ON VARIETIES IN WHICH SOLUBLE GROUPS ARE TORSION-BY-NILPOTENT

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We characterize the varieties in which all soluble groups are torsion-by-nilpotent as well as the varieties in which all soluble groups are locally finite-by-nilpotent.

1. Introduction and main results

Let $\mathcal{B}_e$ be the variety of groups of exponent dividing e and $\mathcal{N}_c$ the variety of nilpotent groups of class at most c. In addition, we denote by $\mathcal{A} = \mathcal{N}_1$ the variety of abelian groups and by $\mathcal{A}_p = \mathcal{A} \cap \mathcal{B}_p$ the variety of elementary abelian p-groups (where p is a prime). The product of varieties is defined in the usual sense. In particular, $\mathcal{A}^d$ is the variety of soluble groups of derived length $\leq d$. Two varieties will play an important role in this paper: the variety $\mathcal{A}_p\mathcal{A}$, defined by the laws $[x,y]^p = [[x,y],[z,t]] = 1$, and the variety $\mathcal{A}\mathcal{A}_p$, defined by the laws $[x^p, y^p] = [x, y, z^p] = [[x,y],[z,t]] = 1$.

Consider a variety $\mathcal{V}$ which does not contain $\mathcal{A}_p\mathcal{A}$, for any prime p. Groves [3] showed that for any positive integer d, there are integers c, e such that $\mathcal{V} \cap \mathcal{A}^d \subseteq \mathcal{N}_c\mathcal{B}_e$. Afterwards, by using deep results of Zel'manov, it has been proved in [2] that c and e may be chosen independent of d.

Similarly, if $\mathcal{V}$ is a variety which does not contain $\mathcal{A}\mathcal{A}_p$ (for any prime p), Groves [3] showed that for any d, there are integers c, e such that $\mathcal{V} \cap \mathcal{A}^d \subseteq \mathcal{B}_e\mathcal{N}_c$. In this case, we show that c may be chosen independent of d, with an integer e whose the set of prime divisors is independent of d (if Π is a set of primes, we say that a positive integer is a Π-number if each prime divisor of this integer belongs to Π). We do not know whether e can be chosen totally independent of d.

Theorem 1.1. *Let $\mathcal{V}$ be a variety of groups. Then the following assertions are equivalent:*

(i) *$\mathcal{V}$ does not contain $\mathcal{A}\mathcal{A}_p$ (for any prime p).*

(ii) *There exist a finite set Π of primes and an integer c such that for any d, we can find a Π-number e satisfying $\mathcal{V} \cap \mathcal{A}^d \subseteq \mathcal{B}_e\mathcal{N}_c$.*

(iii) *Each soluble group in $\mathcal{V}$ is torsion-by-nilpotent.*

(iv) *In each soluble group in $\mathcal{V}$, the elements of finite order form a subgroup.*

Now consider a variety $\mathcal{V}$ in which the subclass of locally nilpotent groups forms a variety. It is easy to see that this property is equivalent to the following one: for each integer n, the nilpotency class of n-generated nilpotent groups of $\mathcal{V}$ is bounded (the bound depending on $\mathcal{V}$ and n only). This property occurs for example in the solution of the restricted Burnside problem, due to Zel'manov [8, 9], where the key result can be stated like this: in the variety of groups of exponent dividing a given prime-power, the locally nilpotent groups form a variety. It turns out that this property is true as well for the variety of m-Engel groups, namely the variety defined by the law $[x, y, \ldots, y] = 1$, where y occurs m times [7, Theorem 2]. That leads to a natural question: what are the varieties in which the locally nilpotent groups form a variety? Our next result characterizes in different ways these varieties.

Theorem 1.2. *Let $\mathcal{V}$ be a variety of groups. Then the following assertions are equivalent:*

(i) *$\mathcal{V}$ contains neither $\mathcal{A}\mathcal{A}_p$ nor $\mathcal{A}_p\mathcal{A}$ (for any prime p).*

(ii) *Each finitely generated soluble group of $\mathcal{V}$ is finite-by-nilpotent.*

(iii) *There exist a function ω, a finite set Π of primes and an integer c such that, for any n, each n-generated soluble group $G \in \mathcal{V}$ is an extension of a finite Π-group of order dividing $\omega(n)$ by a nilpotent group of class $\leq c$.*

(iv) *The subclass of locally nilpotent groups of $\mathcal{V}$ forms a variety.*

Varieties containing neither $\mathcal{A}\mathcal{A}_p$ nor $\mathcal{A}_p\mathcal{A}$ occur already in Groves' works [3]. Also note another characteristic property of these varieties: the locally nilpotent groups of a variety $\mathcal{V}$ form a variety if and only if there are constants c and e such that the class of nilpotent groups of $\mathcal{V}$ is included in $\mathcal{N}_c\mathcal{B}_e \cap \mathcal{B}_e\mathcal{N}_c$ [2].

