On groups in which every subgroup is subnormal of defect at most three

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Abstract

In this paper we study groups in which every subgroup is subnormal of defect at most 3. Let G be a group which is either torsion-free or of prime exponent different from 7. We show that every subgroup in G is subnormal of defect at most 3 if and only if G is nilpotent of class at most 3. When G is of exponent 7 the situation is different. While every group of exponent 7, in which every subgroup is subnormal of defect at most 3, is nilpotent of class at most 4, there are examples of such groups with class exactly 4. We also investigate the structure of these groups.

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1 Introduction

Let G be a group. A subgroup H in G is said to be subnormal, if there exists a finite series $H = H_0, H_1, \ldots, H_{n-1}, H_n = G$, such that

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G.$$

The length of the shortest such series is called the subnormal defect of H in G. Now let H be an arbitrary subgroup of G. We define the series, $(H^{(G,i)})_{i=1}^{\infty}$, of successive normal closures by induction as follows:

$$H^{(G,0)} = G, \quad H^{(G,i+1)} = H^{H^{(G,i)}}$$

where H^K denotes as usual the normal closure of H in K. It is easy to see that H is subnormal in G if and only if $H^{(G,r)} = H$ for some r, and the smallest integer n such that $H^{(G,n)} = H$ is the subnormal defect of H in G.

An element x in G is called a left Engel element if for each $g \in G$ there exists a positive integer $n(g)$ such that

$$[\cdots[[g, \underbrace{x], x], \cdots, x}_{n(g)}] = 1. \tag{1}$$

If $n = n(g)$ in (1) can be chosen independently of g, then we say that x is a left n-Engel element. We define right Engel elements similarly. An element x in G is called a right Engel element if for each $g \in G$ there exists a positive integer $n(g)$ such that

$$[\cdots[[x, \underbrace{g], g], \cdots, g}_{n(g)}] = 1. \tag{2}$$

If $n = n(g)$ in (2) can be chosen independently of g, then we say that x is a right n-Engel element. If every $x \in G$ is a left Engel element, we say that G is an Engel group and if furthermore every $x \in G$ is a left n-Engel element, we say that G is a n-Engel group.

It is a well known fact that for a finite group G the following are equivalent:

(1) G is nilpotent.
(2) Every subgroup of G is subnormal.
(3) Every cyclic subgroup of G is subnormal.
(4) G is an Engel group.

The only difficult part is that the first statement follows from the last. This was proved originally by M. Zorn [19].

For infinite groups these properties need not be equivalent although it is easy to see that (1)⇒(2)⇒(3)⇒(4). It is known that no two of these properties are equivalent. An example of a non-nilpotent group satisfying (2) was constructed by Heineken and Mohamed in 1968 [10]. It is much easier to find an example of a group G satisfying (3) but not (2). One could for example take G to be the standard wreath product of the group of order 2 with the countably infinite elementary abelian 2-group A. This group satisfies (2) but A is not subnormal. By Golod's example [4] we have finitely generated Engel groups that are not nilpotent. However every group satisfying property (3) is locally nilpotent (every subnormal locally nilpotent subgroup is contained in the Hirsch-Plotkin radical and since every cyclic subgroup is subnormal, the Hirsch-Plotkin radical is the whole group) so (4) does not imply (3).

The groups satisfying (3) are called Baer groups. If every cyclic subgroup in G is subnormal of defect at most n then we say that G is an n-Baer group or a B_n-group. Under the stronger hypothesis that every subgroup of G is subnormal of defect at most n, we say that G is a U_n-group. By a theorem of Roseblade [14], every U_n-group is nilpotent and the nilpotency class is bounded by a function only depending on n. This function is however still not well understood. It is easy to see that every group of class n is a U_n-group and of course every U_n-group is a B_n-group. It is also not difficult to see that every B_n-group is an $(n+1)$-Engel group.

Much is known about these classes for some small values of n. It is obvious that a group is a B_1-group if and only if it is a U_1-group. These groups are called Dedekind groups and their structure is well known [2,3]. A group G is a Dedekind group if and only if G is either abelian or the direct product of a quaternion group of order 8 and an abelian torsion group without elements of order 4.

2-Engel groups are also well understood. A group is a 2-Engel group if and only if the normal closure x^G of an arbitrary element is abelian. Moreover every 2-Engel group is nilpotent of class at most 3 [12]. 2-Baer groups are closely related to 2-Engel groups. It follows from Levi's result that every 2-Engel group is a 2-Baer group. Furthermore Heineken [9] and Mahdavianary [13] have shown that if G is a B_2-group then G is centre-by-2-Engel and nilpotent of class at most 3.

3-Engel groups are much harder. Heineken [8] has proved that a 3-Engel group is nilpotent of class at most 4 if it has no element of order 2 or 5. There are 3-Engel 2-groups and 5-groups that are not nilpotent. In fact there is a 3-Engel 5-group that is not soluble [1], whereas Gupta [6] has shown that 3-Engel 2-groups are soluble. In 1972 L. Kappe and W. Kappe [11] gave a characterisation of 3-Engel groups which is analogous to Levi's theorem on 2-Engel groups. They showed that the following are equivalent:

(1) G is a 3-Engel group
(2) x^G is a 2-Engel group for all $x \in G$
(3) for all $x \in G$ we have that x^G is nilpotent of class at most 2.

Property (3) implies that a 3-Engel group with r generators has nilpotency class at most $2r$. Gupta and Newman [6] have shown that $2r - 1$ is the best upper bound when $r \geq 3$. It also follows from property (3) that every 3-Engel group is a 3-Baer group.

Relatively little is known about 4-Engel groups. It is even still an open question whether they are locally nilpotent. Some partial results can be found in [16] and [18]. In this paper we will be looking at the class of U_3-groups. In general, the class of U_3-groups is contained in the class of 4-Engel groups. We will however see later that if one adds the further restriction on a U_3-group that it is either torsion free or of prime exponent, then the group is a 3-Engel group. Our main results are the following.

Theorem 1 *Let G be a 2-torsion free 3-Engel group in U_3. Then G is nilpotent of class at most 4.*

Theorem 2 *Let G be a group in U_3 that is either torsion free or of exponent p where p is a prime not equal to 7. Then G is nilpotent of class at most 3.*

Since every group that is nilpotent of class at most 3 is in U_3, these two conditions are actually equivalent. The prime 7 turns out to be exceptional. For this prime we get the following structure theorem.

Theorem 3 *Let G be a group of exponent 7 and nilpotency class 4 in U_3. Then G is a 3-Engel group which satisfies the following properties:*
(1) $\gamma_4(G)$ is cyclic of order 7.
(2) The left 2-Engel elements of $G/\gamma_4(G)$ form a subgroup $H/\gamma_4(G)$ of index 7. Furthermore H is a characteristic subgroup subgroup and nilpotent of class 2.
(3) The left 2-Engel elements of G form a characteristic subgroup

which is equal to $Z^3(G)$.
Conversely we have that every 3*-Engel group of exponent* 7 *that satisfies (1), (2) and (3) is in* U_3.

The groups of exponent 7 can also be described in terms of certain groups $G(r, 7)$ in U_3 that will be constructed later.

Theorem 4 *Let* $r \geq 3$ *and let* G *be an* r *generator group of exponent* 7 *in* U_3 *that is nilpotent of class* 4. *Then* G *is a homomorphic image of* $G(r, 7)$.

When $r = 3$ we will see that the situation is quite simple.

Theorem 5 *There is exactly one group of exponent* 7 *in* U_3 *that has* 3 *generators and nilpotency class* 4. *This is the group* $G(3, 7)$.

We will also give a complete classification of all groups G of exponent 7 in U_3 that are nilpotent of class 4 and which are minimal with respect to that property in the following sense: every proper quotient of G is nilpotent of class at most 3. In particular we will see that there are no such groups of even rank and for an odd integer $n \geq 3$ there are exactly $(n-1)/2$ such groups of rank n.

2 Upper bounds for nilpotency classes

As we pointed out in the introduction, every 3-Engel group is a 3-Baer group. The converse holds when G is either torsion free or of prime exponent.

