On 4-Engel groups

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1 Introduction

We will be using Heineken's notation in [10] and will denote the commutator of two group elements a and b, $a^{-1}b^{-1}ab$, by $a \circ b$.

A group G is an Engel group if for each ordered pair (x, y) of elements in G there is a positive integer $n(x, y)$ such that

$$\underbrace{x \circ (\cdots (x \circ (x}_{n(x,y)} \circ y))) = 1. \qquad (1)$$

We will be using bracketing from right. But since

$$(((y \circ \underbrace{x) \circ x) \cdots) \circ x}_{m} = (\underbrace{x^{-1} \circ (\cdots (x^{-1} \circ (x^{-1}}_{m} \circ y))))^{x^m}$$

it does not matter whether we use bracketing from right or from left in the definition.

The origin of Engel groups lies in the theory of Lie algebras. In fact they are a group theoretic analog of Engel Lie algebras. It is therefore hardly surprising that many results in the theory of Engel Lie algebras can be translated into theorems on Engel groups. As an example, one of the basic classical results for Engel Lie algebras is Engel's Theorem. It states that every finite dimensional Engel Lie algebra over a field is nilpotent. In 1936 Zorn [19] proved a corresponding theorem for Engel groups.

Zorn's Theorem. A finite Engel group is nilpotent.

So for finite groups the Engel condition is equivalent to nilpotency, but it is much weaker in general. It is clear from the definition that every locally nilpotent group is an Engel group. But the converse does not hold in general since Golod[3] has constructed finitely generated Engel groups that are not nilpotent. For some special classes of groups we do have that the Engel condition is equivalent to local nilpotency. Gruenberg[5] showed that this is true for soluble groups and in [2] Baer shows that this also holds for groups with the maximal condition.

Now suppose $n = n(x, y)$ in (1) can be chosen independently of x and y. We then say that G is an n-Engel group. Now turn back for a moment to Engel Lie algebras. We have the following two results of E. I. Zel'manov.

Theorem Z1.[15] Every n-Engel Lie algebra over a field k with $\operatorname{char} k = 0$ is nilpotent.

Theorem Z2.[17,18] An n-Engel Lie algebra over an arbitrary field is locally nilpotent.

Now consider the corresponding statements for Engel groups.

Q1. Is every torsion free n-Engel group nilpotent?

Q2. Is every n-Engel group locally nilpotent?

The answer to both of these questions is known to be positive for $n \leq 3$ but the questions remain open for $n \geq 4$. This is of course obvious for $n = 1$, since 1-Engel groups are exactly the abelian groups. In 1942 Levi[12] solved the problem for $n = 2$. In fact he proved that a group G is a 2-Engel group if and only if the normal closure x^G of an arbitrary element x is abelian. Furthermore we have that every 2-Engel group is nilpotent of class at most 3. For 3-Engel groups the problem is much harder. Heineken[10] gave a positive answer for both questions in 1961. He proved that every 3-Engel group G is nilpotent of class at most 4 if G has no elements of order 2 or 5. There are 3-Engel 2-groups and 5-groups that are not nilpotent. In fact there is a 3-Engel 5-group that is not soluble [1] but N. Gupta[6] has shown that 3-Engel 2-groups are soluble. In 1972 L. Kappe and W. Kappe[11] gave a characterization of 3-Engel groups which is analogous to Levi's theorem on 2-Engel groups. They showed that the following are equivalent:

(1) G is a 3-Engel group,

(2) x^G is a 2-Engel group for all $x \in G$,

(3) For all $x \in G$ we have that x^G is nilpotent of class at most 2.

It follows from property 3 that a 3-Engel group with r generators has nilpotency class at most $2r$. We do not have a corresponding characterization for 4-Engel groups. N. Gupta and F. Levin[7] have constructed a 4-Engel group with an element x such that the nilpotency class of x^G is greater than 3. In [1] it is shown that for all $n \geq 2$ there is a r-generator 3-Engel group which is nilpotent of class at least $2r - 1$. Then in [8] Newman and Gupta show that $2r - 1$ is the correct upper bound when $n \geq 2$. In that paper they also get some further information about the structure of 3-Engel groups.

In this paper we will be looking at 4-Engel groups. Our main results are the following.

Theorem 1 *Let G be a 4-Engel group. The torsion elements form a subgroup $\tau(G)$ and $\tau(G)/Z(\tau(G))$ is a direct product of p-groups.*

Remark. Notice that it does not follow that 4-Engel torsion groups are locally nilpotent since there are infinite p-groups which are not locally nilpotent. But this theorem implies that 4-Engel groups are locally nilpotent if and only if torsion free 4-Engel groups are locally nilpotent and for each prime p we have that 4-Engel p-groups are locally finite.

