A constructive approach to Zel'manov's global nilpotency theorem for n-Engel Lie algebras over a field of characteristic zero

Gunnar Traustason

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Abstract

We obtain an explicit bound for the nilpotency class of n-Engel Lie algebras of characteristic zero.

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1 Introduction

A Lie algebra L is called an Engel Lie algebra if for each ordered pair (x, y) there is an integer $n(x, y)$ such that

$$(((y \underbrace{x)x) \cdots)x}_{n(x,y)} = 0. \tag{1}$$

One of the basic classical results for Engel Lie algebras is Engel's Theorem. It states that every finite dimensional Engel Lie algebra over a field is nilpotent. So for finite dimensional Lie algebras the Engel condition is equivalent to nilpotency. This is however not true in general and there exist Engel Lie algebras that are not locally nilpotent [4]. Now suppose $n = n(x, y)$ in (1) can be chosen independently of x and y. We then say that L is a n-Engel Lie algebra. The subject of this article is a theorem due to Zel'manov [17] which states that, for every positive integer n, every n-Engel Lie algebra over a field of characteristic zero is nilpotent. Zel'manov's non-constructive proof depends on a result of Kostrikin [8,9] that there always exists a non-trivial abelian ideal in a n-Engel Lie algebra over a field that is of characteristic $p > n$ or 0. From this result it is natural to define inductively the following sequence of ideals:

$$I_0 = \{0\}, \quad I_{i+1}/I_i = \text{sum of all abelian ideals in } L/I_i.$$

Zel'manov shows that the nilpotency of L/I_{j+1} implies that L/I_j is nilpotent. Kostrikin's proof was written in non-constructive way and it is not clear that the sequence of ideals above has to reach L. Zel'manov's proof in fact uses transfinite induction. However, as was demonstrated by Adian and Razborov [1], Kostrikin's proof is essentially constructive. In their paper, Adian and Razborov rewrite Kostrikin's arguments in a constructive way and show that the sequence (I_j) reaches L eventually and obtain an upper bound for the length of the sequence depending only on n.

Our aim in this article is to give an upper bound for the nilpotency class of n-Engel Lie algebras. In order to do so it turns out to be useful to replace the sequence of ideals above by a certain sequence of verbal ideals. Let L be a n-Engel Lie algebra over a field of characteristic zero. The following construction of a sequence of words is implicit in Kostrikin's work. He constructs a sequence of words $f_1, \ldots, f_s$ such that, for each i, every value of f_{i+1} in $L/v(f_1, \ldots, f_i)$ generates an abelian ideal, where $v(f_1, \ldots, f_i)$ is the verbal ideal generated by $f_1, \ldots, f_i$. We will call such a sequence a good sequence. According to Kostrikin's notation, this means that the values of f_{i+1} are sandwiches of infinite thickness in $L/v(f_1, \ldots, f_i)$. If we let $J_0 = \{0\}$ and $J_i = v(f_1, \ldots, f_i)$ for $1 \le i \le s$ then the sequence $J_0, J_1, \ldots, J_s$ has the property that J_{i+1}/J_i is a sum of abelian ideals in L/J_i. From having looked into the paper of Adian and Razborov it seems clear to the author that the bound they obtain for the length of their sequence of ideals is in fact a bound for the length of sequence of verbal ideals as described above. For later arguments we need furthermore that the words in the good sequence are homogenous in each variable. This is not the case with all the words constructed in Kostrikin's work although this can be overcome without too much difficulty. In [11] the author used Kostrikin's work, with few improvements in order to reduce the bounds, to obtain such a sequence. Since Kostrikin's proof is very long and since bounds are implicit in the work of Kostrikin and are in fact for the most part apparent in the paper of Adian and Razborov, we have decided to omit the proof here. So we only state the result. Before we do so we introduce some notation. Define a function $T : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by induction in the following way: $T(m, 1) = m$, $T(m, s+1) = m^{T(m,s)}$.