Lemma 1 *Let* G *be a* n*-Baer group which is either torsion free or of prime exponent. Then* G *is a* n*-Engel group.*

Proof Let us first introduce some notation. Let $x, y \in G$. We define the commutators $[x,_n y]$ and $[y_n, x]$ inductively as follows: $[x,_0 y] = [y_0, x] = x$, $[x,_{n+1} y] = [[x,_n y], y]$ and $[y_{n+1}, x] = [y, [y_n, x]]$. The commutator $[x,_n y] \in \langle y \rangle^{(G,n)}$. Since G is a n-Baer group, we have $\langle y \rangle^{(G,n)} = \langle y \rangle$ and hence $[x,_n y] = y^r$ for some integer r. Since every n-Baer group is a $(n+1)$-Engel group, we can then infer that

$$[x,_{n-1} y, [x,_n y]] = y^{r^2},$$

and by induction that

$$y^{r^m} = [[x,_{n-1}\, y]_m, y].$$

But Baer groups are locally nilpotent and as a consequence we must have $y^{r^m} = 1$ for some positive integer m. Since G is either torsion free or of prime exponent this implies that $y^r = 1$. Hence $[x,_n y] = 1$. □

In this section we will be looking at U_3-groups which are either torsion free or of prime exponent. By Lemma 1 we have that these groups are 3-Engel groups.

Lemma 2 *Let G be a 3-Engel group. If $u \in Z^4(G)$, the 4th term of the upper central series, then*

$$\begin{aligned}
[u,v,c,c]^{12} &= [u,[v,c,c]]^{-8} \\
[u,c,v,c]^{6} &= [u,[v,c,c]]^{-2} \\
[u,c,c,v]^{4} &= [u,[v,c,c]]^{4}.
\end{aligned}$$

for all $v, c \in G$.

Proof From the 3-Engel identity, we have

$$\begin{aligned}
1 &= [u,vc,vc,vc][u,vc^{-1},vc^{-1},vc^{-1}] \\
&= [u,v,c,c]^2[u,c,v,c]^2[u,c,c,v]^2
\end{aligned}$$

and

$$\begin{aligned}
1 &= [v,uc,uc,uc][v,uc^{-1},uc^{-1},uc^{-1}] \\
&= [v,u,c,c]^2[v,c,u,c]^2[v,c,c,u]^2 \\
&= [u,v,c,c]^{-6}[u,c,v,c]^6[u,c,c,v]^{-2}.
\end{aligned}$$

Also

$$[u,[v,c,c]] = [u,v,c,c][u,c,v,c]^{-2}[u,c,c,v].$$

We can now derive the lemma from these three equations. □

Lemma 3 *Let G be a 3-Engel group that is nilpotent of class at most 4 and has no element of order 2 or 3. If $[v,c,c] \in Z(G)$ then every commutator $[x_1,x_2,x_3,x_4]$ of weight 4 with v occurring at least once and c at least twice, is trivial.*

Proof Follows immediately from Lemma 2. □

The next lemma is crucial for later arguments.

Lemma 4 *Let G be a group in U_3 which is either torsion free or of prime exponent. If $u \in Z^4(G)$ and $[u,[v,c,c]] \neq 1$ for some $v, c \in G$, then*

$$[u,[v,c,c]] = [v,c,c]^{\beta}[c,v,v]^{\gamma}$$

for some integers β and γ where $[c,v,v]^{\gamma} \neq 1$. Furthermore, if we let $v_1 = c^{-\beta}v^{\gamma}$ then

$$[u,[v,c,c]]^{\gamma} = [u,[v_1,c,c]] = [c,v_1,v_1].$$

When G is of prime exponent then we can take our power indices from $\mathbb{Z}_p$. If we let $v_2 = c^{-\beta/\gamma}v$ then

$$[u,[v,c,c]] = [u,[v_2,c,c]] = [c,v_2,v_2]^{\gamma}.$$

Proof We have that

$$[u,[v,c,c]] = [u,v,c,c][u,c,v,c]^{-2}[u,c,c,v].$$

Since all the factors on the right hand side are in $\langle v,c\rangle^{(G,3)}$, the same is true for $[u,[v,c,c]]$. But since G is in U_3, $\langle v,c\rangle^{(G,3)} = \langle v,c\rangle$ and we conclude that $[u,[v,c,c]] \in \langle v,c\rangle$. Since $\langle v,c\rangle$ is a 2-generator 3-Engel group without involutions it is nilpotent of class at most 3 [8]. Therefore

$$[u,[v,c,c]] = v^r c^s [v,c]^{\alpha}[v,c,c]^{\beta}[c,v,v]^{\gamma}$$

for some integers $r, s, \alpha, \beta, \gamma$. Since $[u,[v,c,c]] \in Z(G)$,

$$1 = [u,[v,c,c],c,c] = [v,c,c]^r$$

and

$$1 = [u,[v,c,c],v,c] = [v,c,c]^{-s}.$$

As $[u,[v,c,c]] \neq 1$, we must have $r = s = 0$ (modulo p when G is of exponent p). Therefore

$$1 = [u,[v,c,c],c] = [v,c,c]^{\alpha}$$

which implies as before that $\alpha = 0$. So

$$[u, [v, c, c]] = [v, c, c]^{\beta}[c, v, v]^{\gamma}.$$

If $\gamma = 0$ then $\beta \neq 0$ (modulo p when G is of exponent p) and $1 = [u, [v, c, c], u] = [u, [v, c, c]]^{-\beta}$ which would imply that $[u, [v, c, c]] = 1$. Hence we must have $\gamma \neq 0$. Simple calculations give the rest of the lemma. $\square$

Proposition 1 *Let G be a group in U_3 that is either torsion free or of prime exponent. Then G is nilpotent of class at most 4.*

Proof This is obviously true when G is of exponent 2. It is also well known that all groups of exponent 3 are nilpotent of class at most 3. We can thus assume that G is not of exponent 2 or 3. Since G is locally nilpotent it is sufficient to show that $\gamma_5(G) \leq \gamma_6(G)$. We can thus assume that G is nilpotent of class at most 5.

We first reduce the problem to showing that $[x, t, [y, z, z]] = 1$ for all $x, t, y, z \in G$. So suppose we have already established this. We calculate modulo $\gamma_6(G)$. Let $a, b, c, d \in G$. Since G is a 3-Engel group by Lemma 1, we have from Lemma 2 that

$$[x, t, y, z, z] = [x, t, z, y, z] = [x, t, z, z, y] = 1 \tag{3}$$

for all $x, t, y, z \in G$. In particular we have

$$1 = [x, z, tz, tz, y] = [x, z, t, z, y][x, z, z, t, y]. \tag{4}$$

Interchanging x and t in this last identity, using (3), gives

$$\begin{aligned} 1 &= [t, z, x, z, y][t, z, z, x, y] \\ &= [x, t, z, z, y]^{-1}[x, z, t, z, y][x, t, z, z, y]^{-1}[x, z, t, z, y]^2[x, z, z, t, y]^{-1} \\ &= [x, z, t, z, y]^3[x, z, z, t, y]^{-1}. \end{aligned}$$

From this and (4) we deduce that $[x, z, t, z, y] = [x, z, z, t, y] = 1$ for all $x, t, y, z \in G$. From (3) we also have $1 = [x, z, zy, t, zy] = [x, z, z, t, y][x, z, y, t, z]$. We have seen that $[x, z, z, t, y] = 1$, hence $[x, z, y, t, z] = 1$ and we have shown that all commutators of weight 5 with repeated entry are trivial. Thus

$$1 = [x, yz, yz, u, v] = [x, y, z, u, v][x, z, y, u, v],$$

which implies that $[x, y, z, u, v] = [z, x, y, u, v]$. Similarly $[z, x, y, u, v] = [y, z, x, u, v]$. From the Hall-Witt identity we then have

$$\begin{aligned} 1 &= [x, y, z, u, v][y, z, x, u, v][z, x, y, u, v] \\ &= [x, y, z, u, v]^3. \end{aligned}$$

Therefore $\gamma_5(G) \leq \gamma_6(G)$ and G is nilpotent of class at most 4. It now only remains to show that G satisfies the identity $[x, t, [y, z, z]] = 1$ for all $x, t, y, z \in G$. We argue by contradiction and assume that $[a, b, [c, d, d]] \neq 1$ for some $a, b, c, d \in G$. By Lemma 4 we can assume that c has been chosen such that

$$[a, b, [c, d, d]] = [d, c, c]. \tag{5}$$

We see next that $[b, [c, d, d]] \in Z(G)$. We apply Lemma 4 again for $H = G/Z(G)$. From that lemma we would have that $[b, [c, d, d]] \notin Z(G)$ implies that $[d, c, c] \notin Z(G)$. But this contradicts (5). Hence $[b, [c, d, d]] \in Z(G)$ and similarly $[a, [c, d, d]] \in Z(G)$. Finally

$$[c, d, d, [a, b]] = [c, d, d, a, b][c, d, d, b, a]^{-1} \in [Z(G), G] = 1$$

which is the contradiction we were looking for. So $[x, t, [y, z, z]] = 1$ for all $x, t, y, z \in G$ and the proposition has been proved. $\square$

Remark From Heineken [8] we already know that every $\{2, 5\}$-torsion free 3-Engel group is nilpotent of class at most 4. The proof of Proposition 1 is thus only needed for groups of exponent 5.