Theorem 2 *Let G be a 4-Engel group and let $rad(G)$ be the locally nilpotent radical of G. If G is a 2-group or 3-group then it is locally finite. If p is a prime and p is not 2 or 3 then $G/rad(G)$ has exponent dividing p.*

Remark. So the question whether 4-Engel torsion groups are locally nilpotent reduces to the "Burnside problem for 4-Engel groups": for what primes

p are all 4-Engel groups of exponent p locally finite.

If we assume that G is locally nilpotent then we can use results on 4-Engel Lie algebras to get the following theorem.

Theorem 3 *Let G be a locally nilpotent 4-Engel group. If G is a p-group where $p \neq 2, 3, 5$ then G is nilpotent of class at most 7. If G is a 5-group then every r-generator subgroup has class at most $6r$.*

2 A reduction theorem

Let G be a 4-Engel group. We have

$$\begin{aligned} 1 &= a \circ (a \circ (a \circ (a \circ b))) \\ &= a \circ [(a \circ (a \circ b))^{-a} \cdot (a \circ (a \circ b))] \\ &= (a \circ (a \circ b))^{-a} \cdot (a \circ (a \circ b))^{a^2} \cdot (a \circ (a \circ b))^{-a} \cdot (a \circ (a \circ b)) \\ &= (a \circ b)^{-a}(a \circ b)^{a^2}(a \circ b)^{-a^3}(a \circ b)^{a^2}(a \circ b)^{-a}(a \circ b)^{a^2}(a \circ b)^{-a}(a \circ b) \end{aligned}$$

which implies that

$$(a \circ (a \circ b))^{a^2} = (a \circ (a \circ b))^{a} \cdot (a \circ (a \circ b))^{-1} \cdot (a \circ (a \circ b))^{a} \tag{2}$$

$$(a \circ b)^{a^3} = (a \circ b)^{a^2}(a \circ b)^{-a}(a \circ b)^{a^2}(a \circ b)^{-a}(a \circ b)(a \circ b)^{-a}(a \circ b)^{a^2}. \tag{3}$$

Also

$$\begin{aligned} 1 &= (((b \circ a) \circ a) \circ a) \circ a \\ &= [((a \circ b)(a \circ b)^{-a})] \circ a] \circ a \\ &= [(a \circ b)^{a}(a \circ b)^{-1}(a \circ b)^{a}(a \circ b)^{-a^2}] \circ a \\ &= (a \circ b)^{a^2}(a \circ b)^{-a}(a \circ b)(a \circ b)^{-a}(a \circ b)^{a^2}(a \circ b)^{-a}(a \circ b)^{a^2}(a \circ b)^{-a^3} \end{aligned}$$

and therefore

$$(a \circ b)^{a^3} = (a \circ b)^{a^2}(a \circ b)^{-a}(a \circ b)(a \circ b)^{-a}(a \circ b)^{a^2}(a \circ b)^{-a}(a \circ b)^{a^2}. \tag{4}$$

Note that $(((b \circ a) \circ a) \circ a) \circ a = (a^{-1} \circ (a^{-1} \circ (a^{-1} \circ (a^{-1} \circ b))))^{a^4}$. So G satisfies the left normed Engel identity if and only if it satisfies the right normed Engel identity.

From (3) and (4) we have

$$(a \circ b)^{a^2}(a \circ b)^{-a}(a \circ b) = (a \circ b)(a \circ b)^{-a}(a \circ b)^{a^2}. \tag{5}$$

But this is equivalent to (easily checked)

$$(a \circ (a \circ b))^a \cdot (a \circ (a \circ b)) = (a \circ (a \circ b))(a \circ (a \circ b))^a$$

which can also be written

$$(a \circ a^b)^a \cdot (a \circ a^b) = (a \circ a^b) \cdot (a \circ a^b)^a. \tag{6}$$

If we replace a by a^b and b by b^{-1}, we have

$$(a \circ a^b)^{a^b} \cdot (a \circ a^b) = (a \circ a^b) \cdot (a \circ a^b)^{a^b}. \tag{7}$$

So we have proved

Lemma 1 *For all $a, b \in G$ we have that $a \circ a^b$ commutes with $(a \circ a^b)^a$ and $(a \circ a^b)^{a^b}$.*

From (2) and (6) we have

$$(a \circ a^b)^{a^2} = (a \circ a^b)^{2a-1}, \quad (a \circ a^b)^{a^{2b}} = (a \circ a^b)^{2a^b-1}. \tag{8}$$

This also implies that

$$(a \circ a^b)^{a^{-1}} = (a \circ a^b)^{-a+2}, \quad (a \circ a^b)^{a^{-b}} = (a \circ a^b)^{-a^b+2}. \tag{9}$$

Lemma 2 *$\langle a, a^b\rangle'$ is generated by $(a \circ a^b)$, $(a \circ a^b)^a$, $(a \circ a^b)^{a^b}$ and $(a \circ a^b)^{aa^b}$.*