2. Preliminaries

As usual, in a group G, the left-normed commutator $[a_1, \ldots, a_n]$ ($a_i \in G$) of weight n is defined inductively by

$$[a_1, \ldots, a_n] = [a_1, \ldots, a_{n-1}]^{-1} a_n^{-1} [a_1, \ldots, a_{n-1}] a_n.$$

We shall denote by $\gamma_n(G)$ ($n \geq 1$) the nth term of the lower central series of G. Recall that this subgroup is generated by the set of left-normed commutators of weight n. If A and B are subgroups of G, we write $[A, B]$ for the subgroup generated by the elements of the form $[a, b]$, with $a \in A,\ b \in B$.

Let $\mathcal{V}$ be a variety of groups. We denote by $F(\mathcal{V})$ the relatively free group in $\mathcal{V}$ of countably infinite rank, freely generated by $S = \{u_{i,j} \,|\, i, j = 0, 1, 2, \ldots\}$ (for convenience sake, we use a double index to write the elements of S). It is easy to see that the derived subgroup $F(\mathcal{V})'$ is generated by the set R of all the left-normed commutators

$$[x_1^{\lambda_1}, \ldots, x_h^{\lambda_h}] \quad (h \geq 2,\ x_1, \ldots, x_h \in S,\ \lambda_1, \ldots, \lambda_h = \pm 1).$$

It follows from [5, 34.21] that $\gamma_n(F(\mathcal{V})')$ is the normal closure of the set of commutators of the form

$$[y_1^{\mu_1}, \ldots, y_n^{\mu_n}] \quad (y_1, \ldots, y_n \in R,\ \mu_1, \ldots, \mu_n = \pm 1).$$

Therefore, one can easily verify that $\gamma_n(F(\mathcal{V})')$ is generated by the commutators

$$[y_1^{\mu_1}, \ldots, y_n^{\mu_n}, x_1^{\nu_1}, \ldots, x_r^{\nu_r}],$$

with $y_1, \ldots, y_n \in R,\ x_1, \ldots, x_r \in S$, and $\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_r = \pm 1$.

In the following, we consider a fixed integer $m \geq 0$. Denote by S_m the set $\{u_{i,0} \,|\, i = 0, 1, \ldots, m\}$ and by R_m the set of elements of the form

$$[x_1^{\lambda_1}, \ldots, x_h^{\lambda_h}] \quad (h \geq 2,\ x_1, \ldots, x_h \in S,\ \lambda_1, \ldots, \lambda_h = \pm 1),$$

such that among the elements $x_1, \ldots, x_h$, at least one belongs to S_m. Let H_n be the subgroup of $\gamma_n(F(\mathcal{V})')$ generated by the elements of the form

$$[y_1^{\mu_1}, \ldots, y_n^{\mu_n}, x_1^{\nu_1}, \ldots, x_r^{\nu_r}] \quad (y_i \in R,\ x_i \in S,\ \mu_i, \nu_i = \pm 1)$$

such that at least one element of S_m occurs in this expression. This means that at least one of the elements $x_1, \ldots, x_r$ belongs to S_m, or at least one of the elements $y_1, \ldots, y_n$ belongs to R_m. It is clear that H_n is normal in $F(\mathcal{V})$. Moreover, we have:

Lemma 2.1. *Let $\Phi : F(\mathcal{V}) \to F(\mathcal{V})$ be the endomorphism defined by $\Phi(u_{i,j}) = 1$ if $u_{i,j} \in S_m$ and $\Phi(u_{i,j}) = u_{i,j}$ otherwise. Then, for any integers $n, n' \geq 1$, we have:*

(i) $H_n = \ker\Phi \cap \gamma_n(F(\mathcal{V})')$.