All the groups in Proposition 1 are 3-Engel groups. As a corollary we get the following generalisation.

Theorem 1 *Let G be a 2-torsion free 3-Engel group in U_3. Then G is nilpotent of class at most 4.*

Proof By Gupta and Newman [7], every 2-torsion free 3-Engel group G has the property that $\gamma_5(G) \cap G^5 = \{1\}$. So any 2-torsion free 3-Engel group is a subdirect product of a nilpotent 3-Engel group of class at most 4 and a 3-Engel group that is of exponent 5. Now let G be a 2-torsion free 3-Engel group in U_3. It follows from Proposition 1 that $\gamma_5(G) \leq G^5$ and since $\gamma_5(G) \cap G^5 = \{1\}$, we deduce that $\gamma_5(G) = \{1\}$.

Remark It is interesting that while the prime 5 is exceptional for 3-Engel groups in general [1], this is no longer the case when we are in the subclass of 3-Engel groups in U_3.

When G is not of exponent 7 more can be said. We will see in the next section that 7 is exceptional.

Theorem 2 *Let G be a group in U_3 that is either torsion free or of exponent p where p is a prime not equal to 7. Then G is nilpotent of class at most 3.*

Proof As we noted in the proof of Proposition 1, this follows easily if G is of exponent 2 or 3. We therefore exclude those possibilities as well as exponent 7. Since 2-Engel groups without elements of order 3 are known to be nilpotent of class at most 2 [12], we only need to show that G satisfies $[y, z, z, x] = 1$ for all $x, y, z \in G$. Proposition 1 tells us that G is nilpotent of class at most 4. As in the proof of Proposition 1, we argue by contradiction and assume that $[a, [b, c, c]] \neq 1$ for some $a, b, c \in G$. By Lemma 4 we can choose a, b, c such that

$$[a, [b, c, c]] = [c, b, b] \neq 1. \tag{6}$$

By Lemma 4 it is also true for all integers r that

$$[a, [b, c, c]] = [a, [b[a, c]^r, c, c]] = [b[a, c]^r, c, c]^\beta [c, b[a, c]^r, b[a, c]^r]^\tau$$

for some integers β and τ where $\tau \neq 0$ (modulo p when G is of exponent p). This time β must be trivial. Otherwise we would have

$$1 = [a, [b, c, c], a] = [c, b, b, a]^\tau [b, c, c, a]^\beta$$

and since, by (6), $[c, b, b] \in Z(G)$ we would get the contradiction that $[a, [b, c, c]] = 1$. So $\beta = 0$ and it follows that $[a, [b, c, c]] \in \langle [c, b[a, c]^r, b[a, c]^r] \rangle$ for all r. In particular

$$1 \neq [c, b[a, c]^r, b[a, c]^r] = [c, b, b][c, b, [a, c]]^r [c, [a, c], b]^r$$

for all r. But

$$([c, b, [a, c]][c, [a, c], b])^6 = [a, c, b, c]^6 [a, c, c, b]^{-12}$$

which by Lemma 2 is $[a,[b,c,c]]^{-14}$. Hence

$$1 \neq [a,[b,c,c]]^{6-14r}$$

for all integers r. Since the group is not of exponent 7 this gives a contradiction when G is of prime exponent. In the case when G is torsion free, we argue similarly. As

$$[a,[b,c,c]]^6 \in \langle [a,[b,c,c]]^{6-14r} \rangle$$

for all r, we conclude that $6-14r$ divides 6 for all integers r which is absurd. □

3 Groups of exponent 7 in U_3

In the previous section we saw that groups of prime exponent p in U_3 are nilpotent of class at most 3 when $p \neq 7$. In this section we will see that there are groups of exponent 7 that are nilpotent of class 4 and we will investigate their structure.

Let G be a group of exponent 7 in U_3 that is nilpotent of class 4. Since 2-Engel groups of exponent 7 are nilpotent of class at most 2 there must exist elements $c, a_1, b_1 \in G$ such that $[a_1,[b_1,c,c]] \neq 1$. Let $T = \langle [a_1,[b_1,c,c]] \rangle$. (We will see later that $T = \gamma_4(G)$).

Lemma 5 *There exist $a, b \in \langle a_1, b_1, c\rangle [G,G]$ such that:*
(1) $\langle [a,[b,c,c]] \rangle = T$;
(2) $[c,h,h] \in T$ for all $h \in \langle a,b\rangle[G,G]$ and $[c,h,h] = 1$ if and only if $h \in [G,G]$;
From (1) and (2) it follows in particular that $[c,a,a]$, $[c,b,b]$ and $[c,a,b][c,b,a]$ are in T.

Proof Lemma 2 gives us that $[b_1,[a_1,c,c]] = [a_1,[b_1,c,c]]^{-1}$. We can then apply Lemma 4 twice to find $a, b \in \langle a_1, b_1, c\rangle[G,G]$ such that $[a,[b,c,c]] = [a_1,[b_1,c,c]]$ and that $[c,a,a], [c,b,b]$ are nontrivial elements in T. As a consequence of this and Lemma 3, every commutator $[x_1,x_2,x_3,x_4]$ with c occurring at least once and either a or b occurring at least twice is trivial. Therefore $[c,h,h,k] = 1$ for all $h, k \in \langle a,b\rangle[G,G]$. Now let $h \in \langle a,b\rangle[G,G] \setminus [G,G]$ and choose $k \in \langle a, b\rangle$ which is linearly independent from h modulo $[G, G]$. Then $[k, [h, c, c]]$ is a nontrivial multiple of $[a, [b, c, c]]$. Lemma 4 tells us further that

$$[k, [h, c, c]] = [h, c, c]^{\beta}[c, h, h]^{\gamma}$$

for some $\beta, \gamma \in \mathbb{Z}_7$ where $[c, h, h]^{\gamma} \neq 1$. Taking the commutator with k on both sides and using the fact that $[c, h, h, k] = 1$ we get $[k, [h, c, c]]^{\beta} = 1$. Therefore $\beta = 0$ which implies that $[c, h, h] \in T$. This proves (2). Since $[c, ab, ab] = [c, a, a][c, b, b][c, a, b][c, b, a]$, it is now clear from (2) that $[c, a, b][c, b, a] \in T$. □

Lemma 6 *Let $d_1 \in G\backslash\langle c, a, b\rangle[G, G]$. Then there is an element $d \in d_1\langle a, b, c\rangle$ such that:*
(1) $\langle[a, [d, c, c]]\rangle = \langle[b, [d, c, c]]\rangle = T$;
(2) $[c, d, d]$ is a nontrivial element in T and $[c, a, d][c, d, a], [c, b, d][c, d, b] \in T$.

Proof Since $[a, [b, c, c]] \neq 1$ there are some $r, s \in \mathbb{Z}_7$ such that $[a, [d_1 a^r b^s, c, c]] \neq 1$ and $[b, [d_1 a^r b^s, c, c]] \neq 1$. Lemma 4 implies that there is some $d \in d_1 a^r b^s \langle c\rangle$ such that $[c, d, d] = [a, [d, c, c]]^{\gamma}$ where $\gamma \neq 0$. G is a 3-Engel group, so clearly $[a, [d, c, c]]$ and $[b, [d, c, c]]$ are nontrivial. From Lemma 4 we have

$$[a, [d, c, c]] = [d, [a, c, c]]^{-1} = [a, c, c]^{\beta_1}[c, a, a]^{\gamma_1},$$

for some $\beta_1, \gamma_1 \in \mathbb{Z}_7$ with $\gamma_1 \neq 0$. By taking the commutator with d on both sides, we see that $\beta_1 = 0$. So $\langle[a, [d, c, c]]\rangle = \langle[c, a, a]\rangle = T$, and similarly $\langle[b, [d, c, c]]\rangle = \langle[c, b, b]\rangle = T$. We now repeat the second part in the proof of Lemma 5 with b replaced by d and conclude that $[c, a, d][c, d, a] \in \langle[a, [d, c, c]]\rangle = T$. Similarly we get $[c, b, d][c, d, b] \in \langle[b, [d, c, c]]\rangle = T$. □

Now let a and b satisfy the conditions given in Lemma 5 and add to them elements d_i, $i \in I$, such that $\{c, a, b, d_i : i \in I\}$ is a minimal set of generators for G with each d_i having the same properties as d in Lemma 6. Let $H = \langle a, b, d_i : i \in I\rangle[G, G]$. The next proposition contains some of the essential properties which will be needed for our first structure theorem.

