Subnormality conditions in non-torsion groups

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According to results of Heineken and Stadelmann, a non-torsion group is a 2-Baer group if and only if it is 2-Engel, and it has all subgroups 2-subnormal if and only if it is nilpotent of class 2. We extend some of these results to values of n greater than 2. Any non-torsion group which is an n-Baer group is an n-Engel group. The converse holds for $n = 3$, and for all n in case of metabelian groups. A non-torsion group without involutions having all subgroups 3-subnormal has nilpotency class 4, and this bound is sharp.

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1 Introduction

Let G be a group. A subgroup H in G is said to be subnormal, if there exists a finite series $H = H_0, H_1, \ldots, H_{k-1}, H_k = G$, such that

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_k = G.$$

If n is the length of the shortest such series, we say H is subnormal of defect n, or n-subnormal, denoted by $H \lhd_n G$.

In a group of nilpotency class n, all subgroups are subnormal of defect at most n. Conversely, Roseblade in [8] has shown that a group G with all subgroups n-subnormal is nilpotent of class $\mu(n)$. However, $\mu(n)$ is not explicitly given in [8] and the exact values are only known for $n = 1$ and 2. So far, the function $\mu(n)$ seems to be not well understood. On the one hand, using direct methods following Roseblade's approach, it appears that $\mu(n)$ is growing very rapidly with n. On the other hand, it can be easily seen that in metabelian groups $\mu(n)$ grows linearly with n, provided the group does not contain elements of small order.

A group with all cyclic subgroups n-subnormal is called an n-Baer group. The class of n-Baer groups will be denoted by B_n. It can be easily seen that any group in B_n is $(n+1)$-Engel, i.e. $[x,_{n+1} y] = 1$ for all x, y in the group, where $[x,_k y] = [[x,_{k-1} y], y]$ and $[x,_1 y] = [x, y] = x^{-1}y^{-1}xy$. Denoting the class of groups with all subgroups n-subnormal by U_n, we obviously have that every U_n-group is a B_n-group. In case $n = 1$, the class of Dedekind groups (see [1], [2] and Theorem 6.1.1 in [9] for easier reference), all subgroups being n-subnormal is equivalent to all cyclic subgroups being n-subnormal. This is no longer the case if $n \geq 2$. For one thing, n-Baer groups, $n \geq 3$, are not necessarily nilpotent. However, for 2-Baer groups this is still true as follows from results of Heineken [5] and Mahdavianary [7], which states that a group with all cyclic subgroups 2-subnormal is nilpotent of class not exceeding 3. As a corollary of this result, it follows that $\mu(2) \leq 3$. This bound is sharp. However not all B_2-groups are U_2-groups.

In this article we will be interested in the groups in U_n and B_n which are non-torsion groups. There is some evidence that the structure of the non-torsion groups is different. A non-torsion group is for example a Dedekind group if and only if it is abelian. Heineken [5] has also shown that a non-torsion group is a 2-Baer group if and only if it is 2-Engel, whereas a result of Stadelmann [10] states that a non-torsion group is a U_2-group if and only if it is nilpotent of class at most 2. Furthermore, in [3] it was shown that for metabelian non-torsion groups, n-Engel and n-Baer are equivalent, provided $n \leq 5$, or the group contains no elements of order $\leq n-1$ in case $n \geq 6$. In this article we will show that this result holds without restrictions on element orders.

It is natural to ask whether one can extend these results. We will see that one implication in Heineken's result can be generalized. We will show that every non-torsion n-Baer group is always an n-Engel group, and in fact, we have that these conditions are equivalent when $n = 3$. One expects the non-torsion groups in U_n have a much simpler structure than the torsion groups in this class. At this time we do not have an analogue of Stadelmann's Theorem for general n. However, we can say a few things about U_3-groups. It follows from a result in [11] that non-torsion groups in U_3 without involution have nilpotency class at most 4. This bound is sharp as can be seen from a family of non-torsion groups in U_3, having nilpotency class 4 and a torsion subgroup of p-power order, $p \neq 2$. For non-torsion groups in U_3 with involutions, the class bound is at least 4 as we will see in another example. We do not know at present if this bound is sharp.