Proof Since $\langle a, a^b\rangle'$ is the normal closure of $(a \circ a^b)$ in $\langle a, a^b\rangle$, it is sufficient to show that the group generated by $(a \circ a^b)$, $(a \circ a^b)^a$, $(a \circ a^b)^{a^b}$ and $(a \circ a^b)^{aa^b}$ is normal in $\langle a, a^b\rangle$. But this follows form (8), (9) and the following relations.

$$\begin{aligned}
(a \circ a^b)^{aa^{-b}} &= (a \circ a^{-b})^{-1}(a \circ a^b)^{a^{-b}a}(a \circ a^{-b}) \\
&= (a \circ a^b)^{a^{-b}}(a \circ a^b)^{-a^b a}(a \circ a^b)^{2a}(a \circ a^b)^{-a^{-b}}
\end{aligned}$$

$$
\begin{aligned}
&= (a\circ a^b)^{a^{-b}}(a\circ a^b)(a\circ a^b)^{-aa^b}(a\circ a^b)^{-1}(a\circ a^b)^{2a}(a\circ a^b)^{-a^{-b}},\\
(a\circ a^b)^{a^ba} &= (a\circ a^b)(a\circ a^b)^{aa^b}(a\circ a^b)^{-1},\\
(a\circ a^b)^{a^ba^{-1}} &= (a^{-1}\circ a^b)(a\circ a^b)^{a^{-1}a^b}(a^{-1}\circ a^b)^{-1}\\
&= (a\circ a^b)^{-a^{-1}}(a\circ a^b)^{-aa^b}(a\circ a^b)^{2a^b}(a\circ a^b)^{a^{-1}},\\
(a\circ a^b)^{aa^ba} &= (a\circ a^b)(a\circ a^b)^{a^2a^b}(a\circ a^b)^{-1}\\
&= (a\circ a^b)(a\circ a^b)^{2aa^b}(a\circ a^b)^{-a^b}(a\circ a^b)^{-1},\\
(a\circ a^b)^{aa^ba^{-1}} &= (a^{-1}\circ a^b)(a\circ a^b)^{a^b}(a^{-1}\circ a^b)^{-1}\\
&= (a\circ a^b)^{-a^{-1}}(a\circ a^b)^{a^b}(a\circ a^b)^{a^{-1}},\\
(a\circ a^b)^{aa^ba^b} &= (a\circ a^{2b})^{-1}(a\circ a^b)^{a^{2b}a}(a\circ a^{2b})\\
&= (a\circ a^b)^{-a^b-1}(a\circ a^b)^{2a^ba}(a\circ a^b)^{-a}(a\circ a^b)^{1+a^b}\\
&= (a\circ a^b)^{-a^b}(a\circ a^b)^{2aa^b}(a\circ a^b)^{-1}(a\circ a^b)^{-a}(a\circ a^b)^{1+a^b}. \quad \square
\end{aligned}
$$

Now $(a\circ a^b)^{\langle a\rangle}$ is an abelian group generated by $(a\circ a^b)$ and $(a\circ a^b)^a$ and then

$$
\begin{aligned}
a^n\circ(a^m\circ(a^r\circ(a\circ b))) &= a^n\circ(a^m\circ[(a\circ(a\circ b))^{(a^{r-1}+\cdots+a+1)}])\\
&= (a\circ a^b)^{(-a^m+1)(-a^n+1)(a^{r-1}+\cdots+1)}.
\end{aligned}
$$

But $(-a^m+1)(-a^n+1)$ is divisible by $(-a+1)^2$ in $\mathbb{Z}\langle a\rangle$ and since $(a\circ a^b)^{(-a+1)^2} = a\circ(a\circ(a\circ(a\circ b))) = 1$ we have

Lemma 3 *$a^n\circ(a^m\circ(a^r\circ(a\circ b))) = 1$ for all $n,m,r\in\mathbb{N}$.*

We have using (8)

$$(a^3\circ a^b) = (a\circ a^b)^{a^2+a+1} = (a\circ a^b)^{3a}.$$

Similarly $(a \circ a^{3b}) = (a \circ a^b)^{3a^b}$ and thus

$$(a^3 \circ a^b) = (a \circ a^b)^{3a}, \quad (a \circ a^{3b}) = (a \circ a^b)^{3a^b}. \tag{10}$$

Lemma 4 *For all $a, b \in G$ we have that $(a \circ a^b)^{aa^b}$ commutes with $(a \circ a^b)^a$ and $(a \circ a^b)^{a^b}$.*

Proof By Lemma 1 we have that $(a \circ a^b)^a$ commutes with $(a \circ a^b)$. Therefore $(a \circ a^b)^{aa^b}$ commutes with $(a \circ a^b)^{a^b}$. Similarly $(a \circ a^b)^a$ commutes with $(a \circ a^b)^{a^b a}$. But

$$(a \circ a^b)^{a^b a} = (a \circ a^b)^{aa^b(a^b \circ a)} = (a \circ a^b) \cdot (a \circ a^b)^{aa^b} \cdot (a \circ a^b)^{-1}$$

and since $(a \circ a^b)^a$ commutes with $(a \circ a^b)$ it commutes with $(a \circ a^b)^{aa^b}$. $\square$