Proposition 2 *H is a 2-Engel group and the following properties are satisfied for all $h \in H$ and $u \in [G, G]$:*
(1) $[c, h, h] \in T$;
(2) $[c, hu, hu] = [c, h, h]$.

Proof We know from Lemma 5 and Lemma 6 that $[c, e, e] \in T$ for all $e \in \{a, b, d_i : i \in I\}$ and that $[c, a, b][c, b, a]$, $[c, a, d_i][c, d_i, a]$ and $[c, b, d_i][c, d_i, b]$ are in T for all $i \in I$. If we can prove that $[c, d_i, d_j][c, d_j, d_i] \in T$ for all $i, j \in I$ and that $[c, h_1, h_2, h_3] = 1$ for all $h_1, h_2, h_3 \in H$, then (1) is true.

As a preliminary step we show that $[c, d_i, d_j][c, d_j, d_i] \in Z(G)$ for all $i, j \in I$. Since $[d_i, [a, c, c]] \neq 1$, there is some $r \in \mathbb{Z}_7$ such that $[d_i, [d_j a^r, c, c]] \neq 1$. Applying Lemma 4 we conclude that

$$\begin{aligned} [d_i, [d_j a^r, c, c]] &= [d_i, [d_i d_j a^r, c, c]] \\ &= [d_i d_j a^r, c, c]^\beta [c, d_i d_j a^r, d_i d_j a^r]^\gamma, \end{aligned}$$

for some $\beta, \gamma \in \mathbb{Z}_7$ with $\gamma \neq 0$. We know that $[c, d_i, d_i]$ and $[c, d_j a^r, d_j a^r]$ are in T. From this and Lemma 3 we infer that $[c, d_i d_j a^r, d_i d_j a^r, d_i] = 1$. So if we take the commutator with d_i on both sides in the equation above we see that we must have $\beta = 0$. Therefore $[c, d_i d_j a^r, d_i d_j a^r]$ is in $\langle [d_i, [d_j a^r, c, c]] \rangle$. Similarly we see that $\langle [d_i, [d_j a^r, c, c]] \rangle = \langle [c, d_i, d_i] \rangle = T$. We conclude from this that

$$[c, d_i d_j a^r, d_i d_j a^r] \in T. \tag{7}$$

By expanding this commutator and using what we have established so far we see that $[c, d_i, d_j][c, d_j, d_i] \in T\gamma_4(G) \leq Z(G)$.

It follows from the previous paragraph that $[c, h, h]$ is in $Z(G)$ for all $h \in H$. We will now use this fact to prove that H is a 2-Engel group and thus nilpotent of class at most 2. We consider a few cases. First assume that $h, k \in H$ satisfy $[h, [k, c, c]] \neq 1$. Since $[c, h, h] \in Z(G)$ we have by Lemma 3 that every commutator $[x_1, x_2, x_3, x_4]$, with c occurring once and h or k occurring twice, is trivial. Therefore $[h, [k, ch^r, ch^r]] = [h, [k, c, c]]$ for all $r \in \mathbb{Z}_7$. We can then apply Lemma 4 to see that

$$[h, [k, c, c]] = [h, [k, ch^r, ch^r]] = [k, ch^r, ch^r]^{\beta_r} [ch^r, k, k]^{\gamma_r}$$

where $\gamma_r \neq 0$. Taking the commutator with h on both sides gives as before that $\beta_r = 0$. Therefore the expression becomes

$$[h, [k, c, c]] = [c, k, k]^{\gamma_r} [h, k, k]^{r\gamma_r}.$$

Hence $[c,k,k]$ and $[h,k,k]$ are both in $\langle [h,[k,c,c]]\rangle$. If $[h,k,k]\neq 1$ then we could choose r such that $[c,k,k][h,k,k]^r=1$ and we would get the contradiction that $[h,[k,c,c]]=1$. This means that we must have $[h,k,k]=1$ whenever $[h,[k,c,c]]\neq 1$. Let us now assume that $[h,[k,c,c]]=1$ but that $[a,[k,c,c]]\neq 1$. Then $[ah,[k,c,c]]$ is also nontrivial and by the previous case we get that

$$[a,k,k]=[ah,k,k]=1.$$

Now Lemma 3 gives us that $[a,k,h,k]=1$ and thus

$$[h,k,k]=[ah,k,k]=1.$$

Next suppose that $[h,[k,c,c]]=1$ but that $[h,[b,c,c]]\neq 1$. Then $[h,[bk,c,c]]$ and $[h,[b^{-1}k,c,c]]$ are nontrivial as well and by the first case we can deduce that

$$[h,b,b]=[h,bk,bk]=[h,b^{-1}k,b^{-1}k]=1.$$

Then from Lemma 3 we also know that $[h,b,k,b]=[h,k,b,b]=1$. Therefore

$$\begin{aligned} 1 &= [h,bk,bk][h,b^{-1}k,b^{-1}k] \\ &= [h,k,k][h,k,b][h,b,k][h,k,b,k][h,b,k,k] \\ &\quad [h,k,k][h,k,b]^{-1}[h,b,k]^{-1}[h,k,b,k]^{-1}[h,b,k,k]^{-1} \\ &= [h,k,k]^2. \end{aligned}$$

Finally we are left with the situation when $[h,[k,c,c]]=[a,[k,c,c]]=[h,[b,c,c]]=1$. But $[a,[b,c,c]]\neq 1$ and since $[a,[k,c,c]]=1$ we can infer from this and our previous case that $[a,k,k]=1$. Since $[ah,[b,c,c]]\neq 1$ and $[ah,[k,c,c]]=1$, we similarly deduce that $[ah,k,k]=1$. Lemma 3 gives us as before that $[a,k,h,k]=1$ and thus

$$1=[h,k,k]=[ah,k,k][a,k,k]^{-1}=1.$$

We have therefore shown that H is a 2-Engel group and thus nilpotent of class at most 2. It follows in particular that $[c,h_1,h_2,h_3]=1$ for all $h_1,h_2,h_3\in H$. Hence we can derive from equation (7) that $[c,d_i,d_j][c,d_j,d_i]\in T$ for all $i,j\in I$. As we said in the beginning of the proof we can conclude from this that statement (1) holds.

We now turn to the second statement. Let $e\in\{a,b,d_i : i\in I\}$ and

$u \in [G,G]$. By Lemma 6 we have that $[e,c,c]$ is not in $Z(G)$. It follows that $[eu^r,c,c]$ is not in $Z(G)$ for all $r \in \mathbb{Z}_7$. Lemma 4 then tells us that $[c,eu^r,eu^r]$ is nontrivial for all $r \in \mathbb{Z}_7$ and by the first part of the proposition we see that $[c,eu^r,eu^r] \in T$ for all $r \in \mathbb{Z}_7$. In particular $[c,eu,eu] = [c,e,e]^\alpha$ for some $\alpha \in \mathbb{Z}_7$ which implies that $[c,e,u][c,u,e] = [c,e,e]^{\alpha-1}$. If $\alpha \neq 1$ then $\alpha - 1$ would have an inverse τ in $\mathbb{Z}_7$ and then $[c,eu^{-\tau},eu^{-\tau}] = [c,e,e][c,e,e]^{-1} = 1$ which gives a contradiction. Hence $\alpha = 1$ and $[c,e,u][c,u,e] = 1$. Since this is true for all $e \in \{a,b,d_i : i \in I\}$ we conclude that $[c,h,u][c,u,h] = 1$ for all $h \in H$. The second statement clearly results from this. □

We are now ready to state and prove the first structure theorem of this section.

Theorem 3 *Let G be a group of exponent 7 and nilpotency class 4 in U_3. Then G is a 3-Engel group which satisfies the following properties:*
(1) $\gamma_4(G)$ is cyclic of order 7.
(2) The left 2-Engel elements of $G/\gamma_4(G)$ form a subgroup $H/gamma_4(G)$ of index 7. Furthermore H is a characteristic subgroup and nilpotent of class 2.
(3) The left 2-Engel elements of G form a characteristic subgroup which is equal to $Z^3(G)$.

Conversely we have that every 3-Engel group of exponent 7 that satisfies (1), (2) and (3) is in U_3.