Theorem 1 *Let k be a field with characteristic p where p is either 0 or a prime number greater than or equal to $[n + n/2]$. There is a good sequence $f_1 = z_1 z_2^n, f_2, \ldots, f_s$ of multihomogenous words in the free Lie algebra on $z_1, z_2, \ldots$ over k, where $s \leq 256 \cdot 10^{46} \cdot n^{40} \cdot 3^{[(n-5)/2]}$, such that every Lie algebra satisfying the identities $f_1 = f_2 = \ldots = f_s = 0$ is nilpotent of class at most 7. Furthermore every word f_i is homogenous of degree at most $T(n, 3)$.*

We will use Theorem 1 and Zel'manov's arguments to give a constructive proof of Zel'manov's Theorem and obtain a bound for the nilpotency class of n-Engel Lie algebras over a field of characteristic zero. In the next section we will prove the following.

Theorem 2 *Suppose $n \geq 5$ and that L is an n-Engel Lie algebra over a field k with characterist either 0 or greater than $T(n, l)$, where $l = 1024 \cdot 10^{46} \cdot n^{40} \cdot 3^{[(n-5)/2]}$. Then L is nilpotent of class at most $T(n, l)$.*

When the underlying field has "small" characteristic the n-Engel Lie algebra does not have to be nilpotent. In [2] the authors construct a 3-Engel Lie algebra over a field of characteristic 5 that is non-solvable. This result was later generalized by Razmyslov [10], who proved that for each prime $p \geq 5$, there exists a non-solvable Lie algebra of characteristic p satisfying the $(p-2)$-Engel identity. However, it follows from Zel'manov's solution to the Restricted Burnside Problem [18,19] that an n-Engel Lie algebra over an arbitrary field is locally nilpotent. In [16], Vaughan-Lee and Zel'manov give upper bounds for the nilpotency class in terms of the number of generators r. It follows from their work that an n-Engel Lie algebra with r generators is nilpotent of class at most $T(r, n^{n^n})$. When the characteristic of the field is greater than n they get smaller bounds. So if $25 \leq n < p$ then L is nilpotent of class at most $T(r, 2^n)$ and when $26 > n < p$ we have that L is nilpotent of class at most $T(r, 3^n)$.

We have more detailed information for small values of n. It is well known that Lie algebras satisfying the 2-Engel identity are nilpotent of class at most

3. In [12] it is shown that 3-Engel Lie algebras with $\operatorname{char} k \neq 2, 5$ are nilpotent of class at most 4 and that when the characteristic is 5 we have that the class is at most $2r$. In [13] it is shown that the class is at most $2(r+1)^6$ when $\operatorname{char} k = 2$. We also have some close information about 4-Engel Lie algebras. For characteristics not equal to one of $2, 3$ or 5 we have that the class c is at most 7 [5,12]. For $\operatorname{char} k = 3$ we have that $c \leq 3r$ [12] and $c \leq 6r$ when $\operatorname{char} k = 5$ [6]. In [13] a polynomial upper bound is also given for c when $\operatorname{char} k = 2$ and $|k| \neq 2$. In the case of 5-Engel Lie algebras calculations are becoming much more difficult and we do not have as good results as for 4-Engel Lie algebras. However, for most characteristics we do have a linear upper bound for the nilpotency class in terms of the number of generators. In [14] it is shown that the nilpotency class is at most $59r$ when the characteristic does not divide $2 \cdot 3 \cdot 5 \cdot 7 \cdot$ and that the nilpotency class is at most $80r$ if the characteristic is 7. Vaughan-Lee [15], has also shown that 6-Engel Lie algebras with r generators over a field with characteristic 7 are nilpotent of class at most $51r^8$.

2 The Proof

In this section we prove Theorem 2. We start with a well known lemma.

Lemma 1 *Let L be an n-Engel Lie algebra over a field k where $\operatorname{char} k > n$. Suppose I is a subset of L and N a subalgebra. Then, if $t \geq n$*

$$I\underbrace{N\cdots N}_{t} \leq \sum_{\substack{r_1,\dots,r_{n-1}\geq 0\\ r_1+\cdots+r_{n-1}=t}} IN^{r_1}\cdots N^{r_{n-1}}.$$

In particular if I and J are ideals of L then

$$I\underbrace{J\cdots J}_{(n-1)(m-1)+1} \leq IJ^m.$$

Proof Consider the subspace

$$\begin{aligned} U &= IN^2\underbrace{N\cdots N}_{t-2} + INN^2\underbrace{N\cdots N}_{t-3} + \cdots + I\underbrace{N\cdots N}_{t-2}N^2 \\ &= \sum_{\substack{r_1,\dots,r_{t-1}\\ r_1+\cdots+r_{t-1}=t}} IN^{r_1}\cdots N^{r_{t-1}}. \end{aligned}$$