In the following calculations, one must be careful with notation. As usually $u^{g_1+g_2}$ is a shorthand notation for $u^{g_1} \cdot u^{g_2}$. This means that $u^{(g_1+g_2)(h_1+h_2)} = u^{(g_1+g_2)h_1} \cdot u^{(g_1+g_2)h_2}$ which does not have to be equal to $u^{g_1(h_1+h_2)} \cdot u^{g_2(h_1+h_2)}$. We also have that $u^{(g_1+g_2)(-h)} = ((u^{g_1} \cdot u^{g_2})^{-1})^h$ which is equal to $u^{-g_2h-g_1h}$. This does not have to be same as $u^{-g_1h-g_2h}$.

Suppose that a has a finite order not divisible by 2. Then $\langle a \rangle$ is generated by a^2. Using Lemma 3 with a replaced by a^2 and also Lemmas 1 and 4 and the identities (8) and (9) we have

$$\begin{aligned}
1 &= a \circ (a \circ (a \circ (a^2 \circ b))) \\
&= a \circ (a \circ (a \circ a^{2b})) \\
&= (a \circ a^b)^{(1+a^b)(-a+1)(-a+1)} \\
&= (a \circ a^b)^{(-a^b a-a+1+a^b)(-a+1)} \\
&= (a \circ a^b)^{(1-aa^b-1-a+1+a^b)(-a+1)} \\
&= (a \circ a^b)^{(1-a-aa^b+a^b)(-a+1)} \\
&= (a \circ a^b)^{-a^b a+aa^b a+a^2-a+1-a-aa^b+a^b} \\
&= (a \circ a^b)^{(1-aa^b-1)+(1+a^2a^b-1)+a^2-2a+1-aa^b+a^b} \\
&= (a \circ a^b)^{1-aa^b+(2a-1)a^b-1-aa^b+a^b} \\
&= (a \circ a^b)^{1+aa^b-a^b-1-aa^b+a^b} \\
&= (a \circ a^b)^{1+aa^b-1-aa^b}.
\end{aligned}$$

So

$$(a \circ a^b) \cdot (a \circ a^b)^{aa^b} = (a \circ a^b)^{aa^b} \cdot (a \circ a^b)$$

and it is clear that $(a \circ a^b)$ lies in the centre of $\langle a, a^b \rangle'$. Next suppose that a has finite order not divisible by 3. In this case $\langle a \rangle$ is generated by a^3. We now apply Lemma 3 again and go through similar calculations using Lemmas 1 and 4 and identities (8) and (9). We have

$$\begin{aligned}
1 &= a \circ (a \circ (a \circ (a^3 \circ b))) \\
&= a \circ (a \circ (a \circ a^{3b})) \\
&= (a \circ a^b)^{(3a^b)(-a+1)(-a+1)} \qquad \text{(using (10))} \\
&= (a \circ a^b)^{(-3a^b a+3a^b)(-a+1)} \\
&= (a \circ a^b)^{(1-3aa^b-1+3a^b)(-a+1)} \\
&= (a \circ a^b)^{-3a^b a+a+3aa^b a-a+1-3aa^b-1+3a^b} \\
&= (a \circ a^b)^{(1-3aa^b-1)+a+(1+3a^2a^b-1)-a+1-3aa^b-1+3a^b} \\
&= (a \circ a^b)^{1-3aa^b+a+(6a-3)a^b-a-3aa^b-1+3a^b} \\
&= (a \circ a^b)^{1+a-3a^b-a-1+3a^b} \\
&= (a \circ a^b)^{a-3a^b-a+3a^b}.
\end{aligned}$$

So $(a \circ a^b)^a$ commutes with $(a \circ a^b)^{3a^b}$. If we conjugate by a we get that $(a \circ a^b)^{2a-1}$ commutes with $(a \circ a^b)^{3a^b a}$. But since $(a \circ a^b)^{2a}$ commutes with $(a \circ a^b)^{3a^b a}$ it follows that $(a \circ a^b)$ commutes with $(a \circ a^b)^{3a^b a} = (a \circ a^b) \cdot (a \circ a^b)^{3aa^b} \cdot (a \circ a^b)^{-1}$. Hence $(a \circ a^b)$ commutes with $(a \circ a^b)^{3aa^b}$. It is easy to see that it follows from this that $(a \circ a^b)$ commutes with all elements in $\langle a^3, a^{3b} \rangle'$. But $\langle a^3, a^{3b} \rangle' = \langle a, a^b \rangle'$. We can now easily prove the following lemma.

