\varepsilon \neq 0.0$). Similar behaviour was observed by Attia [20].

Numerical values of the local rate of heat transfer, for $\varepsilon = 0.0$ and $\varepsilon = 2.0$ and different values of m for the fluid with $Pr = 0.72$, are displayed in *table I*. From *table I* it can be seen that the effect of magnetic field parameter m on the heat transfer rate in the small time dominated regime is negligible for both values of the viscosity variation parameter ε. At the large time dominated regime, the imposed magnetic field on the flow affects the rate of heat transfer significantly for both uniform and variable viscosity. An increase in the viscosity variation parameter ε leads to a decrease in the values of local heat transfer rate for each values of m far from the disc surface.

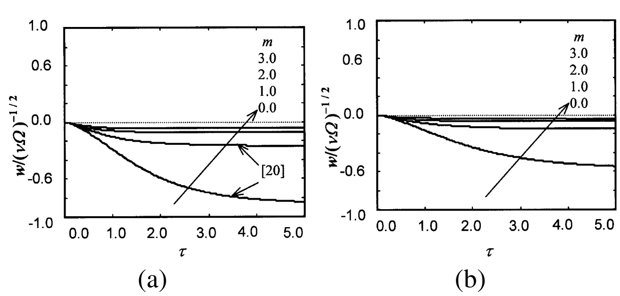


Figure 2. Time development of the axial velocity for (a) $\varepsilon = 0.0$ and (b) $\varepsilon = 2.0$ at infinity with different values of the magnetic field parameter m.

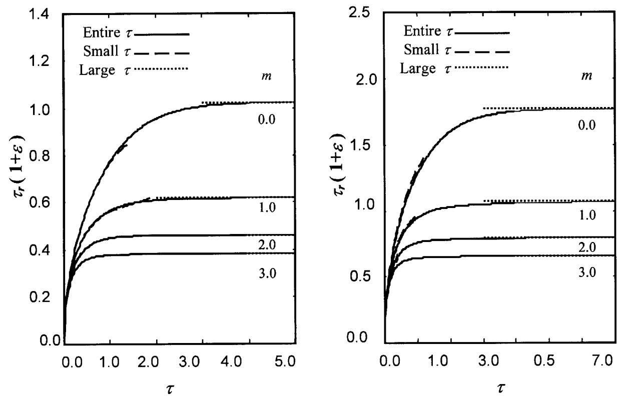
The perturbation solutions for small τ, asymptotic solutions for large τ and the finite difference solutions for the entire τ regime are illustrated in *figures 3* and *4* for comparison. Comparison between these solutions shows excellent agreement in the respective regimes, i.e. for small and large times.

The effects of magnetic field parameter $m = 0.0$, 1.0, 2.0 and 3.0 on the dimensionless radial skin friction, as well as tangential skin friction, for the fluid with $Pr = 0.72$, are depicted in *figures 3(a), 4(a)* and *3(b), 4(b)* for $\varepsilon = 0.0$ and $\varepsilon = 2.0$, respectively. From *figure 3*, it can be seen that an increasing value of m causes a decrease in radial skin friction, whereas the radial skin friction increases monotonically as τ increases and eventually reaches a constant value. Further inspection of *figure 3* reveals the fact that the approach of the radial

TABLE I
Numerical values of the local rate of heat transfer q obtained by different methods for $Pr = 0.72$ with $\varepsilon = 0.0$ and 2.0 and the magnetic field parameter $m = 0.0$ and 1.0.