Proof Lemma 1 gives us that G must be a 3-Engel group. Let $\{c,a,b,d_i : i \in I\}$ be a minimal set of generators. Suppose furthermore that c is chosen such that $[a_1,[b_1,c,c]] \neq 1$ for some a_1, b_1; that a and b are then chosen as in Lemma 5 and that all the d_i are chosen with the same properties as d in Lemma 6. Let $H = \langle a,b,d_i : i \in I\rangle[G,G]$. By Proposition 2 we have that H is a 2-Engel group and thus nilpotent of class at most 2 by Levi [12]. We also have from Proposition 2 that $[c,h,h] \in \gamma_4(G)$ for all $h \in H$ and therefore $H/\gamma_4(G)$ is a set of left 2-Engel elements in $G/\gamma_4(G)$. Clearly H is of index 7. Also since H is nilpotent of class at most 2, $[a,ch,ch,b] = [a,c,c,b] \neq 1$ for all $h \in H$. Then $H/\gamma_4(G)$ is the set of all left 2-Engel elements and we have proved (2).

Consider some commutator of weight 4 in the generators. If c occurs three times then the commutator is trivial since G is a 3-Engel group. If c occurs at most once then the commutator is again trivial because $\gamma_3(H) = 1$. So if the commutator is nontrivial c occurs necessarily exactly twice. So we have a commutator of c, h_1,h_2 with c occurring twice and where h_1,h_2 are in H. By Lemma 2, the commutator is in $\langle [h_1, [h_2, c, c]\rangle$. If $[h_1, [h_2, c, c]] \neq 1$ then Lemma 4 tells us that

$$[h_1, [h_2, c, c]] = [h_2, c, c]^{\beta}[c, h_2, h_2]^{\gamma}$$

where $[c, h_2, h_2]^{\gamma} \neq 1$. Taking the commutator with h_1 on both sides and using the fact that $\gamma_3(H) = 1$, we see that $[h_1, [h_2, c, c]]^{-\beta} = 1$. Therefore $\beta = 0$ and $[h_1, [h_2, c, c]] \in \langle [c, h_2, h_2]\rangle$ which is contained in $\langle [a, [b, c, c]]\rangle$ by Proposition 2. So we have proved that $\gamma_4(G) = \langle [a, [b, c, c]]\rangle$, and it is thus cyclic of order 7.

To prove (3) we apply Proposition 2 again. We define a map q_c from $H/[G, G]$ to $\gamma_4(G)$ as follows:

$$q_c(h[G, G]) = [c, h, h].$$

By Proposition 2 this is well defined and since $\gamma_4(G)$ is cyclic, this gives a quadratic form on the vector space $H/[G, G]$. We first choose an orthogonal basis for $\langle a, b\rangle[G, G]/[G, G]$ with respect to q_c and then expand it to an orthogonal basis for the whole vector space $H/[G, G]$. Without loss of generality we can assume that $\bar{a} = a[G, G]$ and $\bar{b} = b[G, G]$ are orthogonal. We use additive notation for the group operations in $H/[G, G]$ and $\gamma_4(G)$. Since -1 is a non-square in $\mathbb{Z}_7$ and since by Lemma 5 we know that $\langle a, b\rangle[G, G]/[G, G]$ is a regular subspace, we can furthermore choose a and b such that $q_c(\bar{b})$ is equal to either $q_c(\bar{a})$ or $-q_c(\bar{a})$. But by Lemma 5 $q_c(x) = 0$ if and only if $x = 0$, and this can only happen in the first case. So we can assume that $q_c(\bar{b}) = q_c(\bar{a})$. Let $\{\bar{a}, \bar{b}, \bar{d}_i = d_i[G, G] : i \in I\}$ be the orthogonal basis for $H/[G, G]$. Our next step is to prove that $q_c(\bar{d}_i) = 0$ for all $i \in I$. Since every element in $\mathbb{Z}_7$ can be written as a sum of two squares, we can find $r, s \in \mathbb{Z}_7$ such that $q_c(r\bar{a} + s\bar{b} + \bar{d}_i) = q_c(r\bar{a} + s\bar{b} - \bar{d}_i) = 0$. Then $[c, a^r b^s d_i, a^r b^s d_i] = [c, a^r b^s d_i^{-1}, a^r b^s d_i^{-1}] = 1$ and by Lemma 4 we must have $[a^r b^s d_i, c, c], [a^r b^s d_i^{-1}, c, c] \in Z(G)$. We multiply these elements together and see that $[a^r b^s, c, c] \in Z(G)$. But this can only happen when $(r, s) = (0, 0)$ and thus $q_c(\bar{d}_i) = 0$. In other words we have shown that $D = \langle d_i : i \in I\rangle[G, G]$ consists of left 2-Engel elements in G. We next show that there are no other left 2-Engel elements in G. Suppose $x = c^r a^s b^t$ is a left 2 Engel element in G. Since $[a, x, x, b] = [a, c, c, b]^r$, we must have $r = 0$. Then $0 = q_c(s\bar{a}+t\bar{b}) = (s^2+t^2)q_c(\bar{a})$ which implies that $(s, t) = (0, 0)$. Therefore D is the set of all left 2-Engel elements of G. Finally let $[d, x_1, x_2, x_3]$ be a commutator such that x_1, x_2, x_3 are in $\{c\} \cup H$ and $d \in D$. Since $[c, d, d] = 1$ it follows from Lemma 4 that $[d, c, c] \in Z(G)$ and thus we have from Lemma 3 that $[d, x_1, x_2, x_3] = 1$ if c occurs at least twice. Since $\gamma_3(H) = 1$ this is also true if c occurs at most once. Hence $d \in Z^3(G)$. Clearly $\langle a, b, c\rangle \cap Z^3(G) \leq [G, G]$ so $D = Z^3(G)$ and we have proved (3).

Now suppose that G is a 3-Engel group of exponent 7 satisfying (1), (2) and (3). We know from Heineken [8] that G is nilpotent and from (1) that G is nilpotent of class 4. To show that G is in U_3 we need to show that $[G, K, K, K] \leq K$ for every subgroup K of G. Since G is nilpotent of class 4 this is equivalent to

$$[u, v_1, v_2, v_3] \in \langle v_1, v_2, v_3\rangle$$

for all $u, v_1, v_2, v_3 \in G$. Suppose $[u, v_1, v_2, v_3] \neq 1$. Since H is nilpotent of class at most 2 it is necessary that at least one of v_1, v_2, v_3 is not in H. Call this element c. Then $\langle v_1, v_2, v_3\rangle = \langle c\rangle(H \cap \langle v_1, v_2, v_3\rangle)$. Since $[u, c, c, c] = 1$ we cannot have that $H \cap \langle v_1, v_2, v_3\rangle \subseteq Z^3(G)$. Let $h \in H \cap \langle v_1, v_2, v_3\rangle \setminus Z^3(G)$. By (3) we have that h is not a left 2-Engel element in G and since $\gamma_3(H) = 1$ this is equivalent to saying that $[c, h, h] \neq 1$. From (1) and (2) we conclude that $\langle [c, h, h]\rangle = \gamma_4(G)$. Therefore $[u, v_1, v_2, v_3] \in \langle [c, h, h]\rangle \subseteq \langle v_1, v_2, v_3\rangle$. □

Remark Let G be as in Theorem 3. It is not difficult to see that $Z^3(G)/Z(G)$ is the set of right 2-Engel elements of $G/Z(G)$ and that $Z^2(G)$ is the set of right 2-Engel elements of G. Notice also that the proof tells us that $Z^3(G)$ has index 49 in H.

Our next result will establish the existence of groups with the properties given in Theorem 3. We will see that for each cardinal $r \geq 3$, there is an r-generator group $G(r, 7)$ of exponent 7 in U_3 that is nilpotent of class 4 and has the further property that every r-generator group of exponent 7 in U_3 that is nilpotent of class 4 is a quotient of $G(r, 7)$.

Let r be a cardinal greater than 2. Let $E(r, 7)$ be the relatively free r-

generator 3-Engel group of exponent 7 with nilpotency class at most 4 and let the free generators be $\{z, x, y, t_i : i \in I\}$ where the cardinal of I is $r - 3$. Let $H_0 = \{x, y, t_i : i \in I\}$ and $D_0 = \{t_i : i \in I\}$. We define $N(r, 7)$ as the normal closure of the set of following elements:

$$[z,x,x][x,[y,z,z]]^{-1} \qquad [z,y,y][x,[y,z,z]]^{-1}; \tag{8}$$
$$[t,z,z,u], \quad [z,t,t] \qquad t \in D_0, u \in H_0; \tag{9}$$
$$[z,u,v][z,v,u] \qquad u,v \in H_0 \text{ and } u \neq v; \tag{10}$$
$$[z,u,v,w] \qquad u,v,w \in H_0; \tag{11}$$
$$[u,v,w] \qquad u,v,w \in H_0. \tag{12}$$

Finally we let $G(r, 7) = E(r, 7)/N(r, 7)$.