Lemma 5 *If a is of finite order, either not divisible by 2 or not divisible by 3, then $\langle a, a^b \rangle$ is metabelian. Furthermore we have that $\langle a, a^b \rangle$ is nilpotent of class at most 4.*

Proof We have seen that $(a \circ a^b)$ is in the centre of $\langle a, a^b \rangle'$. But the centre is characteristic in $\langle a, a^b \rangle'$ which is normal in $\langle a, a^b \rangle$. Therefore the centre is normal in $\langle a, a^b \rangle$. But since $\langle a, a^b \rangle'$ is the normal closure of $(a \circ a^b)$ in $\langle a, a^b \rangle$ we have that it is abelian. Therefore $\langle a, a^b \rangle$ is metabelian. Now $(a \circ a^b)^{(-a+1)^2} = (a \circ a^b)^{(-a^b+1)^2} = 1$. It follows that

$$x_1 \circ (x_2 \circ (x_3 \circ (a \circ a^b))) = 1$$

when $x_1, x_2, x_3 \in \{a, a^b\}$. So $\langle a, a^b \rangle$ is nilpotent of class at most 4. $\Box$

Lemma 6 *Let p and q be different prime numbers. If a is a p-element and that b is a q-element in an 4-Engel group then a and b commute.*

Proof We have

$$a^{-1}a^b = (a \circ b) = b^{-a}b.$$

From Lemma 5 we have that $\langle a, a^b \rangle$ and $\langle b, b^a \rangle$ are both nilpotent. Therefore $(a \circ b)$ is both a p-element and a q-element which forces it to be the identity. $\Box$

We can now strengthen Lemma 5 so that it includes all elements of finite order.

Proposition 1 *If a is of finite order then $\langle a, a^b \rangle$ is metabelian and nilpotent of class at most 4.*

Proof Suppose

$$a = \prod_p a_p$$

is the decomposition of a into a product of p-elements. By Lemma 6 the groups $\langle a_p, a_p^b \rangle$ commute with each other. Then

$$\langle a, a^b \rangle \subseteq \prod_p \langle a_p, a_p^b \rangle.$$

And $\langle a, a^b \rangle' \subseteq \prod_p \langle a_p, a_p^b \rangle'$ which is abelian by Lemma 5. Hence $\langle a, a^b \rangle$ is metabelian. That $\langle a, a^b \rangle$ is nilpotent of class 4 follows as in the proof of Lemma 5. $\Box$

Lemma 7 *Let a be an element of an 4-Engel group G which has order q. The exponent of $\langle a, a^b \rangle$ divides q^2 if q is an odd number and it divides $2q^2$ if q is an even number.*

Proof From the 4-Engel identity we have

$$1 = a^q \circ (a \circ a^b) = [a \circ (a \circ a^b)]^q = (a \circ a^b)^{-qa+q}. \tag{11}$$

Let first assume that q is an odd number. From Proposition 1 we have that $\langle a, a^b \rangle$ is metabelian and by Lemma 2 $\langle a, a^b \rangle'$ is generated by $(a \circ a^b)$, $(a \circ a^b)^a$, $(a \circ a^b)^{a^b}$ and $(a \circ a^b)^{aa^b}$. To show that $\langle a, a^b \rangle$ has exponent q^2 it is therefore sufficient to show that $(a \circ a^b)^q = 1$. Then, using the fact that $(a \circ a^b)^{a^2} = (a \circ a^b)^{2a-1}$, we have

$$\begin{aligned} 1 &= a^q \circ a^b \\ &= (a \circ a^b)^{a^{q-1}+a^{q-2}+\cdots+a+1} \\ &= (a \circ a^b)^{(q-1)a-(q-2)+(q-2)a-(q-3)+\cdots+2a-1+a+1} \\ &= (a \circ a^b)^{\frac{q(q-1)}{2}a-q(\frac{q-1}{2}-1)} \\ &= (a \circ a^b)^q \qquad \text{(by (11) since } q \text{ is odd.)} \end{aligned}$$

Now suppose q is an even number. By going through the same calculations we get

$$\begin{aligned} 1 &= a^{2q} \circ a^b \\ &= (a \circ a^b)^{\frac{2q(2q-1)}{2}a-2q(\frac{2q-1}{2}-1)} \\ &= (a \circ a^b)^{q(2q-1)a-q(2q-3)} \\ &= (a \circ a^b)^{2q} \qquad \text{(by (11).)} \end{aligned}$$

It follows that in this case we have that the exponent divides $2q^2$. □

Remark. From Proposition 1 we have that $\langle a, a^b \rangle$ has nilpotency class 4. If q is a power of p, where $p \neq 2, 3$, then it is well known that $\langle a, a^b \rangle$ is a regular p-group since the class is less than p. The exponent of a regular group is the maximum order of the generators. It follows that in this case we have the stronger result that the exponent is q.

Proposition 2 *If for some prime p we have that $a_1, a_2, \ldots, a_r$ are p-elements in a 4-Engel group G then $\langle a_1, a_2, \ldots, a_r \rangle / Z(\langle a_1, a_2, \ldots, a_r \rangle)$ is a p-group of finite exponent.*

Proof Notice that we are not assuming that $\langle a_1, \cdots, a_r \rangle$ is a p-group. In Section 3 we will turn our attention to p-groups.