(ii) $[\gamma_n(F(\mathcal{V})'), H_{n'}] \leq H_{n+n'}$; *in particular,* $[H_n, H_{n'}] \leq H_{n+n'}$ *(hence* $(H_n)_{n\geq 1}$ *is a central series of* H_1*).*

Proof. (i). Let K_n be the subgroup of $\gamma_n(F(\mathcal{V})')$ generated by the elements of the form

$$[y_1^{\mu_1}, \dots, y_n^{\mu_n}, x_1^{\nu_1}, \dots, x_r^{\nu_r}] \quad (y_i \in R,\ x_i \in S,\ \mu_i, \nu_i = \pm 1)$$

such that no element of S_m occurs in this expression. Clearly, $\gamma_n(F(\mathcal{V})') = H_n K_n$; besides, for any $a \in H_n$ (resp. $b \in K_n$), we have $\Phi(a) = 1$ (resp. $\Phi(b) = b$). Let z be an element in $\ker\Phi \cap \gamma_n(F(\mathcal{V})')$. There exist elements $a \in H_n, b \in K_n$ such that $z = ab$; it follows

$$1 = \Phi(z) = \Phi(a)\Phi(b) = b,$$

hence $z = a \in H_n$. Thus we have shown the inclusion $\ker\Phi \cap \gamma_n(F(\mathcal{V})') \leq H_n$. Since the converse inclusion is clear, (i) is proved.
(ii). This is an easy consequence of (i) and the well-known inclusion

$$[\gamma_n(F(\mathcal{V})'), \gamma_{n'}(F(\mathcal{V})')] \leq \gamma_{n+n'}(F(\mathcal{V})').$$

□

Lemma 2.2. *Suppose that for some integers* $c, e \geq 1$, *we have* $\mathcal{V} \cap \mathcal{A}^2 \subseteq \mathcal{B}_e\mathcal{N}_c$. *Then, for any integer* $k \geq 1$, *we have*

$$\left(\prod_{i=0}^{m}[u_{i,0}, u_{i,1}, u_{i,2}, \dots, u_{i,kc}]\right)^{e^k} \in H_{k+1}.$$

Proof. We prove by induction on the integer t ($1 \leq t \leq k$) that the element

$$w_t = \left(\prod_{i=0}^{m}[u_{i,0}, u_{i,(k-t)c+1}, \dots, u_{i,kc}]\right)^{e^t}$$

belongs to H_{t+1} (the conclusion is the case $t = k$). By assumption, $F(\mathcal{V})/\gamma_2(F(\mathcal{V})')$ is in $\mathcal{B}_e\mathcal{N}_c$ and so

$$w_1 = \left(\prod_{i=0}^{m}[u_{i,0}, u_{i,(k-1)c+1}, \ldots, u_{i,kc}]\right)^e$$

lies in $\gamma_2(F(\mathcal{V})')$. Besides, if Φ is the endomorphism defined in Lemma 2.1, we have $\Phi(w_1) = 1$. Hence, by Lemma 2.1(i), w_1 belongs to H_2.

Now suppose that for some t (with $1 < t \leq k$), w_{t-1} belongs to H_t. Let $\Psi : F(\mathcal{V}) \to F(\mathcal{V})$ be the endomorphism defined by

$$\Psi(u_{i,0}) = [u_{i,0}, u_{i,(k-t)c+1}, \ldots, u_{i,(k-t+1)c}]$$

if $0 \leq i \leq m$ (that is, $u_{i,0} \in S_m$) and $\Psi(u_{i,j}) = u_{i,j}$ otherwise. We have $w_t = \Psi(w_{t-1}^e)$. By the inductive hypothesis, w_{t-1} is a product of elements of the form

$$[y_1^{\mu_1}, \ldots, y_t^{\mu_t}, x_1^{\nu_1}, \ldots, x_r^{\nu_r}]^{\pm 1},$$

where at least one element of S_m occurs in this commutator. Therefore, since H_t/H_{t+1} is abelian, w_{t-1}^e can be expressed modulo H_{t+1} as a product of elements of the form

$$[y_1^{\mu_1}, \ldots, y_t^{\mu_t}, x_1^{\nu_1}, \ldots, x_r^{\nu_r}]^{\pm e},$$

where at least one element of S_m occurs in this commutator. Moreover, by using Lemma 2.1(i), it is easy to see that $\Psi(H_n) \leq H_n$ for any positive integer n. Consequently, in order to prove that w_t belongs to H_{t+1}, it is enough to prove that

$$\Psi\left([y_1^{\mu_1}, \ldots, y_t^{\mu_t}, x_1^{\nu_1}, \ldots, x_r^{\nu_r}]^e\right)$$

belongs to H_{t+1}. We consider two cases.