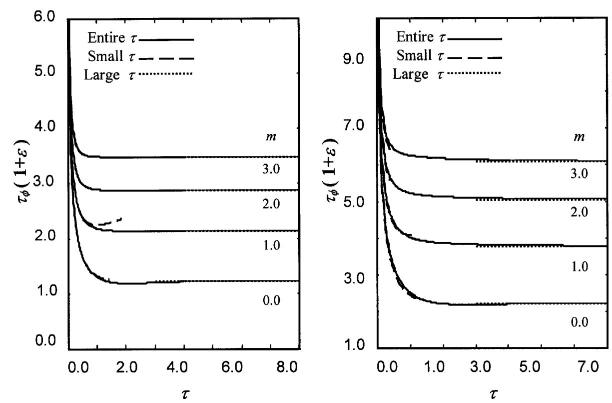
τ	$\varepsilon = 0.0$		$\varepsilon = 2.0$	
	Series & asymptotic	Keller box	Series & asymptotic	Keller box
		$m = 0.0$		
0.01	0.9591^s	0.95918	0.95910^s	0.95918
0.10	3.03443^s	3.04144	3.03428^s	3.04128
0.20	2.14881^s	2.15348	2.14840^s	2.15299
0.30	1.75878^s	1.76228	1.75802^s	1.76131
0.40	1.52834^s	1.53106	1.52716^s	1.52949
0.50	1.37295^s	1.37507	1.37132^s	1.37277
0.60	1.25999^s	1.26156	1.25784^s	1.25843
0.70	1.17381^s	1.17482	1.17110^s	1.17077
0.80	1.10586^s	1.10627	1.10255^s	1.10120
0.90	1.05102^s	1.05073	1.04706^s	1.04457
1.00	1.00599^s	1.00489	1.00136^s	0.99757
∞	0.65719^a	0.65719	0.57543^a	0.57543
		$m = 1.0$		
0.0001	95.91019^s	95.9186	95.91019^s	95.9186
0.10	3.03443^s	3.04138	3.03428^s	3.04122
0.20	2.14881^s	2.15312	2.14840^s	2.15264
0.30	1.75878^s	1.76131	1.75802^s	1.76039
0.40	1.52834^s	1.52914	1.52716^s	1.52769
0.50	1.37295^s	1.37184	1.37132^s	1.36978
0.60	1.25999^s	1.25665	1.25784^s	1.25393
0.70	1.17381^s	1.16788	1.17110^s	1.16447
0.80	1.10586^s	1.09696	1.10255^s	1.09281
0.90	1.05102^s	1.03873	1.04706^s	1.03384
1.00	1.00599^s	0.98990	1.00136^s	0.98426
∞	0.33580^a	0.33580	0.27330^a	0.27330

^{a} for large τ and ^{s} for small τ.



(a) (b)

Figure 3. Nondimensional radial skin friction $\tau_r(1+\varepsilon)$ against τ for different values of magnetic field parameter m, while $Pr = 0.72$: (a) $\varepsilon = 0.0$ and (b) $\varepsilon = 2.0$.



(a) (b)

Figure 4. Nondimensional tangential skin friction $\tau_\phi(1+\varepsilon)$ against τ for different values of magnetic field parameter m, while $Pr = 0.72$: (a) $\varepsilon = 0.0$ and (b) $\varepsilon = 2.0$.

skin friction to the asymptotic state becomes slower as ε increases.

From *figure 4*, it is observed that the tangential skin friction decreases with a decrease in the values of magnetic field parameter m. Further, it may be seen that the tangential skin friction decreases monotonically as τ increases up to a steady-state value. Increasing value of ε causes slower approach of tangential skin friction to the asymptotic state.

The effect of increase in the viscosity variation parameter, $\varepsilon = 0.0$, 2.0 and 4.0, for different values of magnetic field parameter m on the dimensionless velocity and temperature profiles as function of η is depicted in *figure 5*. From *figure 5(a)*, it can be seen that an increase in the value of ε leads to a decrease in values of dimensionless radial velocity. Further inspection of *figure 5(a)* reveals that an increase in the magnetic field parameter m causes a significant decrease in radial velocities and thinning of the momentum boundary layer. In the absence of a magnetic field ($m = 0.0$) and for constant viscosity the maximum radial velocity appears at $\eta = 0.45$, whereas for $m = 1.0$ the maximum radial velocity appears at $\eta = 0.30$, i.e. the point of maximum radial velocity moves closer to the surface of the disc. In *figure 5(b)*, we see that an increase in ε also leads to a decrease in tangential velocities in the boundary layer for both values of the magnetic field parameter ($m = 0.0$ and 1.0). In *figure 5(c)*, the nondimensional (negative) axial velocity decreases negatively from 0.22 to 0.11 with increase in the value of ε for $m = 0.0$. In presence of a magnetic field parameter ($m = 1.0$) the axial velocity more rapidly approaches the steady state situation with increase in the viscosity variation ε. From *figure 5(d)*, it may be observed that temperature profiles and the thermal bound-