If $b \in I$ and $a_1, \ldots, a_t \in N$ then $ba_1 \cdots a_{i-1}(a_i a_{i+1})a_{i+2} \cdots a_t \in U$ for all $i \in \{1, \cdots, t-1\}$. Therefore modulo U

$$ba_1 \cdots a_t = 1/n! \sum_{\sigma \in \mathrm{Sym}(n)} ba_{\sigma(1)}a_{\sigma(2)} \cdots a_{\sigma(n)}a_{n+1} \cdots a_t = 0,$$

so $I\underbrace{N \cdots N}_{t} \leq U$. Using this same trick again we can show that each summand of U lies in

$$\sum_{\substack{r_1, \cdots, r_{t-2} \\ r_1 + \cdots + r_{t-2} = t}} IN^{r_1} \cdots N^{r_{t-2}}.$$

Continuing like this we finally get the required inequality. □

Now let k be a field of characteristic either 0 or a prime number $p > n + [n/2]$ and let F be the free Lie algebra over k freely generated by $z_1, z_2, \ldots$. By Theorem 1 there is a good sequence $f_1 = z_1 z_2^n, f_2, \ldots, f_s$ of multihomogenous words in F such that

(1) every Lie algebra over k satisfying the identities $f_1 = \cdots = f_s = 0$ is nilpotent of class at most 7;

(2) f_i is homogeneous of degree at most $T(n, 3)$ for $i = 1, 2, \ldots, s$.

In this section we will prove that (1) and (2) imply that if L is an n-Engel Lie algebra over a field k, and if $\mathrm{char}\, k$ is either 0 or a prime number greater than $T(n, 4(s-1)+2)$ then L is nilpotent of class at most $T(n, 4(s-1)+2)$. First we establish a preliminary lemma used by Zel'manov.

Lemma 2 *Let L be an n-Engel Lie algebra over a field k where $\mathrm{char}\, k > n$. Let L be $\mathbb{Z}_2$-graded, so that*

$$L = L_0 \oplus L_1$$

with $L_i L_j \leq L_{i+j}$ for $i, j \in \mathbb{Z}_2$. If the subalgebra L_0 is nilpotent of class $m > 2$ then L is nilpotent of class at most $n^{nm} - 1$.

Proof With $I = L$ and $N = L_0$, Lemma 1 tells us that $\mathrm{ad}(L_0)^{(n-1)m+1} = \{0\}$. Then we have

$$L^{(1)} = L_0 L_0 + L_1 L_1 + L_1 L_0 \leq L_0 + L_1 L_0,$$

and by induction

$$L^{(r)} \leq L_0 + L_1 \underbrace{L_0 \cdots L_0}_{r} \qquad r = 1, 2, \ldots .$$

Therefore $L^{((n-1)m+1)} \leq L_0$. Since L_0 is nilpotent of class m, it is solvable of derived length at most $m-1$ and so L is solvable of derived length at most nm. By Higgins' Theorem [7] this implies that L is nilpotent of class at most

$$\frac{n^{nm}-1}{n-1} < n^{nm}.$$

□

Let $f(x)$ be a word in F that is homogenous in some variable x. Suppose it has weight l in x. Now let $x_1, \ldots, x_l$ be some variables not occurring in f and replace x by $x_1 + \cdots + x_l$. The resulting word $f(x_1 + \cdots + x_l)$ can be written uniquely as a sum of multihomogenous terms. The term of multiweight $(1 \underbrace{, \ldots ,}_{l} 1)$ in $x_1, \ldots, x_l$ is called the linearization of f in x. Now suppose f is a multihomogenous word in some variables $x_1, \ldots, x_s$. We define a sequence $g_1, g_2, \ldots, g_s$ inductively as follows. We let g_1 be the linearization of f in x_1 and for each $2 \leq i \leq s$, we let g_i be the linearization of g_{i-1} in x_i. The multilinear word $g = g_s$ is called a full linearization of f.

Now suppose that k is a field with $\operatorname{char} k > T(n, 4(s-1)+2)$ and that $f_1, f_2, \ldots f_s$ is a sequence of words satisfying (1) and (2) above. We prove by induction that a Lie algebra over k which satisfies the identity $f_1 = 0$ is nilpotent of class at most $T(n, 4(s-1)+2)$.