Case 1. Suppose that at least one of the elements $x_1, \ldots, x_r$, say x_q, belongs to S_m. Then $\Psi(x_q)$ lies in H_1. Since $[\gamma_t(F(\mathcal{V})'), H_1] \leq H_{t+1}$ (Lemma 2.1(ii)), the element

$$\begin{aligned} & \Psi\left([y_1^{\mu_1}, \ldots, y_t^{\mu_t}, x_1^{\nu_1}, \ldots, x_q^{\nu_q}, \ldots, x_r^{\nu_r}]^e\right) \\ = \quad & [\Psi(y_1)^{\mu_1}, \ldots, \Psi(y_t)^{\mu_t}, \Psi(x_1)^{\nu_1}, \ldots, \Psi(x_q)^{\nu_q}, \ldots, \Psi(x_r)^{\nu_r}]^e \end{aligned}$$

belongs to H_{t+1}.

Case 2. Suppose that none of the elements $x_1, \ldots, x_r$ belongs to S_m. In this case, there exists an integer q $(1 \leq q \leq t)$ such that y_q belongs to the set R_m defined above. This means that y_q may be written in the form

$$y_q = [x_1^{\lambda_1}, \ldots, x_p^{\lambda_p}, \ldots, x_h^{\lambda_h}] \quad (\lambda_1, \ldots, \lambda_h = \pm 1),$$

where $x_p = u_{d,0} \in S_m$ $(0 \leq d \leq m)$.

Since $(H_n)_{n \geq 1}$ is a central series of H_1 (Lemma 2.1) and since

$$\Psi(x_p) = [u_{d,0}, u_{d,(k-t)c+1}, \ldots, u_{d,(k-t+1)c}]$$

belongs to H_1, it follows from well-known commutator identities (see for example [6, 5.1.5]) the relations:

$$\begin{aligned} \Psi(y_q)^e &\equiv [\Psi(x_1^{\lambda_1}), \ldots, \Psi(x_p^{\lambda_p}), \ldots, \Psi(x_h^{\lambda_h})]^e \\ &\equiv [\Psi(x_1^{\lambda_1}), \ldots, \Psi(x_p^{\lambda_p})^e, \ldots, \Psi(x_h^{\lambda_h})] \quad \text{modulo } H_2. \end{aligned}$$

But as we have seen in the case $t = 1$, we remark that the element

$$\Psi(x_p^{\lambda_p})^e = [u_{d,0}, u_{d,(k-t)c+1}, \ldots, u_{d,(k-t+1)c}]^{e\lambda_p}$$

belongs to H_2, and so does $\Psi(y_q)^e$. By using again the commutator identities, we can write:

$$\begin{aligned} &\Psi\left([y_1^{\mu_1}, \ldots, y_q^{\mu_q}, \ldots, y_t^{\mu_t}, x_1^{\nu_1}, \ldots, x_r^{\nu_r}]^e\right) \\ \equiv\ & [\Psi(y_1^{\mu_1}), \ldots, \Psi(y_q^{\mu_q}), \ldots, \Psi(y_t^{\mu_t}), \Psi(x_1^{\nu_1}), \ldots, \Psi(x_r^{\nu_r})]^e \\ \equiv\ & [[\Psi(y_1^{\mu_1}), \ldots, \Psi(y_q^{\mu_q}), \ldots, \Psi(y_t^{\mu_t})]^e, \Psi(x_1^{\nu_1}), \ldots, \Psi(x_r^{\nu_r})] \\ \equiv\ & [\Psi(y_1^{\mu_1}), \ldots, \Psi(y_q^{\mu_q})^e, \ldots, \Psi(y_t^{\mu_t}), \Psi(x_1^{\nu_1}), \ldots, \Psi(x_r^{\nu_r})] \quad \text{modulo } H_{t+1}. \end{aligned}$$