We prove the following stronger assertion. If a is a p-element then there is an integer s such that g^s commutes with a for all g in G.

Suppose a has order $q = p^i$. Now $g \circ a \in \langle a, a^g \rangle$. By Lemma 7 we have that $(g\circ a)^{q^3}=1$. Using Lemma 7 again we have $(g\circ(g\circ a))^{q^9}=1$. Let $n=q^{10}=p^{10i}$. We have

$$\begin{aligned} g^n\circ(g\circ a) &= (g\circ g^a)^{g^{n-1}+g^{n-2}+\cdots+1} \\ &= (g\circ g^a)^{\frac{n(n-1)}{2}g-n(\frac{n-1}{2}-1)} \\ &= (g\circ(g\circ a))^{q^9(\frac{q(n-1)}{2}g-\frac{q(n-3)}{2})} \\ &= 1. \end{aligned}$$

So g^n commutes with $(g\circ a)$.Now $(g^n\circ a)^{q^3}=1$ by Lemma 7. Then

$$\begin{aligned} g^{q^{13}}\circ a &= g^{q^3 n}\circ a \\ &= (g\circ a)^{g^{q^3n-1}+g^{q^3n-2}+\cdots+g^{(q^3-1)n}} \\ &\cdot\ (g\circ a)^{g^{(q^3-1)n-1}+\cdots+g^{(q^3-2)n}} \\ &\vdots \\ &\cdot\ (g\circ a)^{g^{n-1}+\cdots+1} \\ &= [(g\circ a)^{g^{n-1}+\cdots+1}]^{q^3} \qquad (\text{since } (g\circ a)^{g^n}=(g\circ a)) \\ &= (g^n\circ a)^{q^3} \\ &= 1. \end{aligned}$$

So $g^{q^{13}}$ commutes with a for all g in G. We have thus shown that for each element a in G there exists some power s_a of p such that a commutes with g^{s_a} for all g in G. Now let $s=max\{s_{a_1},s_{a_2},\ldots,s_{a_r}\}$. Then $\langle a_1,a_2,\ldots,a_r\rangle/Z(\langle a_1,a_2,\ldots,a_r\rangle)$ is of exponent s. □

Lemma 8 *If a and b are p-elements in an 4-Engel group then ab is of finite order.*

Proof Let us first consider the case when $p=2$. Suppose $a^{2^i}=1$. We then have

$$(ab)^{2^i}=a^{-2^i}(ab)^{2^i}=b^{a^{2^i-1}}b^{a^{2^i-2}}\cdots b^a b.$$

Since $\langle u,u^x\rangle$ is nilpotent for all $u,x\in\langle a,b\rangle$ when u is of finite order, we have that $u^x u$ is a p-element whenever u is a p-element. Therefore $b^a b$ is a 2-element and then also $b^{a^3}b^{a^2}b^a b=(b^a b)^{a^2}(b^a b)$. By induction we get that

$$b^{a^{2^i-1}}b^{a^{2^i-2}}\cdots b^a b=(ab)^{2^i}$$

is a 2-element and hence ab is a 2 element.

Now assume p is an odd number. Suppose $a^{p^i} = b^{p^i} = 1$. It follows from last proposition that there is an integer $s \geq i$ such that $(ab)^{p^s}$ commutes with a and b. We then have

$$(ab)^{p^s} = a^{-p^s}(ab)^{p^s} = b^{a^{p^s-1}} b^{a^{p^s-2}} \cdots b^a b.$$

Since $2 \not\equiv 0 \pmod{p}$ we have that 2 is in the group of units modulo p^s. So there is a positive integer m such that $2^m \equiv 1 \pmod{p^s}$. ($m = (p-1)p^{s-1}$ for example). Suppose $2^m = p^s r + 1$. By the argument above we get that

$$c := b^{a^{2^m-1}} b^{a^{2^m-2}} \cdots b^a b = b(b^{a^{p^s-1}} \cdots b^a b)^r = b(ab)^{p^s r}$$

is a p-element. Suppose $c^{p^l} = 1$ where $l \geq i$. Since $(ab)^{p^s r}$ commutes with b, we get

$$1 = [b(ab)^{p^s r}]^{p^l} = (ab)^{p^{s+l} r} \qquad \Box$$

Theorem 1 *Let G be an* 4*-Engel group. The torsion elements form a group $\tau(G)$ and $\tau(G)/Z(\tau(G))$ is a direct product of p-groups.*