We use reverse induction on i and assume that if L is a Lie algebra over k satisfying the identities $f_1 = 0, f_2 = 0, \ldots, f_i = 0$, where $1 < i \leq s$, then L is nilpotent of class at most $T(n, 4(s-i)+2)$. We want to prove that if L is a Lie algebra over k satisfying the identities $f_1 = 0, f_2 = 0, \ldots, f_{i-1} = 0$ then L is nilpotent of class at most $T(n, 4(s-(i-1))+2)$.

For each $j = 1, 2, \ldots, s$ we let g_j be a full linearization of f_j. Since $\operatorname{char} k$ is greater than the degree of f_j, the identity $f_j = 0$ is equivalent to the identity $g_j = 0$. Also, since f_j has degree not more than $T(n, 3)$, it is not difficult to see that every value of g_j is a linear combination of at most $2^{T(n,3)}-1$ values of f_j. Since every value of f_j in $L/v(f_1,\ldots,f_{j-1})$ generates an abelian ideal we have that this implies

(3) if L is a Lie algebra over k which satisfies the identities $f_1=0, f_2=0, \ldots, f_{j-1}=0$ for some $1<j\leq s$, then every value of g_j in L generates an ideal which is nilpotent of class at most $2^{T(n,3)}-1$.

Let V be the variety of Lie algebras over k determined by the identities $f_1=0, f_2=0,\ldots,f_{i-1}=0$, and let A be the free Lie algebra of the variety V freely generated by $x_1,x_2,\ldots$. Note that since the identities $f_1=0,f_2=0,\ldots,f_{i-1}=0$ are equivalent to multilinear identities, the Lie algebra A is multigraded. Let

$$m=T(n,4(s-i)+2),\quad M=n^{nm}\ \text{ and }\ K=((M-1)!\cdot 2^{T(n,3)})^{2^M}.$$

Also let $g_i=g=g(z_1,z_2,\ldots,z_e)$. Note that $M=n^{nm}>n^m\geq T(n,3)$. If we can show that $(A^M)^K=\{0\}$ then it follows from Lemma 1, when $I=A^M$ and $J=A$, that the nilpotency class of A is not more than $M+((n-1)(M-1)+1)(K-1)$. But

$$\begin{aligned} M+((n-1)(M-1)+1)(K-1) &< nMK < (M!\,2^{T(n,3)})^{2^M} \\ &< (M!\,2^M)^{2^M} < (M^M)^{2^M} = n^{nmM2^M} < n^{M^2 2^M} < n^{n^M} = n^{n^{n^{nm}}} \\ &< n^{n^{n^{n^m}}} = T(n,4(s-(i-1))+2). \end{aligned}$$

So we need to show that $(A^M)^K=\{0\}$. The proof of this is for the most part an argument due to Zel'manov. The argument is based on a beautiful application of the representation theory of the symmetric groups. We decribe the represention theory that is needed below but we refer to [3] for more detailed discussion.

Let $N=MK$. Since $\operatorname{char} k$ does not divede $N!$, we have that $k\mathrm{Sym}(N)$ is a semisimple algebra. We can thus write it as a direct sum of simple left ideals. The number of simple ideals (up to isomorphism) is equal to the number of conjugacy classes of $\mathrm{Sym}(N)$. That is, it is equal to the number of partitions $(m_1,\ldots,m_t)$ of N with $m_1+\cdots+m_t=N$ and $m_1\geq m_2\geq\cdots\geq m_t>0$.

To each such partition we associate a Young diagram, which is an array of N boxes arranged in t rows, with m_i boxes in the i-th row. The boxes are arranged so that the j-th column of the array consists of the j-th boxes out of the rows which have length j or more. We obtain a Young tableau from a Young diagram by filling in the N boxes of the diagram with $1, \ldots, N$ in some order.

1	6	9	12
8	14	2	5
13	4	10	
7	11		
3			

A Young tableau

We then let $R \leq \mathrm{Sym}(N)$ be the stabilizer of the rows in the Young tableau and $C \leq \mathrm{Sym}(N)$ be the stabilizer of the columns. If we let

$$e = \sum_{\pi \in R, \tau \in C} \mathrm{sign}(\pi)\pi\tau$$

then e generates a simple left ideal. The element e is called the Young symmetrizer associated with the tableau. Different Young diagrams give rise to non-isomorphic simple left ideals. It then follows that we can write $S = k\,\mathrm{Sym}(N)$ as a direct sum of some left ideals $Se_1, Se_2, \ldots, Se_m$ where each e_i is a Young symmetrizer.