Since $\Psi(y_q^{\mu_q})^e$ belongs to H_2, by applying Lemma 2.1(ii), we obtain

$$\Psi\left([y_1^{\mu_1}, \ldots, y_q^{\mu_q}, \ldots, y_t^{\mu_t}, x_1^{\nu_1}, \ldots, x_r^{\nu_r}]^e\right) \in H_{t+1},$$

as required. □

Lemma 2.3. *Suppose that for some integers $c, e \geq 1$, we have $\mathcal{V} \cap \mathcal{A}^2 \subseteq \mathcal{B}_e\mathcal{N}_c$. Then, for any integer $k \geq 1$, we have $\mathcal{V} \cap (\mathcal{N}_k\mathcal{A}) \subseteq \mathcal{B}_{e^k}\mathcal{N}_{kc}$.*

Proof. By Lemma 2.2, we have

$$\left(\prod_{i=0}^{m}[u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,kc}]\right)^{e^k} \in H_{k+1}$$

and so

$$\left(\prod_{i=0}^{m}[u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,kc}]\right)^{e^k} \in \gamma_{k+1}(F(\mathcal{V})')$$

since H_{k+1} is a subgroup of $\gamma_{k+1}(F(\mathcal{V})')$. But $F(\mathcal{V})$ is freely generated by $S = \{u_{i,j} \mid i, j = 0, 1, 2, \ldots\}$ and $\gamma_{k+1}(F(\mathcal{V})')$ is a fully-invariant subgroup of $F(\mathcal{V})$. Therefore we have in fact

$$\left(\prod_{i=0}^{m}[v_{i,0}, v_{i,1}, v_{i,2}, \ldots, v_{i,kc}]\right)^{e^k} \in \gamma_{k+1}(F(\mathcal{V})')$$

for all elements $v_{i,j} \in F(\mathcal{V})$. Clearly, this implies that $F(\mathcal{V})/\gamma_{k+1}(F(\mathcal{V})')$ belongs to $\mathcal{B}_{e^k}\mathcal{N}_{kc}$ and so $\mathcal{V} \cap (\mathcal{N}_k\mathcal{A}) \subseteq \mathcal{B}_{e^k}\mathcal{N}_{kc}$. □

The two next propositions are key results in the proof of Theorem 1.1 and may be of independent interest.

Proposition 2.1. *Let $\mathcal{V}$ be a variety of groups. Suppose there exist integers $c, e \geq 1$ such that $\mathcal{V} \cap \mathcal{A}^2 \subseteq \mathcal{B}_e\mathcal{N}_c$. Then, for any integer $d \geq 2$, we have $\mathcal{V} \cap \mathcal{A}^d \subseteq \mathcal{B}_{e'}\mathcal{N}_{c'}$, with $e' = e^{1+c+c^2+\cdots+c^{d-2}}$ and $c' = c^{d-1}$.*

Proof. It suffices to prove by induction on d that

$$\mathcal{V} \cap \mathcal{A}^d \subseteq \mathcal{B}_{e_0}\mathcal{B}_{e_1}\ldots\mathcal{B}_{e_{d-2}}\mathcal{N}_{c^{d-1}}, \text{ with } e_n = e^{c^n}.$$

The case $d = 2$ follows from hypothesis of the proposition. Thus suppose that the result is true for $d - 1$ ($d \geq 3$) and consider a group $G \in \mathcal{V} \cap \mathcal{A}^d$. Since the derived subgroup G' belongs to $\mathcal{V} \cap \mathcal{A}^{d-1}$, we have by induction

$G' \in \mathcal{B}_{e_0} \dots \mathcal{B}_{e_{d-3}} \mathcal{N}_{c^{d-2}}$, and so $G \in \mathcal{B}_{e_0} \dots \mathcal{B}_{e_{d-3}} \mathcal{N}_{c^{d-2}} \mathcal{A}$. In other words, G contains a normal subgroup $H \in \mathcal{B}_{e_0} \dots \mathcal{B}_{e_{d-3}}$ such that $G/H \in \mathcal{N}_{c^{d-2}}\mathcal{A}$. Lemma 2.3 yields the inclusion $\mathcal{V} \cap \mathcal{N}_{c^{d-2}}\mathcal{A} \subseteq \mathcal{B}_{e_{d-2}}\mathcal{N}_{c^{d-1}}$. Consequently, the quotient G/H belongs to $\mathcal{B}_{e_{d-2}}\mathcal{N}_{c^{d-1}}$. This implies that G belongs to $\mathcal{B}_{e_0} \dots \mathcal{B}_{e_{d-3}} \mathcal{B}_{e_{d-2}} \mathcal{N}_{c^{d-1}}$, as required. □