Proof Suppose a_p and b_p are two p-elements in G. By Lemma 8 we have that $a_p b_p$ is of finite order. It can then be written in the form

$$a_p b_p = (\prod_{q \neq p} c_q) c_p,$$

where for each prime q we have that c_q is a q-element of G. Suppose q is some prime number different from p then by Lemma 6 we have that c_q commutes with all r-elements when r is a prime different from q. We want to show that it commutes also with all q-elements. So suppose d is a q-element. It then commutes with all elements of the right hand side except possibly c_q. But d commutes with $a_p b_p$ and therefore it must also commute with c_q. Hence c_q commutes with all elements of finite order. This is true for all $q \neq p$ and therefore

$$a_p b_p = c_p d_p \tag{12}$$

where d_p commutes with all elements of finite order and is of finite order itself. Now let a and b be two elements of finite order. Suppose $a = \prod_p a_p$

and $b = \prod_p b_p$ are their decomposition into a product of p-elements. From Lemma 6 and (12) we then have

$$ab = \prod_p a_p b_p = (\prod_p c_p)d.$$

Where each c_p is a p element and d is a torsion element that commutes with all torsion elements. By Lemma 6 the factors commute pairwise and hence ab is of finite order. So the torsion elements form a group $\tau(G)$. By (12) it follows that product of p-elements in $\tau(G)/Z(\tau(G))$ is a p-element and it follows from this and Lemma 6 that $\tau(G)/Z(\tau(G))$ is a direct product of p-groups. □

3 4-Engel p-groups

In this section we will reduce our problem of finding out whether 4-Engel p-groups are locally nilpotent to the class of 4-Engel groups of exponent p and we will show that 4-Engel 2-groups and 4-Engel 3-groups are locally nilpotent.

We remind the reader of the fact that every group G has a maximal normal locally nilpotent subgroup called the locally nilpotent radical. We will denote it by $\mathrm{rad}(G)$. It contains all the normal locally nilpotent subgroups of G.

Lemma 9 *Let G be an 4-Engel group. If $a \in G$ has order 2^i where $i \geq 3$ then*

$$a^{2^{i-1}} \circ (a^{2^{i-1}} \circ b) = 1$$

for all $b \in G$.

Proof From the Engel-4 identity we have that $(a \circ a^b)^{(a-1)^2} = 1$. It follows that

$$(a \circ a^b)^{(a^m-1)^2} = 1$$

for all $m \in \mathbb{N}$ since $(a-1)^2 | (a^m - 1)^2$ in $\mathbb{Z}\langle a \rangle$. (We are repeatedly using the fact that $\langle a, a^b \rangle$ is metabelian). It follows that

$$(a \circ a^b)^{1+a^{2m}} = (a \circ a^b)^{1+(2a^m-1)} = (a \circ a^b)^{2a^m} \tag{13}$$

for all $m \in \mathbb{N}$. Let $m = 2^{i-3}$. Since $a^{8m} = 1$, we have

$$\begin{aligned} 1 &= a^{8m} \circ a^b \\ &= (a^{4m} \circ a^b)^{a^{4m}+1} \\ &= (a^{4m} \circ a^b)^{2a^{2m}} \qquad (by(13)). \end{aligned}$$

This implies that $(a^{4m} \circ a^b)^2 = 1$. But then

$$\begin{aligned} a^{4m} \circ (a^{4m} \circ b) &= a^{4m} \circ a^{4mb} \\ &= (a^{4m} \circ a^{2mb})^{1+a^{2mb}} \\ &= (a^{4m} \circ a^{2mb})^{2a^{mb}} \qquad \text{(by (13))} \\ &= (a^{4m} \circ a^b)^{2a^{mb}(1+a^b+\cdots+a^{(2m-1)b})} \\ &= 1. \qquad \square \end{aligned}$$

Lemma 10 *Let G be a 4-Engel group and let p be an odd prime number. If $a \in G$ has order p^i where $i \geq 2$ then*

$$a^{p^{i-1}} \circ (a^{p^{i-1}} \circ b) = 1$$

for all $b \in G$

Proof Let $m = p^{i-1}$. We have that

$$\begin{aligned} a^m \circ (a^m \circ b) &= a^m \circ a^{mb} \\ &= (a \circ a^b)^{(a^{m-1}+\cdots+1)(a^{(m-1)b}+\cdots+1)} \\ &= (a \circ a^b)^{(\frac{m(m-1)}{2}a - m(\frac{m-1}{2}-1))(\frac{m(m-1)}{2}a^b - m(\frac{m-1}{2}-1))}. \end{aligned}$$

From the proof of Lemma 7 we have that $(a \circ a^b)^{pm} = 1$. Since pm divides m^2 it follows from the calculations above that $a^m \circ (a^m \circ b) = 1$. $\square$

Proposition 3 *Let G be a 4-Engel group. If $a \in G$ is a 2-element then $a^4 \in rad(G)$. If $a \in G$ is a p-element, where p is an odd prime number, then $a^p \in rad(G)$.*

Proof Let p be an odd prime number and let a be a p-element. We prove by induction on the order of a that $a^p \in \text{rad}(G)$. This is obvious when a has order p. Now suppose this is true when a has order p^{i-1} where $i \geq 2$. Let

$a \in G$ be an element of order p^i. By the induction hypothesis $a^{p^2} \in \mathrm{rad}(G)$. It then follows from Lemma 10 that $a^p \circ (a^p \circ b) = 1$ modulo $\mathrm{rad}(G)$. This implies that the normal closure of a^p is abelian modulo $\mathrm{rad}(G)$ and thus locally soluble. By a theorem of Gruenberg [5] it follows that the normal closure of a^p is locally nilpotent and therefore that $a^p \in \mathrm{rad}(G)$.