Let J be a subset of $\{1, \ldots, N\}$ and let H_J be the subgroup in $\mathrm{Sym}\,(N)$ consisting of the elements that fixes $\{1, \ldots, N\} \setminus J$ elementwise. We let

$$S_J = \sum_{\pi \in H_J} \pi, \quad A_J = \sum_{\pi \in H_J} (\mathrm{sign}(\pi))\pi.$$

We call the former element a symmetrization of J or less precisely a $|J|$-symmetrization where $|J|$ is the number of elements in J. We call the latter element similarly a skewsymmetrization of J or $|J|$-skewsymmetrization. Now let J be a subset of some column in the tableau. Then $H_J \leq C$. Let $g_1, \ldots, g_l$ be some left coset representatives of H_J in C. Then

$$e = \sum_{\tau \in R} \sum_{i=1}^{l} \mathrm{sign}(\tau)\tau g_i \sum_{\pi \in H_J}$$

so e is in the module span of the $|J|$-symmetrizations. Similarly if J is a subset of some row of the tableau then e is in the module span of the $|J|$-skewsymmetriztions . Since every Young tableau must have a column or a row of length at least $[\sqrt{N}]$, it follows from the discussion above that all the Young symmetrizators are in the module span of the set of $[\sqrt{N}]$-skewsymmetrizations and $[\sqrt{N}]$-symmetrizations. Hence $k\,\mathrm{Sym}(N)$ is generated as a left ideal by these elements. We now finish the proof of Theorem 2 by proving the following proposition.

Proposition 1 $(A^M)^N = \{0\}$

Proof Let $N = MK$ and let $h(x_1, \dots, x_N)$ be some word in A which is linear in the variables $x_1, x_2, \dots x_N$ and does not include other variables. There is a natural action from the left by elements of $k\,\mathrm{Sym}(N)$ arising from $\tau h(x_1, \dots, x_N) = h(x_{\tau(1)}, \dots, x_{\tau(N)})$ for all $\tau \in \mathrm{Sym}(N)$. For $r \in \{1, 2, \dots, M\}$ and $J \subseteq \{1, 2, \dots, K\}$ let S^r_J be the symmetrization of the set $\{r + (j-1)M : j \in J\}$. Similarly let A^r_J be the skewsymmetrization of this set. $S^r_J h$ is the symmetrization of h in the variables $x_{r+(j-1)M}$, $j \in J$ and $A^r_J h$ is the skewsymmetrization of h in these variables.

To show that $(A^M)^K = 0$, it is sufficient to prove that

$$H := (x_1 x_2 \cdots x_M)(x_{M+1} x_{M+2} \cdots x_{2M}) \cdots (x_{N-M+1} x_{N-M+2} \cdots x_N) = 0.$$

We argue by contradiction and assume that this product is not zero. We have $K = d^{2^M}$ where $d = (M-1)! \cdot 2^{T(n,3)}$. Now consider the variables $x_1, x_{1+M}, \dots, x_{1+(K-1)M}$. From the discussion about the representation theory of the symmetric groups above it follows that if $\mathrm{char}\, k > K$ then the product H can be written as a sum of words such that each word is either a symmetrization or a skew-symmetrization in $\sqrt{K}$ of these variables. Therefore if $H \neq 0$ then there is either a symmetrization or skew-symmetrization in $d^{2^{M-1}}$ of these variables which is non-zero. So we have

$$Q^1_{J_1} H \neq 0.$$

Where $Q^1_{J_1}$ is either $S^1_{J_1}$ or $A^1_{J_1}$, and $J_1 \subseteq \{1, 2, \dots, K\}$ with $|J_1| = d^{2^{M-1}}$. Now look at the variables $x_{2+(j-1)M}$, $j \in J_1$. We use the same argument again and we see that there is either a symmetrization or a skew-symmetrization of $Q^1_{J_1}H$ in $d^{2^{M-2}}$ of these variables which is non-zero. So we have