Proposition 2.1 can be considered as an extension of the well-known following result (see for example [4, 3.30]):

Corollary 2.1. *Let $\mathcal{V}$ be a variety of groups such that $\mathcal{V} \cap \mathcal{A}^2 \subseteq \mathcal{N}_c$ for some integer $c \geq 1$. Then, for any integer $d \geq 2$, we have $\mathcal{V} \cap \mathcal{A}^d \subseteq \mathcal{N}_{c'}$, with $c' = c^{d-1}$.*

The bound $c' = c^{d-1}$ obtained here improves slightly the bound given in [4, 3.30]. Notice that if $\mathcal{V}$ is a variety such that each group in $\mathcal{V} \cap \mathcal{A}^2$ is nilpotent, then there exists necessarily an integer c such that $\mathcal{V} \cap \mathcal{A}^2 \subseteq \mathcal{N}_c$ (we can take for c the nilpotency class of the relatively free group of countably infinite rank in $\mathcal{V} \cap \mathcal{A}^2$).

The next lemma is a consequence of [1, Theorem 1].

Lemma 2.4. *Let G be a nilpotent group of class k generated by a subset $S \subseteq G$. Let e be an integer such that $x^e = 1$ for each product x of at most k elements of S. Then we have $x^e = 1$ for all $x \in G$.*

Proposition 2.2. *Let $\mathcal{V}$ be a variety of groups. Then the following two assertions are equivalent:*

(i) *$\mathcal{V}$ does not contain $\mathcal{A}^2$.*

(ii) *There exist a finite set Π of primes and an integer c such that for any k, we can find a Π-number $\sigma(k)$ satisfying $\mathcal{V} \cap \mathcal{N}_k \subseteq \mathcal{B}_{\sigma(k)}\mathcal{N}_c$.*

Proof. (i)⇒(ii). Since $\mathcal{V}$ does not contain $\mathcal{A}^2$, there exist a finite set of primes Π and an integer c such that each nilpotent group of $\mathcal{V}$ without non-trivial Π-element belongs to $\mathcal{N}_c$ [2, Corollary 1]. Let k be a positive integer.

Denote by Γ the relatively free group of rank $k(c+1)$ in the variety $\mathcal{V} \cap \mathcal{N}_k$, freely generated by $\{u_{i,j} \mid i = 1, \ldots k,\ j = 1, \ldots, c+1\}$. The set H of Π-elements of Γ is obviously a normal subgroup of Γ and the nilpotency class of Γ/H is at most c. Consequently, the product

$$[u_{1,1}, \ldots, u_{1,c+1}] \times \cdots \times [u_{k,1}, \ldots, u_{k,c+1}]$$

is a Π-element, of order say $\sigma(k)$. In particular, in any group $G \in \mathcal{V} \cap \mathcal{N}_k$, we have the relation

$$([x_{1,1}, \ldots, x_{1,c+1}] \times \cdots \times [x_{k,1}, \ldots, x_{k,c+1}])^{\sigma(k)} = 1$$

for all $x_{i,j} \in G$ $(i = 1, \ldots k,\ j = 1, \ldots, c+1\})$. Since $\gamma_{c+1}(G)$ is nilpotent of class $\leq k$ and generated by the elements of the form $[y_1, \ldots, y_{c+1}]$ $(y_i \in G)$, it follows from Lemma 2.4 that $\gamma_{c+1}(G)^{\sigma(k)} = \{1\}$. Hence $G \in \mathcal{B}_{\sigma(k)}\mathcal{N}_c$, as desired.
(ii)$\Rightarrow$(i). Let p be a prime which is not in the set Π. Then the nilpotency class of each nilpotent p-group $G \in \mathcal{V}$ is at most c. Consider the wreath product $G = (\mathbb{Z}/p\mathbb{Z}) \wr (\mathbb{Z}/p^n\mathbb{Z})$, where n is an integer such that $p^n - 1 > c$. Clearly, G is a nilpotent p-group (of class say k) which is in $\mathcal{A}^2$; moreover, we have $k > p^n - 1$ [7, Result 2.2]. If $\mathcal{V}$ contains $\mathcal{A}^2$, then G belongs to $\mathcal{V}$ and so $k \leq c$, a contradiction. Therefore the variety $\mathcal{V}$ does not contain $\mathcal{A}^2$ and this completes the proof. $\square$