2 Non-torsion groups with every cyclic subgroup n-subnormal

In this section we prove our first main result, that every non-torsion n-Baer group is an n-Engel group, extending Heineken's result for $n = 2$ [5]. We begin with an elementary lemma for nilpotent groups.

Lemma 2.1 *Let G be a nilpotent group of class c in which the torsion subgroup $\tau(G)$ has finite exponent r. Then $[\tau(G), G^{r^{c-1}}] = 1$.*

Proof We show by induction that

$$[\tau(G),_{c-i} G, G^{r^{i-1}}] = 1,$$

for $i = 1, 2, \ldots, c$. Since G is nilpotent of class c this is clearly true for $i = 1$. For the induction step we assume that this is true for $i = 1, \ldots, k$ for some $1 \leq k < c$. Let $x \in [\tau(G),_{c-(k+1)} G]$ and let $g \in G$. From the induction hypothesis we have that

$$\begin{aligned} [x, ug^{r^{k-1}}] &= [x, g^{r^{k-1}}][x, u][x, u, g^{r^{k-1}}] \\ &= [x, g^{r^{k-1}}][x, u] \end{aligned}$$

for all $u \in G$. Using this repeatedly we get

$$[x, g^{r^k}] = [x, g^{(r^{k-1})r}] = [x, g^{r^{k-1}}]^r = 1,$$

since $\tau(G)^r = 1$. This finishes the proof of the inductive hypothesis. In particular we have $[\tau(G), G^{r^{c-1}}] = 1$. $\square$

Our aim is to show that every element in a non-torsion n-Baer group is a left n-Engel element. In our next lemma we show this for the elements of infinite order.

Lemma 2.2 *Let G be an n-Baer group. Then every element of infinite order is a left n-Engel element.*

Proof Suppose $x, y \in G$ with $|y| = \infty$. Since G is an n-Baer group, we have $\langle y \rangle^{(G,n)} = \langle y \rangle$. It follows that $[y, [x,_{n-1} y]] = y^r$ for some $r \in \mathbb{Z}$. Since $G \in B_n$, we have that G is $(n+1)$-Engel. Hence

$$1 = [y,_{n+1} [x,_{n-1} y]] = y^{r^{n+1}}.$$

Since $|y| = \infty$, we must have $r = 0$. $\Box$

Theorem 2.3 *Let G be a non-torsion n-Baer group. Then G is an n-Engel group.*

Proof By the last lemma, every element of infinite order is a left n-Engel element. We thus only have to show that every element of finite order is a left n-Engel element. We first show that every element of infinite order is a right n-Engel element. Let $z, x \in G$ such that $|z| = \infty$ and $|x| < \infty$. Then $H = \langle z, x \rangle$ is a finitely generated nilpotent group and thus $\tau(\langle z, x \rangle)$ is finite. By Lemma 2.1 we have that $H/C_H(\tau(H))$ is of finite exponent, say m. Then, since $|xz^m| = \infty$, we have that

$$\begin{aligned} 1 &= [z,_n xz^m] \\ &= [[z, x][z, x, z^m],_{n-1} xz^m] \\ &= [z, x,_{n-1} xz^m] \\ &= [z,_n x]. \end{aligned}$$

So we have shown that every element of infinite order is a right n-Engel element. In particular, if $|y|, |x| < \infty$ and $|z| = \infty$ then $1 = [yz^m,_n x]$. Since $\langle x, y, z \rangle$ is nilpotent, we can apply Lemma 2.1 again to find another m such that z^m commutes with x and y. But then

$$1 = [yz^m,_n x] = [y,_n x],$$

and we have proved the theorem. $\Box$

Corollary 2.4 *A non-torsion group is a* 3*-Engel group if and only if it is a* 3*-Baer group.*

Proof By Theorem 2.3, a non-torsion group in B_3 is 3-Engel. Conversely, by [6] we have for any group G the conditions G being 3-Engel and the normal closure x^G of every element x in G having nilpotency class 2 are equivalent. Hence $\langle x \rangle \lhd_2 x^G$. Since $x^G \lhd G$, it follows that $\langle x \rangle \lhd_3 G$, the desired result. □

The result for $n = 2$, corresponding to the above corollary, is due to Heineken [5]. The result of the next corollary appears in [3] for $n \leq 5$ and for $n \geq 6$, provided the group contains no elements of order $\leq n - 1$.