Theorem 4 *$G(r, 7)$ is in U_3 and is nilpotent of class 4. Furthermore, if G is an r-generator group of exponent 7 in U_3 that is nilpotent of class 4 then G is a homomorphic image of $G(r, 7)$.*

Proof Let G be an r-generator group of exponent 7 in U_3 that is nilpotent of class 4. Then G is a 3-Engel group and we saw in the proof of Theorem 3 that G has a set of generators $\{c, a, b, d_i : i \in I\}$ with all d_i in $Z^3(G)$ and such that the following properties hold:

$$[c,a,a] = [c,b,b] \in \langle [a,[b,c,c]] \rangle; \tag{13}$$
$$[c,d_i,d_i] = 1 \qquad \text{for all } i \in I; \tag{14}$$
$$[c,u,v][c,v,u] = 1 \qquad \text{for all } u,v \in \{a,b,d_i : i \in I\} \text{ where } u \neq v. \tag{15}$$

Let $H = \langle a, b, d_i : i \in I \rangle [G, G]$. The proof of Theorem 3 implies that $\gamma_3(H) = 1$. Therefore we have:

$$[c,u,v,w] = 1 \qquad \text{for all } u,v,w \in \{a,b,d_i : i \in I\}; \tag{16}$$
$$[u,v,w] = 1 \qquad \text{for all } u,v,w \in \{a,b,d_i : i \in I\}. \tag{17}$$

Suppose that $[c, a, a] = [c, b, b] = [a, [b, c, c]]^{\tau}$. Then

$$[c^{\tau},a,a] = [c^{\tau},b,b] = [a,[b,c^{\tau},c^{\tau}]].$$

We define a homomorphism from $E(r,7)$ to G by mapping (z,x,y) to (c^τ,a,b) and each t_i to d_i. Since $N(r,7)$ is the normal closure of the elements (8)-(12) it follows from the relations above that $N(r,7)$ is in the kernel of this homomorphism and thus G is a homomorphic image of $G(r,7)$. So it only remains to show that $G(r,7)$ has the properties in the statement of the theorem.

Let $c = zN(r,7)$, $a = xN(r,7)$, $b = yN(r,7)$ and $d_i = t_iN(r,7)$ for all $i \in I$. Then let $H = \langle a,b,d_i : i \in I\rangle[G(r,7),G(r,7)]$ and $D = \langle d_i : i \in I\rangle[G(r,7),G(r,7)]$. It clearly results from relations (8)-(12) and Lemma 2 that H is nilpotent of class at most 2, and that $\gamma_4(G(r,7)) = \langle[a,[b,c,c]]\rangle$. It is also clear from these relations that the elements of $H/\gamma_4(G(r,7))$ are left 2-Engel elements in $G/\gamma_4(G(r,7))$ and that the elements of D are left 2-Engel elements of $G(r,7)$. Lemma 2, (9) and (11) also give that $D \leq Z^3(G)$. To finish the proof we need to establish two things. That $[a,[b,c,c]] \neq 1$ and that all left 2-Engel elements in $G(r,7)$ are contained in D. From $[a,[b,c,c]] \neq 1$ we deduce then that $Z^3(G) \leq D$ and Theorem 3 implies that $G(r,7)$ has the claimed properties. It will follow from the next theorem that $[a,[b,c,c]] \neq 1$. We will now assume this and show that D contains all the left 2-Engel elements.

Let

$$g = c^u a^s b^t [c,a]^l [c,b]^m [a,b]^n d$$

where $d \in \langle d_i : i \in I\rangle^{(G(r,7)}\gamma_3(G(r,7))$. Clearly every element in $G(r,7)$ can be written in this form. Suppose that g is a left 2-Engel element then

$$1 = [b,g,g,a] = [b,c,c,a]^u$$

and thus $u = 0$ (since we are assuming that $[a,[b,c,c]] \neq 1$). From this and the fact that $\gamma_3(H) = 1$ we can infer that g is a left 2-Engel element if and only if $[c,g,g] = 1$. Now

$$\begin{aligned}[c,g,g] &= [c,a,a]^{s^2}[c,b,b]^{t^2}[c,a,[c,b]]^{sm} \\ &\quad [c,[c,b],a]^{sm}[c,b,[c,a]]^{tl}[c,[c,a],b]^{tl}.\end{aligned}$$

But $[c,b,[c,a]][c,[c,a],b] = [a,c,b,c][a,c,c,b]^{-2}$ which is, by Lemma 2, equal to $[a,[b,c,c]]^{-7/3} = 1$. Similarly we have that $[c,a,[c,b]][c,[c,b],a] = 1$ (it is here that we need the exponent to be 7). Therefore we have

$$1 = [c,a,a]^{s^2}[c,b,b]^{t^2} = [a,[b,c,c]]^{(s^2+t^2)},$$

and since -1 is not a square in $\mathbb{Z}_7$ we must have that $(s,t) = (0,0)$. Hence $g = [c,a]^l[c,b]^m[a,b]^n d \in D$ and we have shown that every left 2-Engel element is in D. It remains to be proved that $\gamma_4(G(r,7)) \neq 1$. Since $G(3,7)$ is a quotient of $G(r,7)$ for all cardinals $r \geq 3$ it is sufficient to show that $\gamma_4(G(3,7)) \neq 1$. This is a consequence of our next result. □

Theorem 5 *There is exactly one group of exponent* 7 *in* U_3 *that has* 3 *generators and nilpotency class* 4. *This is the group* $G(3,7)$.

Let $a = xN(3,7)$, $b = yN(3,7)$ and $c = zN(3,7)$. Then it is clear from the relations (8)-(12) that

$$\begin{aligned} [c,a,a] &= [a,[b,c,c]]; \\ [c,b,b] &= [a,[b,c,c]]; \\ [c,a,b][c,b,a]; &= 1 \\ [a,b,b] &= 1; \\ [b,a,a] &= 1. \end{aligned}$$

From the Hall-Witt identity and Lemma 2 we have:

$$\begin{aligned} 1 &= [a,b,c^a][c,a,b^c][b,c,a^b] \\ &= [a,b,c][c,a,b][b,c,a][c,a,[b,c]] \\ &= [a,b,c][c,a,b]^2[a,[b,c,c]]^{-1}. \end{aligned}$$

It follows that

$$[a,b,c] = [c,a,b]^{-2}[a,[b,c,c]]. \tag{18}$$

From Lemma 2 we have the extra relations

$$\begin{aligned} [a,b,c,c] &= [a,[b,c,c]]^{-3} && (19) \\ [a,c,b,c] &= [a,[b,c,c]]^2 && (20) \\ [a,c,c,b] &= [a,[b,c,c]]. && (21) \end{aligned}$$

Let $a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = [b,a]$, $a_5 = [c,a]$, $a_6 = [c,b]$, $a_7 = [c,a,b]$, $a_8 = [a,c,c]$, $a_9 = [b,c,c]$ and $a_{10} = [a,[b,c,c]]$. We can deduce from the relations above that $G(r,3)$ has a power-commutator presentation with generators $a_1,\ldots,a_{10}$ and the following relations:

$$\begin{aligned}
&a_1^7 = a_2^7 = \cdots = a_{10}^7 = 1,\\
&[a_2,a_1] = a_4, \quad [a_3,a_1] = a_5, \quad [a_3,a_2] = a_6,\\
&[a_4,a_1] = 1, \quad [a_4,a_2] = 1, \quad [a_4,a_3] = a_7^2 a_{10}^6,\\
&[a_5,a_1] = a_{10}, \quad [a_5,a_2] = a_7, \quad [a_5,a_3] = a_8^6,\\
&[a_5,a_4] = 1, \quad [a_6,a_1] = a_7^6, \quad [a_6,a_2] = a_{10},\\
&[a_6,a_3] = a_9^6, \quad [a_6,a_4] = 1, \quad [a_6,a_5] = a_{10}^6,\\
&[a_7,a_3] = a_{10}^5, \quad [a_7,a_i] = 1 \text{ if } i \neq 3\\
&[a_8,a_2] = a_{10}, \quad [a_8,a_i] = 1, \text{ if } i \neq 2\\
&[a_9,a_1] = a_{10}^6, \quad [a_9,a_i] = 1 \text{ if } i \neq 1\\
&[a_{10},a_i] = 1 \text{ for all } i.
\end{aligned}$$

We refer to [17] for a discussion of power-commutator presentations. One can check that this power commutator-presentation is consistent so $G(3,7)$ has order 7^{10} and class 4. It is also easy to see that $Z(G) = \gamma_4(G) = \langle a_{10}\rangle$. This implies that every quotient of $G(3,7)$ is nilpotent of class at most 3. By Theorem 4 we have that every 3-generator group of exponent 7 in U_3 that is nilpotent of class 4 is a quotient of $G(3,7)$, therefore $G(3,7)$ is the only such group. □

When $r \geq 4$ the situation is much more complicated and we will not attempt here to obtain a detailed classification of the r-generator groups of exponent 7 in U_3. We will however proceed a bit further and analyse closely a certain subclass. The group $G(3,7)$ has the property that every proper quotient is nilpotent of class at most 3. Our next result gives a complete classification of all finitely generated groups of exponent 7 in U_3 with this property.