$$Q^2_{J_2}Q^1_{J_1}H \neq 0,$$

where $Q^2_{J_2}$ is either $S^2_{J_2}$ or $A^2_{J_2}$, and $J_2 \subseteq J_1$ with $|J_2| = d^{2^{M-2}}$. Continuing like this we finally get

$$Q^M_{J_M}Q^{M-1}_{J_{M-1}}\cdots Q^1_{J_1}H \neq 0.$$

Where $J_M \subseteq J_{M-1} \subseteq \cdots \subseteq J_1 \subseteq \{1,2,\ldots,K\}$, $|J_r| = d^{2^{M-r}}$ for $r = 1,2,\ldots,M$ and $Q^r_{J_r}$ is either $S^r_{J_r}$ or $A^r_{J_r}$. Let $J = J_M$ then $|J| = d$. We will now show that this leads to a contradiction.

Give each of the free generators $x_1, x_2, \ldots$ of A a weight 0 or 1 in the following way. Let $l \in \{1,2,\ldots,M\}$. If $Q^l_{J_l} = A^l_{J_l}$ then we give x_r weight 1 for all r such that $r = l$ modulo M and if $Q^l_{J_l} = S^l_{J_l}$ then we give x_r weight 0 for all r such that $r = l$ modulo M. Since A is multigraded, every product of the generators can also be given a weight, where the weight of a product is the sum of the weights of the generators occurring in the product (counting multiplicities). So we can express A as a direct sum $A = A_0 \oplus A_1$, where A_0 is spanned by the products of even weight and A_1 is spanned by the products of odd weight, and this defines a $\mathbb{Z}_2$-grading on A. Consider the ideal I of A generated by all values $g(a_1, a_2, \ldots, a_e)$ with $a_1, a_2, \ldots, a_e \in A_0$. Then, $I = I_0 \oplus I_1$, with $I_0 = I \cap A_0$ and $I_1 = I \cap A_1$. Furthermore, I_0 is an ideal of the subalgebra A_0, and the quotient algebra satisfies the identity $g_i = 0$. By the induction hypothesis, A_0/I_0 is then nilpotent of class not more than m and then Lemma 2 implies that $A/I = A_0/I_0 \oplus A_1/I_1$ is nilpotent of class not more than $n^{nm} - 1 = M - 1$. Therefore $x_1x_2\cdots x_M \in I$. This implies that

$$x_1x_2\cdots x_M = \sum_{r=1}^{l} h_r.$$

Where each each h_r has the form

$$h_r = g(y_{1r}, y_{2r}, \ldots, y_{er})u_r.$$

Where $y_{1r}, y_{2r}, \ldots, y_{er} \in A_0$, and $u_r \in \hat{A}$. Where $\hat{A}$ is the associative algebra generated by $\mathrm{ad}(A)$. Since g is multilinear, we can assume that each h_r is linear in the variables $x_1, x_2, \ldots, x_M$, that y_{jr} for $j = 1, 2, \ldots, r$ are products of these variables and that u_r is a product of adjoints of these variables such that no two products involve the same variables. Since the dimension of the subspace generated by all products of $x_1, x_2, \ldots, x_M$ (in some order) is at most $(M-1)!$ we may also assume that $l \le (M-1)!$. Let $\tau : A \to A$ be given by $x_j\tau = x_{j+M}$ for $j = 1, 2, \ldots$. Note that τ preserves the $\mathbb{Z}_2$-grading on A. Then

$$x_{M+1}x_{M+2}\cdots x_{2M} = (x_1x_2\cdots x_M)\tau = \sum_{r=1}^{l} h_r\tau$$

and generally

$$x_{jM+1}x_{jM+2}\cdots x_{(j+1)M} = \sum_{r=1}^{l} h_r\tau^j.$$

Because

$$Q^M_{J_M}\cdots Q^1_{J_1}H = Q^M_{J_M}\cdots Q^1_{J_1}(\sum h_r)(\sum h_r\tau)\cdots(\sum h_r\tau^{K-1})$$

is not zero, we have

$$W := Q^M_{J_M}Q^{M-1}_{J_{M-1}}\cdots Q^1_{J_1}(h_{r_1})(h_{r_2}\tau)\cdots(h_{r_K}\tau^{K-1}) \neq 0$$