Corollary 2.5 *A metabelian non-torsion group is an n-Engel group if and only if it is an n-Baer group.*

Proof Since G is a non-torsion group in B_n, it follows by Theorem 2.3 that G is n-Engel. By Lemma 2.6 in [3] we have that any metabelian n-Engel group is an n-Baer group, the desired result. □

3 Non-torsion groups with every subgroup 3-subnormal

As mentioned in the introduction, Stadelmann [10] has shown that a non-torsion group is a U_2-group if and only if it is nilpotent of class at most 2. The topic of this section is the investigation of bounds for the nilpotency class of non-torsion groups in U_3. A subgroup H of a group G is 3-subnormal in G if and only if $[G, H, H, H] \leq H$. Thus any group of class 3 is in U_3. However, as we will see in the next theorem, the converse is not true, even in the case of non-torsion groups.

Theorem 3.1 *A non-torsion group in U_3 without involutions has nilpotency class at most four. There exist non-torsion groups without involutions in U_3 having nilpotency class precisely four.*

Proof Let G be a group as in the hypothesis. Then G is 3-Engel by Theorem 2.3. Since G has no involutions, it follows by Theorem 1 in [11] that the nilpotency class of G does not exceed four. Consider the groups of Example 3.2. They have nilpotency class precisely four and are U_3-groups by Proposition 3.3. Thus the bound on the nilpotency class is sharp. □

In the following, we construct a non-torsion group with a torsion subgroup being a p-group, p an odd prime, which has the properties as claimed in Theorem 3.1.

Example 3.2 Let p be an arbitrary prime different from 2. Take the relatively free group $\langle x, y, z\rangle$ in the variety of 3-Engel groups that are nilpotent of class at most 4 and let $G(p)$ be the quotient satisfying the extra relations:

$$\begin{array}{llll} y^{p^2}=1, & [y,z]=y^{2p}, & [x,z,z]=1, & z^{p^2}=[y,x,z,x], \\ [z,x]^p=1, & [x,y,y]=1, & [y,x]^p=[y,x,z]. & \end{array}$$

Let $a_1 = x$, $a_2 = z$, $a_3 = y$, $a_4 = x^p$, $a_5 = z^p$, $a_6 = y^p$, $a_7 = [z, x]$, $a_8 = [y, x]$, $a_9 = [y, x]^p$, $a_{10} = [z, x, x]$, $a_{11} = [y, x, x]$ and $a_{12} = [y, x, z, x]$. We can deduce from these relations that $G(p)/\langle x^{p^2}\rangle$ has power-commutator presentation with generators $a_1, \ldots, a_{12}$ and the following relations:

$$\begin{array}{l}
a_1^p=a_4,\ \ a_2^p=a_5,\ \ a_3^p=a_6,\ \ a_4^p=1,\ \ a_5^p=a_{12},\ \ a_6^p=1,\\
a_7^p=1,\ \ a_8^p=a_9,\ \ a_9^p=1,\ \ a_{10}^p=1,\ \ a_{11}^p=a_{12},\ \ a_{12}^p=1,\\
[a_2,a_1]=a_7,\ \ [a_3,a_1]=a_8,\ \ [a_3,a_2]=a_6^2,\ \ [a_4,a_1]=1,\ \ [a_4,a_2]=1,\\
[a_4,a_3]=a_9^{p-1}a_{12}^{(p+1)/2},\ \ [a_5,a_i]=1,\ \ [a_6,a_1]=a_9,\ \ [a_6,a_i]=1 \text{ if } i\neq 1,\\
[a_7,a_1]=a_{10},\ \ [a_7,a_2]=1,\ \ [a_7,a_3]=a_9^{p-1}a_{12}^4,\ \ [a_7,a_i]=1 \text{ if } i\geq 4,\\
[a_8,a_1]=a_{11},\ \ [a_8,a_2]=a_9,\ \ [a_8,a_3]=1,\ \ [a_8,a_4]=a_{12},\ \ [a_8,a_5]=1,\\
[a_8,a_6]=1,\ \ [a_8,a_7]=a_{12}^4,\ \ [a_9,a_1]=a_{12},\ \ [a_9,a_i]=1 \text{ if } i\neq 1,\\
[a_{10},a_3]=a_{12}^3,\ \ [a_{10},a_i]=1 \text{ if } i\neq 3,\ \ [a_{11},a_2]=a_{12}^{p-3},\ \ [a_{11},a_i]=1 \text{ if } i\neq 2,\\
[a_{12},a_i]=1.
\end{array}$$

We refer to [12] for a discussion of power-commutator presentations. One can check that this power-commutator presentation is consistent. It follows that $G(p)$ has class 4.