Remark 2.1. In the precedent statement, we cannot hope replace $\sigma(k)$ by a constant independent of k. Indeed, consider for example the variety $\mathcal{V} = \mathcal{A}\mathcal{A}_{\surd}$, where p is a given prime. Evidently, $\mathcal{V}$ does not contain $\mathcal{A}^2$. Suppose that there are integers c, e such that $\mathcal{V} \cap \mathcal{N}_k \subseteq \mathcal{B}_e\mathcal{N}_c$ for all integers k. Then the wreath product $\mathbb{Z} \wr (\mathbb{Z}/p\mathbb{Z})$, which belongs to $\mathcal{V}$, would be in $\mathcal{B}_e\mathcal{N}_c$ since it is residually nilpotent. But this group does not contain a non-trivial normal torsion subgroup and is not nilpotent, a contradiction.

3. Proof of the theorems

Proof of Theorem 1.1. (i)⇒(ii). By the result of Groves already mentioned [3, Theorem C(ii)], there exist two positive integers e_1, c_1 such that $\mathcal{V} \cap \mathcal{A}^2 \subseteq \mathcal{B}_{e_1}\mathcal{N}_{c_1}$. Denote by Π_1 the set of primes dividing e_1. By Proposition 2.1, there are functions θ and τ such that each group $G \in \mathcal{V} \cap \mathcal{A}^d$ belongs to $\mathcal{B}_{\theta(d)}\mathcal{N}_{\tau(d)}$ (also note that $\theta(d)$ is a Π_1-number). Let H be a normal subgroup of G satisfying $H \in \mathcal{B}_{\theta(d)}$ and $G/H \in \mathcal{N}_{\tau(d)}$. Since $\mathcal{A}\mathcal{A}_p \subseteq \mathcal{A}^2$, the variety $\mathcal{V}$ does not contain $\mathcal{A}^2$. Hence, by Proposition 2.2, there exist a finite set Π_2 of primes and an integer c_2 (depending on $\mathcal{V}$ only) such that G/H belongs to $\mathcal{B}_{\sigma(\tau(d))}\mathcal{N}_{c_2}$, where $\sigma(\tau(d))$ is a Π_2-number depending on d and $\mathcal{V}$. It follows that G belongs to $\mathcal{B}_{\theta(d)}\mathcal{B}_{\sigma(\tau(d))}\mathcal{N}_{c_2}$. Now put $\Pi = \Pi_1 \cup \Pi_2$, $e = \theta(d)\sigma(\tau(d))$ and $c = c_2$. Then e is a Π-number and we have $\mathcal{V} \cap \mathcal{A}^d \subseteq \mathcal{B}_e\mathcal{N}_c$, as required.
(ii)⇒(iii) and (iii)⇒(iv) are clear.
(iv)⇒(i). Suppose that $\mathcal{V}$ contains $\mathcal{A}\mathcal{A}_p$ for some prime p. The restricted wreath product $G = \mathbb{Z} \wr (\mathbb{Z}/p\mathbb{Z})$ belongs to $\mathcal{A}\mathcal{A}_p$ and so would be in $\mathcal{V}$. But G is a metabelian group in which the elements of finite order do not form a subgroup. Since that contradicts (iv), the implication is proved. □

To prove Theorem 1.2, we shall use the following result, which is an immediate consequence of Lemma 2 and Theorem 2 of [2].

Lemma 3.1. *Let $\mathcal{V}$ be a variety of groups which does not no contain $\mathcal{A}_p\mathcal{A}$ (for any prime p). Then there is a function ρ such that, for any positive integer n, the derived length of every n-generated soluble group of $\mathcal{V}$ is at most $\rho(n)$.*

Proof of Theorem 1.2. (i)⇒(ii). Let G be a finitely generated soluble group of a variety $\mathcal{V}$, where $\mathcal{V}$ contains neither $\mathcal{A}\mathcal{A}_p$ nor $\mathcal{A}_p\mathcal{A}$ (for any prime p). Then G is torsion-by-nilpotent by Theorem 1.1. But G is polycyclic since it is nilpotent-by-finite [2, Theorem 2]. Hence G is finite-by-nilpotent.
(ii)⇒(iii). Suppose that $\mathcal{V}$ is a variety whose finitely generated soluble groups are finite-by-nilpotent. The wreath products $\mathbb{Z} \wr (\mathbb{Z}/p\mathbb{Z})$ and $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$ are finitely generated soluble groups which belong to $\mathcal{A}\mathcal{A}_p$ and $\mathcal{A}_p\mathcal{A}$ respectively.