By using Lemma 9 one proves similarly that $a^4 \in \mathrm{rad}(G)$ whenever a is a 2-element. □

Let G be a p-group. It follows from Proposition 3 that $G/\mathrm{rad}(G)$ has exponent dividing p if p is an odd prime number and exponent dividing 4 if $p = 2$. Since groups of exponent 4 [13] and groups of exponent 3 are known to be locally finite we have as an immediate corollary.

Theorem 2 *Let G be a 4-Engel group and let rad(G) be the locally nilpotent radical of G. If G is a 2-group or a 3-group then it is locally finite. If p is a prime and p is not 2 or 3 then G/rad(G) has exponent dividing p.*

Remark It follows that for and odd prime p we have that local finiteness of 4-Engel groups of exponent p implies local finiteness of 4-Engel p-groups. It also follows from Proposition 2 and Proposition 3 that 4-Engel groups generated by 2-elements or generated by 3-elements are locally nilpotent. It follows that a product of 3-elements (2-elements) is a 3-elements (2-element) in every 4-Engel group.

4 Locally nilpotent 4-Engel groups

In the introduction we mentioned the connection between Engel groups and Engel Lie algebras. In this last section we want to take that discussion a bit further. We want to see what knowledge of 4-Engel Lie algebras tells us about 4-Engel groups.

Let us first see how we can associate to every group a Lie ring. Let G be a group with lower central series

$$\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_i \geq \cdots$$

and let $L_i = \gamma_i/\gamma_{i+1}$. Because L_i is abelian we have a $\mathbb{Z}$-module

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_i \oplus \cdots .$$

We write the group operation in L_i additively: $x\gamma_{i+1} + y\gamma_{i+1} := xy\gamma_{i+1}$. We define a multiplication on L as follows: if $a = x\gamma_{i+1} \in L_i$ and $b = y\gamma_{j+1} \in L_j$ then let

$$ab = (x \circ y)\gamma_{i+j+1} \in L_{i+j},$$

and extend this product linearly to all of L. It is not difficult to show that L with this multiplication, is a Lie ring. Fur thermore if G is nilpotent then L is also nilpotent with same nilpotency class.

Now let G be an 4-Engel group. Then the associated Lie ring satisfies the following two identities (see [16]):

$$(1) \qquad \sum_{\sigma \in \mathrm{Sym}(4)} (((xx_{\sigma(1)})x_{\sigma(2)})x_{\sigma(3)})x_{\sigma(4)} = 0,$$

$$(2) \qquad (((xy)y)y)y = 0 \quad \text{if } y \text{ is a product of generators.}$$

By the generators in (2) we mean the elements $a\gamma_2$ with $a \in G$. Now suppose L is a Lie algebra over a field with characteristic not equal to 2 or 3 that satisfies these two identities. It is not difficult to see that (1) is now equivalent to the 4-Engel identity: $(((xy)y)y)y = 0$ for all $x, y \in L$. Now it can be shown that every 4-Engel Lie algebra over a field with characteristic not equal to $2, 3$ or 5, is nilpotent of class at most 7 (see [4] and [14]).

So suppose that G is a locally nilpotent 4-Engel p-group, where p is a prime not equal to $2, 3$ or 5. Let H be any subgroup with 8 generators. Then H is a finite p-group. Suppose it has exponent p^i.Let L be the associated Lie ring of H. Then L/pL is a Lie algebra over the field $\mathbb{Z}_p$ and is therefore nilpotent of class at most 7. This implies that $L^8 = 0$. It follows that the commutator of arbitrary 8 elements in G is 1 and hence G is nilpotent of class at most 7. We have thus shown that every locally finite 4-Engel p-group is nilpotent of class at most 7 if p is not equal to 2, 3 or 5. Every 4-Engel Lie algebra over a field of characteristic 5 with r generators is nilpotent of class at most $6r$ (see [9]). It follows by a same kind of argument as above, that every nilpotent 4-Engel 5-group with r generators has nilpotency class at most $6r$. Let us now summarize what we have shown.

Theorem 3 *Let G be a locally nilpotent* 4*-Engel group. If G is a p-group where $p \neq 2, 3, 5$ then G is nilpotent of class at most* 7*. If G is* 5 *group then every r generator subgroup has class at most $6r$.*

By Zorn's Theorem we have the following corollary.

Corollary 1 *If G is a finite* 4*-Engel group with no elements of order* 2*,* 3 *or* 5 *then G is nilpotent of class at most* 7*.*

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