This concludes the construction of the example. In the next proposition we will establish that all subgroups of $G(p)$ are 3-subnormal.

Proposition 3.3 *For each prime $p \neq 2$ we have that $G(p)$ is in U_3.*

Proof Let $g, h_1, h_2, h_3 \in G(p)$ and $H = \langle h_1, h_2, h_3\rangle$. It is sufficient to show that $[g, h_1, h_2, h_3] \in H$. For ease of notation write $G(p) = G$. We consider several cases. First assume that H contains an element of finite order that is not contained in $\langle x, y\rangle[G, G]G^p$. Then H has an element of the form $zy^r u$, where r is some integer and $u \in \langle a_6, a_7, \ldots, a_{12}\rangle$. From the presentation above one sees that $\langle a_6, a_9, a_{12}\rangle \lhd G$ and that $\langle a_2, a_3, a_6, \ldots, a_{12}\rangle/\langle a_6, a_9, a_{12}\rangle$ is abelian. Since $y^p, u^p \in \langle a_6, a_9, a_{12}\rangle$, we thus have that

$$(zy^r u)^p = z^p v,$$

with $v \in \langle a_6, a_9, a_{12}\rangle$. The group $\langle a_5, a_6, a_9, a_{12}\rangle$ is abelian and $a_6^p = a_9^p = a_{12}^p = 1$. Therefore

$$(zy^r u)^{p^2} = (z^p v)^p = z^{p^2} = [y, x, z, x],$$

and H contains $\gamma_4(G)$. In particular $[G, H, H, H] \leq H$.

We can thus assume that H has no torsion element outside $\langle x, y\rangle[G, G]G^p$. Therefore $\dim(H[G, G]G^p/[G, G]G^p)$ is at most 2. Suppose first that the dimension is 2. Then H contains elements of the form $x^m y^r z^s u$, yv, where m, r, s are integers with p not dividing m and $u, v \in [G, G]G^p$. From the presentation we have that

$$[yv, x^m y^r z^s, x^m y^r z^s]^p = [y, x, x]^{pm^2} = [y, x, z, x]^{m^2} \neq 1,$$

and H again contains $[G, H, H, H]$.

Finally we can assume that H is contained in $\langle t\rangle[G, G]G^p$ for some $t \in G$. From the presentation we see that G has nilpotency class 4 and that $\gamma_4(G)$ is cyclic of order p. This together with the fact that G is 3-Engel implies $[G, H, H, H] \leq \langle [g, t, t, t] \mid g \in G\rangle = 1$, the desired result. □

The result of the next proposition shows that the class bound for a group in U_3 with involutions is at least four.

Proposition 3.4 *The relatively free* 3*-Engel group of rank* 2 *is a* U_3*-group.*

Proof Let $G = \langle x, y\rangle$ be the free 3-Engel group of rank 2. Let H be an arbitrary subgroup of G. We want to show that $[G, H, H, H] \leq H$. First suppose that $H[G, G]G^2 = G$. Then H contains elements of the form xu, yv with $u, v \in [G, G]G^2$. From [4], we know that G has nilpotency class 4 and that $\gamma_4(G) = \langle [x, y, y, x]\rangle$ is cyclic of order 2. But then $[xu, yv, yv, xu] = [x, y, y, x]$, and H therefore contains $\gamma_4(G)$. In particular, $[G, H, H, H] \leq H$. We can thus assume that $H \leq \langle t\rangle[G, G]G^2$ for some $t \in G \setminus [G, G]G^2$. Using again the fact that $\gamma_5(G) = 1$ and $\gamma_4(G)^2 = 1$, we have $[G, H, H, H] \leq \langle [g, t, t, t] \mid g \in G\rangle = 1$. □

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