Since these groups are not finite-by-nilpotent, $\mathcal{V}$ contains neither $\mathcal{A}\mathcal{A}_p$ nor $\mathcal{A}_p\mathcal{A}$. Now Consider the set Π and the integer c given by Theorem 1.1, and the function ρ given by Lemma 3.1. If n is a positive integer, denote by Γ the relatively free group of rank n in the variety $\mathcal{V} \cap \mathcal{A}^{\rho(n)}$. By Theorem 1.1, Γ contains a normal Π-subgroup H such that $G/H \in \mathcal{N}_c$. Moreover, Γ is polycyclic since it is nilpotent-by-finite [2, Theorem 2]. Consequently, H is a finite Π-group, and so Γ is an extension of a finite Π-group (of order say $\omega(n)$) by a nilpotent group of class $\leq c$. Since each n-generated soluble group $G \in \mathcal{V}$ belongs to $\mathcal{V} \cap \mathcal{A}^{\rho(n)}$, G is a homomorphic image of Γ, and the result follows.
(iii)$\Rightarrow$(iv). It suffices to prove that the nilpotency class of every n-generated nilpotent group $G \in \mathcal{V}$ is bounded by a function of $\mathcal{V}$ and n. Such a group G is an extension of a finite group of order dividing $\omega(n)$ by a nilpotent group of class $\leq c$. Then, if $\omega(n) = p_1^{\alpha_1} \dots p_m^{\alpha_m}$ is the factorization of $\omega(n)$ into a product of prime numbers, the nilpotency class of G is clearly bounded by $c + \alpha_1 + \cdots + \alpha_m$.
(iv)$\Rightarrow$(i). If a variety $\mathcal{V}$ contains $\mathcal{A}\mathcal{A}_p$ for some prime p, then the restricted wreath product $G = \mathbb{Z} \wr (\mathbb{Z}/p\mathbb{Z})$ (which is in $\mathcal{A}\mathcal{A}_p$) belongs to $\mathcal{V}$. Since the group G is finitely generated, residually nilpotent but is not nilpotent, the class of locally nilpotent groups of $\mathcal{V}$ is not a variety. We obtain the same conclusion when $\mathcal{V}$ contains $\mathcal{A}_p\mathcal{A}$, by considering the group $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$. Therefore, if the class of locally nilpotent groups of $\mathcal{V}$ is a variety, then $\mathcal{V}$ contains neither $\mathcal{A}\mathcal{A}_p$ nor $\mathcal{A}_p\mathcal{A}$. □

References

1. G. Endimioni, *On conditions for a group to satisfy given laws*, J. Group Theory **2** (1999), 191–198.

2. G. Endimioni, *Bounds for nilpotent-by-finite groups in certain varieties*, J. Austral. Math. Soc. **73** (2002), 393–404.

3. J.R.J. Groves, *Varieties of soluble groups and a dichotomy of P. Hall*, Bull. Austral. Math. Soc. **5** (1971), 391–410.

4. E. I. Khukhro, *p-automorphisms of finite p-groups*, London Math. Soc. Lecture Note Ser. **246**, Cambridge Univ. Press, 1998.

5. H. Neumann, *Varieties of groups*, Springer-Verlag, Berlin, 1967.

6. D. J. S. Robinson, *A course in the theory of groups*, Springer-Verlag, New York, 1982.

7. J.S. Wilson, *Two-generator conditions for residually finite groups*, Bull. London Math. Soc. **23** (1991), 239–248.

8. E.I. Zel'manov, *Solution of the restricted Burnside problem for groups of odd exponent*, Izv. Akad. Nauk. SSSR Ser. Mat. **54** (1990), 42–59.

9. E.I. Zel'manov, *Solution of the restricted Burnside problem for 2-groups*, Mat. Sb. **182** (1991), 568–592.