We let r and s be some nonnegative integers and as before we let $E(3+2r+2s,7)$ be the relatively free $(3+2r+2s)$-generator 3-Engel group of exponent 7 with nilpotency class at most 4 and with free generators $\{z,x,y,t_i : 1 \leq i \leq 2r+2s\}$. We recall that $G(3+2r+2s,7)$ was defined as $E(3+2r+2s,7)/N(3+2r+2s,7)$, where $N(3+2r+2s,7)$ was the normal closure of the relations (8)-(12). We rename the generators $t_1,t_2,\ldots,t_{2r+2s}$ as follows:

$$f_i = t_i, \quad g_j = t_{j+2r},$$

for $1 \leq i \leq 2r$ and $1 \leq j \leq 2s$. We extend $N(3+2r+2s,7)$ to a normal subgroup $N(r,s,7)$ which is defined to be the normal closure of the set of elements (8)-(12) together with the set of following elements:

$$[f_i, x],\quad [f_i, y],\quad [f_i, z, z] \qquad 1 \leq i \leq 2r \tag{22}$$

$$[g_i, x],\quad [g_i, y],\quad [g_i, z] \qquad 1 \leq i \leq 2s \tag{23}$$

$$[f_i, f_{2r+1-i}][x, y, z]^{-1} \qquad 1 \leq i \leq r \tag{24}$$

$$[g_i, g_{2s+1-i}][x, [y, z, z]]^{-1} \qquad 1 \leq i \leq s \tag{25}$$

$$[f_i, f_j] \qquad 1 \leq i < j \leq 2r \text{ and } i+j \neq 2r+1 \tag{26}$$

$$[g_i, g_j] \qquad 1 \leq i < j \leq 2s \text{ and } i+j \neq 2s+1 \tag{27}$$

$$[f_i, g_j] \qquad 1 \leq i \leq 2r \text{ and } 1 \leq j \leq 2s. \tag{28}$$

We define $G(r,s,7)$ as $E(3+2r+2s,7)/N(r,s,7)$.

Theorem 6 *For each pair of integers $r, s \geq 0$, we have that the group $G(r,s,7)$ is nilpotent of class 4 and has the property that every proper quotient has class at most 3. Conversely, if G is a finitely generated group of exponent 7 in U_3 which is nilpotent of class 4 and of which every proper quotient has class at most 3 then G is isomorphic to $G(r,s,7)$ for some integers $r, s \geq 0$.*

Proof Let G be a group of exponent 7 in U_3 that is nilpotent of class 4 but of which every proper quotient has class at most 3. This clearly happens if and only if $Z(G) = \gamma_4(G)$, since we have seen that $\gamma_4(G)$ has order 7. Let H be as in Theorem 3 and $D = Z^3(G)$. Choose some $c \in G \setminus H$ and $a, b \in H \setminus D$ such that $[c,a,a] = [c,b,b] = [a,[b,c,c]]$. We then have that $Z(G) = \langle [a,[b,c,c]] \rangle$.

We define a map f from $D/[G,G] \times D/[G,G]$ to $Z(G)$ as follows:

$$f(d[G,G], e[G,G]) = [c,d,e].$$

We can think of f as an antisymmetric bilinear form on the vector space $V = D/[G,G]$. According to the classical theory of antisymmetric bilinear forms, we can find a basis $U_1 = d_1[G,G], \ldots, U_{2r} = d_{2r}[G,G], V_1 = e_1[G,G], \ldots, V_t = e_t[G,G]$ such that (with additive notation)

$$f(U_i, U_j) = \begin{cases} [a,[b,c,c]] & \text{if } i+j = 2r+1 \text{ and } i < j \\ 0 & \text{otherwise,} \end{cases}$$

and such that

$$f(U_i, V_j) = f(V_k, V_j) = 0$$

for all $1 \leq i \leq 2r$ and $1 \leq j, k \leq t$. It follows that $[d_i, d_j]$ commutes with c when $i + j \neq 2r + 1$ and that $[e_i, d_j]$ and $[e_i, e_j]$ commute with c for all i and j. Since $\gamma_3(H) = 1$, all these elements are in $Z(G)$. Suppose

$$\begin{aligned} [d_i, d_j] &= [c, d_{2r+1-j}, d_j]^{2r_{i,j}} && \text{when } i + j \neq 2r + 1; \\ [e_i, d_j] &= [c, d_{2r+1-j}, d_j]^{s_{i,j}} && 1 \leq i \leq t \text{ and } 1 \leq j \leq 2r. \end{aligned}$$

If we let

$$\begin{aligned} \tilde{e_i} &= e_i \prod_{j=1}^{2r} [c, d_j]^{-s_{i,2r+1-j}} \\ \tilde{d_i} &= d_i \prod_{j=1}^{2r} [c, d_j]^{-r_{i,2r+1-j}}, \end{aligned}$$

then one easily computes that $\tilde{e_i}$ commutes with $\tilde{d_j}$ for all i and j. One also has

$$\begin{aligned} [\tilde{d_i}, \tilde{d_j}] &= [d_i, d_j][c, d_{2r+1-j}, d_j]^{-r_{i,j}}[c, d_{2r+1-i}, d_i]^{r_{j,i}} \\ &= [d_i, d_j]^{1-1/2-1/2} \\ &= 1. \end{aligned}$$

Without loss of generality we can thus assume that all the indices $r_{i,j}$ and $s_{i,j}$ are 0. By the choice of basis with respect to the antisymmetric form f it is also true that $[d_i, d_{2r+1-i}, c]$ is a nontrivial element of $Z(G)$ for all $1 \leq i \leq r$. Suppose that $[d_i, d_{2r+1-i}, c] = [a, b, c, c]^{r_i}$. Let t_i be the inverse of r_i modulo 7. By replacing d_i with $d_i^{t_i}$, we can assume that $[d_i, d_{2r+1-i}, c] = [a, b, c, c]$. Then $[d_i, d_{2r+1-i}][a, b, c]^{-1}$ is in $Z(G)$. Suppose

$$[d_i, d_{2r+1-i}] = [a, b, c][c, d_i, d_{2r+1-i}]^{-l_i}.$$

Then $[d_i[c, d_i]^{l_i}, d_{2r+1-i}] = [a, b, c]$. We can thus choose d_i such that $[d_i, d_{2r+1-i}] = [a, b, c]$. Let us summarise: We can choose $d_1, \ldots, d_{2r}, e_1, \ldots, e_t$ in D such that $d_1[G, G], \ldots, d_{2r}[G, G], e_1[G, G], \ldots, e_t[G, G]$ form a basis for $D/[G, G]$ and such that

$$[e_i, d_j] = 1 \qquad \text{for all } i, j; \tag{29}$$

$$[d_i, d_j] = 1 \qquad \text{when } i + j \neq 2r + 1; \tag{30}$$

$$[d_i, d_{2r+1-i}] = [a, b, c] \qquad \text{for } i = 1, \ldots, r. \tag{31}$$

Let $g \in D\setminus[G,G]$. Since $D = Z^3(G)$, $[g,a,c]$, $[g,b,c]$ and $[g,c,c]$ are in $Z(G)$. Suppose

$$\begin{aligned} [g,a,c] &= [c,b,a,c]^{-r} \\ [g,b,c] &= [c,a,b,c]^{-s} \\ [g,c,c] &= [a,b,c,c]^{-t}. \end{aligned}$$

Then $g_1 = g[c,b]^r[c,a]^s[a,b]^t$ satisfies $[g_1,a,c] = [g_1,b,c] = [g_1,c,c] = 1$. By Theorem 3, $\gamma_3(H) = 1$ and we conclude that $[g_1,a]$ and $[g_1,b]$ are in $Z(G)$. Suppose

$$\begin{aligned} [g_1,a] &= [b,c,c,a]^{-m} \\ [g_1,b] &= [a,c,c,b]^{-n}. \end{aligned}$$

Then $g_2 = g_1[b,c,c]^m[a,c,c]^n$ satisfies $[g_2,a] = [g_2,b] = 1$. Since $[g_2,a,c] = [g_2,c,a]^2$, we also see that $[g_2,c,a] = 1$. Similarly $[g_2,c,b] = 1$.