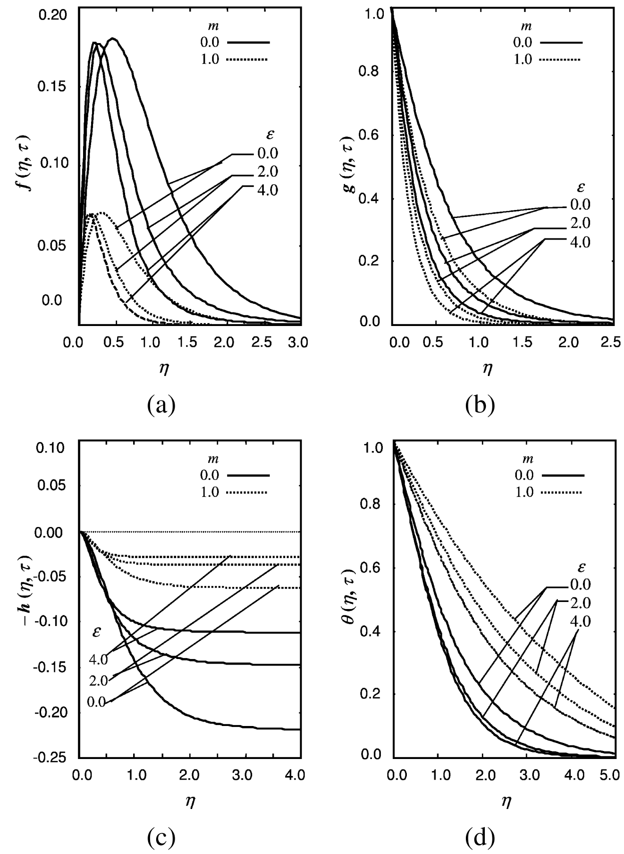


Figure 5. The dimensionless (a) radial velocity profile $f(\eta,\tau)$, (b) tangential velocity profile $g(\eta,\tau)$, (c) axial velocity profile $-h(\eta,\tau)$ and (d) temperature profile $\theta(\eta,\tau)$ against η for different values of $\varepsilon = 0.0$, 2.0, 4.0 and with $m = 0.0$, 1.0 for $Pr = 0.72$.

ary layer thickness increase with increasing value of ε for both $m = 0.0$ and $m = 1.0$.

7. CONCLUSIONS

In this paper, the effects of imposed magnetic field and temperature-dependent viscosity on the behaviour of unsteady flow of an incompressible, viscous and electrically conducting fluid due to an impulsively started rotating disc have been investigated. The local nonsimilarity equations governing the unsteady flow and heat transfer are developed for small time and large time regimes as well as in the entire time regime. Different solution methodologies have been employed for the complete integration of the resulting nonsimilarity equations, namely, (i) perturbation solutions, (ii) asymptotic solutions and (iii) implicit finite difference methods with the Keller box elimination technique, for small time and large time regimes as well as in the entire time regime, as appropriate.

From the present investigation, we can draw the following conclusions:

1. The solutions obtained for the cases of small time regime and large time regime are found to be in excellent agreement with that for the entire time regime, at every selected value of the magnetic field parameter m over the range of $0 \leq \tau \leq 8$ with $\varepsilon = 0.0$ and 2.0 and $Pr = 0.72$.

2. At the surface of the disc, the local radial skin friction increases, whereas the local tangential skin friction decreases with increasing values of the time dependent rotating parameter τ in both the presence and absence of a magnetic field until the steady flow limit is reached.

3. Increasing the value of the viscosity variation parameter $\varepsilon = 0.0$, 2.0. and 4.0 leads to decrease in the values of radial and tangential velocity profile in both the presence and absence of magnetic field for the fixed value of Prandtl number $Pr = 0.72$.

4. Increasing the viscosity variation parameter $\varepsilon =$ 0.0, 2.0. and 4.0 reduces the axial velocity towards the disc surface for both cases $m = 0.0$ and $m = 1.0$.

5. The effect of increasing the value of the viscosity variation parameter $\varepsilon = 0.0$, 2.0. and 4.0 on the dimensionless radial, tangential and axial velocity profiles is to reduce the momentum boundary layer thickness, for both cases when $m = 0.0$ and 1.0 for Prandtl number equal to 0.72.

6. As the value of the viscosity variation parameter ε increases, values of dimensionless temperature also increase for both cases $m = 0.0$ and 1.0. This effect causes a small increase in the thermal boundary layer thickness in the absence of a magnetic field, but for $m = 1.0$ there is a greater increase in the thermal boundary layer thickness for Prandtl number $Pr = 0.72$.

7. The effect of magnetic field parameter $m = 0.0$ and 1.0 on the local rate of heat transfer is negligible near the disc surface for $\varepsilon = 0.0$ and 2.0, but at the outer edge of the disc surface significant effects are found on the heat transfer rate for both values of m, again for $Pr = 0.72$.

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