for some $r_1, r_2, \ldots, r_K \in \{1, 2, \ldots, l\}$. We are interested in the subset $J = J_M \subseteq \{1, 2, \ldots, K\}$. Since $d = |J| = (M-1)!2^{T(n,3)} > (M-1)!\,(2^{T(n,3)}-1) \ge l(2^{T(n,3)}-1)$, we have that some $2^{T(n,3)}$ of the indexes r_j, $(j \in J)$ must be equal. Suppose these are $r_{j_1} = r_{j_2} = \cdots = r_{j_c} = r$, where $j_1, j_2, \ldots, j_c \in J$ and $c = 2^{T(n,3)}$, and where $j_1 < j_2 < \cdots < j_c$. Let $\bar{g} = g(y_{1r}, \ldots, y_{er})$ then

$$(h_{r_1})(h_{r_2\tau})\cdots(h_{r_K}\tau^{K-1}) = (\bar{g}\tau^{j_1-1})V_1(\bar{g}\tau^{j_2-1})V_2\cdots(\bar{g}\tau^{j_c-1})V_c$$

for some $V_1, V_2, \ldots, V_c \in \hat{A}$. We have by property (3) that the ideal generated by an arbitrary value of g is nilpotent of class at most $c-1$. We then have that

$$g(u_1, \ldots, u_e)V_1g(u_1, \ldots, u_e)V_2\ldots g(u_1, \ldots, u_e)V_c = 0$$

for all $u_1, u_2, \ldots, u_e \in A$ and all $V_1, V_2, \ldots, V_c \in \hat{A}$. If we replace each u_t by $u_{1t} + u_{2t} + \cdots + u_{ct}$ we get the multilinear version of this identity

$$\sum_{\sigma_1}\cdots\sum_{\sigma_e}(g(u_{\sigma_1(1)1}, \ldots, u_{\sigma_e(1)e})V_1\cdots(g(u_{\sigma_1(c)1}, \ldots, u_{\sigma_e(c)e})V_c = 0$$

where $\sigma_1, \sigma_2, \ldots, \sigma_e$ run independently over the symmetric group on $\{1, 2, \ldots, c\}$. Now if we let $u_{qt} = y_{tr}\tau^{j_q-1}$ we get $U = 0$ where

$$U := \sum_{\sigma_1} \cdots \sum_{\sigma_e} g(y_{1r}\tau^{j_{\sigma_1(1)}-1}, \ldots, y_{er}\tau^{j_{\sigma_e(1)}-1})V_1 \cdots g(y_{1r}\tau^{j_{\sigma_1(c)}-1}, \ldots, y_{er}\tau^{j_{\sigma_e(c)}-1})V_c,$$

where $\sigma_1, \sigma_2, \ldots \sigma_e$ run over $\mathrm{Sym}(c)$. Now the word W is symmetric in all $\{x_{j+(j_1-1)M}, \ldots, x_{j+(j_c-1)M}\}$ for those j in $\{1, \ldots, M\}$ where x_j has weight 0. But it is antisymmetric when x_j has weight 1. Since each y_{tr} has weight 0 there is an even number of x_j in y_{tr} such that we have antisymmetry in $\{x_{j+(j_1-1)M}, \ldots, x_{j+(j_c-1)M}\}$. This implies that if we get w' by interchanging $y_{tr}\tau^{j_q-1}$ and $y_{tr}\tau^{j_{q'}-1}$ in $w = (h_{r_1})(h_{r_2}\tau)\cdots(h_{r_K}\tau^{K-1})$, where $1 \leq t \leq e$ and $1 \leq q, q' \leq c$, then

$$Q^M_{J_M} \cdots Q^1_{J_1} w' = Q^M_{J_M} \cdots Q^1_{J_1} w = W.$$

Therefore for all the $(c!)^e$ summands $\tilde{w}$ of U we have $Q^M_{J_M} \cdots Q^1_{J_1}\tilde{w} = W$ and therefore

$$\begin{aligned} 0 &= Q^M_{J_M} Q^{M-1}_{J_{M-1}} \cdots Q^1_{J_1} U \\ &= c!^e W. \end{aligned}$$

And because $\mathrm{char}\, k > c$ we have W=0, which is the contradiction we were seeking. □

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Christ Church
Oxford OX1 1DP
England
traustas@ermine.ox.ac.uk