The previous paragraph allows us furthermore to assume that the generators $d_1,\ldots,d_{2r}$ can be chosen such that we also have

$$[d_i,a] = [d_i,b] = [d_i,c,a] = [d_i,c,b] = [d_i,c,c] = 1. \tag{32}$$

We can suppose that the same equations hold for the e_i. Since we have seen that $[e_i,e_j]$ commutes with c for all i and j, it follows from (29) and (32) that $[e_i,c] \in Z(G)$. Suppose that

$$[e_i,c] = [a,b,c,c]^{-r_i}.$$

Then $e_i[a,b,c]^{r_i}$ commutes with c. It is now clear that we can choose $e_1,\ldots,e_t$ such that:

$$[e_i,a] = [e_i,b] = [e_i,c] = 1. \tag{33}$$

Let $E = Z^2(G)$. From (29)-(33) we can infer that $E = \langle e_1,\ldots,e_t\rangle[D,G]$ so $e_1[D,G],\ldots,e_t[D,G]$ is a basis for $E/[D,G]$. We define a map g from $E/[D,G] \times E/[D,G]$ to $Z(G)$ by:

$$g(e[D,G],h[D,G]) = [e,h].$$

We can think of g as an antisymmetric bilinear form on the vector space $E/[D,G]$. As before we apply the classical theory of antisymmetric forms and get a basis $W_1 = k_1[D,G], \ldots, W_{2s} = k_{2s}[D,G], X_1 = h_1[D,G], \ldots, X_l = h_l[D,G]$ for $E/[D,G]$ such that

$$g(W_i, W_j) = \begin{cases} [a,[b,c,c]] & \text{if } i+j = 2s+1 \text{ and } i<j \\ 0 & \text{otherwise,} \end{cases}$$

and such that

$$g(X_i, W_j) = g(X_i, X_j) = 0$$

for all i and j. But now we must have $l = 0$: each $h_i \in Z(G) = \gamma_4(G)$ and thus cannot be in $E \setminus [D,G]$. So $t = 2s$ and without loss of generality we can assume that $k_i = e_i$ for $1 \le i \le 2s$. So we can choose e_i such that

$$[e_i, e_{2s+1-i}] = [a,[b,c,c]] \qquad \text{for } i = 1, \ldots, s; \tag{34}$$

$$[e_i, e_j] = 1 \qquad \text{when } i+j \neq 2s+1. \tag{35}$$

Comparing (29)-(35) with (22)-(28), we see that G is a homomorphic image of $G(r,s,7)$.

We finish the proof by showing that $G(r,s,7)$ has the claimed properties stated in the theorem. Let $a = xN(r,s,7)$, $b = yN(r,s,7)$, $c = zN(r,s,7)$, $d_i = f_iN(r,s,7)$, $e_i = g_iN(r,s,7)$. Then it is clear from the relations that $\langle a,b,c\rangle$ is isomorphic to $G(3,7)$. Therefore $G(r,s,7)$ has nilpotency class 4. It now only remains to be shown that every proper quotient is nilpotent of class at most 3.

Let H and D be as before. Let N be a proper normal subgroup of $G(r,s,7)$. We want to show that $[a,[b,c,c]] \in N$. We argue by contradiction and assume this is not the case. Suppose first that N is not contained in H. Then $c^r h \in N$ for some $r \neq 0$ and some $h \in H$. But then $[a,b,c,c]^{r^2} = [a,b,c^r h, c^r h] \in N$, and we arrive at the contradiction that $[a,[b,c,c]] \in N$. Thus N must be contained in H. Similarly it is easy to see that N must be contained in D. Suppose that

$$x = ([a,b]^r [c,a]^s [c,b]^t \prod_i d_i^{r_i}) u \in N$$

for some $u \in Z^2(G)$. Then $[a,b,c,c]^r = [x,c,c]$, $[c,a,c,b]^s = [x,c,b]$ and $[c,b,c,a]^t = [x,c,a]$ are in N so we must have $r = s = t = 0$. Also $[c,d_j,d_{2r+1-j}]^{r_j} = [c,x,d_{2r+1-j}]$. Therefore $[a,[b,c,c]]^{r_j} \in N$ for all $1 \le$

$j \leq 2r$ and we must have that all these indices are 0. Hence N must be contained in $Z^2(G)$. Clearly $Z^2(G) = \langle e_1, \ldots, e_{2s}, [c, d_1], \ldots, [c, d_{2r}]\rangle \gamma_3(G)$. Suppose that

$$x = u \prod_{i=1}^{2s} e_i^{r_i} \prod_{j=1}^{2r} [c, d_i]^{s_i}$$

for some $u \in \gamma_3(G)$. Then $[a, [b, c, c]]^{r_j} = [e_j, e_{2s+1-j}]^{r_j} = [x, e_{2s+1-j}] \in N$ which implies that $r_j = 0$ for all $1 \leq j \leq 2s$. Also $[a, [b, c, c]]^{s_j} = [c, d_j, d_{2r+1-j}]^{s_j} = [x, d_{2r+1-j}] \in N$ and this necessarily implies that $s_j = 0$ for all $1 \leq j \leq 2r$. Therefore $N \subseteq \gamma_3(G)$. So if x is a nontrivial element in N it must be of the form

$$x = [c, a, b]^u [a, c, c]^v [b, c, c]^w [a, [b, c, c]]^l$$

for some $u, v, w, l \in \mathbb{Z}_7$. But then $[c, a, b, c]^u = [x, c]$, $[a, c, c, b]^v = [x, b]$ and $[b, c, c, a]^w = [x, a]$ are in N and thus $u = v = w = 0$. Hence we must have $N = \langle [a, [b, c, c]] \rangle$ which contradicts our assumption that $[a, [b, c, c]] \notin N$. $\Box$

Remarks (1) Every element x in $G(r, s, 7)$ can be written uniquely of the form

$$\begin{aligned} x &= c^u a^v b^w [c, a]^l [c, b]^m [a, b]^n \\ &\quad [a, b, c]^\alpha [a, c, c]^\beta [b, c, c]^\gamma [a, [b, c, c]]^\tau \\ &\quad \prod_{i=1}^{2r} d_i^{r_i} [c, d_i]^{s_i} \prod_{i=1}^{2s} e_i^{t_i}. \end{aligned}$$

Therefore $G(r, s, 7)$ has rank $3 + 2r + 2s$ and order $7^{10+4r+2s}$.

(2) Notice that $G(r, s, 7)$ always has odd rank. For a given odd integer $n \geq 3$ there are exactly $(n - 1)/2$ groups of exponent 7 in U_3 which have rank n, are nilpotent of class 4 and of which every proper quotient is of class at most 3. These groups have orders $7^{7+n}, 7^{9+n}, \ldots, 7^{4+2n}$.

(3) Suppose that $n \geq 3$. Let N be a maximal element in

$$\{U \trianglelefteq G(n, 7) : G(n, 7)/U \text{ is nilpotent of class 4 and has rank } n\}.$$

It follows from Theorem 6 that $G(n, 7)/N \cong G(r, s, 7) \times M(t)$ where $3 + 2r + 2s + t = n$ and $M(t)$ is an elementary abelian group of exponent 7.

We have that $Z^3(G(n,7)) = \langle t_1, \dots, t_{n-3}\rangle [G(n,7), G(n,7)]$. By the proof of Theorem 6 there are elements $f_1, \dots, f_{2r}, g_1, \dots, g_{2s}, k_1, \dots, k_t \in Z^3(G) \setminus [G(n,7), G(n,7)]$ such that N is the normal closure of the union of the set of the elements in (8)-(12),(22)-(28) and following set of elements:

$$\{[k_i, k_m], [k_i, d_j], [k_i, e_l], [k_i, a], [k_i, b], [k_i, c] : i, j, l, m\}$$

where $1 \leq i, m \leq t, 1 \leq j \leq 2r$ and $1 \leq l \leq 2s$.

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