A note on the local nilpotence of 4-Engel groups

Gunnar Traustason
Centre for Mathematical Sciences, Lund University,
Box 118, SE-221 00 Lund, Sweden

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Recently Havas and Vaughan-Lee proved that 4-Engel groups are locally nilpotent. Their proof relies on the fact that a certain 4-Engel group T is nilpotent and this they prove using a computer and the Knuth-Bendix algorithm. In this paper we give a short hand-proof of the nilpotency of T.

1 Introduction

Recently, Havas and Vaughan-Lee have proved that 4-Engel groups are locally nilpotent [1]. Their proof uses repeatedly the fact that a certain 3-generator 4-Engel group T is nilpotent. This group $T = \langle a, b, c\rangle$ is the largest 4-Engel group satisfying the extra properties that

$\langle a, b\rangle$ is abelian,
$\langle a, c\rangle$ is nilpotent of class at most two,
$\langle b, c\rangle$ is nilpotent of class at most three.

The largest quotient of exponent 5 of this group also played a role in the proof of Vaughan-Lee that 4-Engel groups of exponent 5 are locally nilpotent [5]. The proof of Havas and Vaughan-Lee is a computer proof using the

Knuth-Bendix algorithm. In Section 2 we will give a short hand proof of the nilpotence of T. This then implies that we have essentially a computer-free proof of the local nilpotence of 4-Engel groups. (Havas and Vaughan-Lee also use the nilpotent quotient algorithm to calculate the nilpotency class of a number of 4-Engel groups, but the class is never very large and so these calculations could be done by hand using commutator calculus or Lie ring methods. Even so these calculations are without doubt better done by computer as this is both quicker and more reliable).

As in the proof of Havas and Vaughan-Lee, we will use the fact that any 2-generator 4-Engel group is nilpotent [4]. In this paper we will denote by $R(G)$, the Hirsch-Plotkin radical of G. It is well known that for any 4-Engel group G, $R(G/R(G)) = \{1\}$. To prove that T is nilpotent we can thus without loss of generality assume that $R(T) = \{1\}$. As any 2-generator 4-Engel group is nilpotent it follows that the torsion elements of T form a normal subgroup that is a direct product of p-groups. By [5], every 4-Engel 2-group and every 4-Engel 3-group is locally nilpotent. Since we are assuming that $R(T) = \{1\}$ we thus have that T has no elements of order 2 and 3. In fact T has no elements of order 5 by Vaughan-Lee's result but we don't want to assume this as we also want to give a computer-free proof of the finiteness of the largest exponent 5 quotient of T.

In turns out to be the case that if one adds the extra condition that $[b, c, c] = 1$ then one doesn't need the 4-Engel identity to prove the nilpotence of T. One only needs the radical properties above. This we will prove in Section 3. We have not been able to do this without this extra condition and it may well be the case that the 4-Engel property is needed for the stronger version.

2 The nilpotence of T

As in the proof of Havas and Vaughan-Lee we use the fact [4] that any 2-generator 4-Engel group is nilpotent. We will also need the following lemmas.

Lemma 2.1 *Let G be any 4-Engel group without 2-elements and let $x, y \in G$.*

(1) If $[y, x, x] = 1$ then $\langle x\rangle^{\langle y\rangle}$ is abelian.
(2) If $[y, x, x, x] = 1$ then $\langle x\rangle^{\langle y\rangle}$ is nilpotent of class at most 2.

Proof. By [4] the group $\langle x, y\rangle$ is nilpotent. The two statements can now be read from a polycyclic presentation of the free 2-generator 4-Engel group. See for example [2]. In fact part (1) follows also directly from the next lemma. $\Box$

Lemma 2.2 *Let G be any 4-Engel group and $x, y, z \in G$. If z commutes with x, x^y and $x^{y^{-1}}$ then z commutes with all elements in $\langle x\rangle^{\langle y\rangle}$.*

Proof Let $u = x^{y^{-1}}$. Then z commutes with u, u^y and u^{y^2} and thus with $u, [u, y], [u, y, y]$. As G is 4-Engel, we have

$$\begin{aligned} 1 &= [u, zy, zy, zy, zy] \\ &= [u, y, zy, zy, zy] \\ &= [u, y, y, zy, zy] \\ &= [u, y, y, y, zy] \\ &= [u, y, y, y, z]^y. \end{aligned}$$

Thus z commutes also with u^{y^3} and as $\langle x\rangle^{\langle y\rangle} = \langle u\rangle^{\langle y\rangle}$ is generated by u, u^y, u^{y^2} and u^{y^3}, the result follows. $\Box$

We now prove that the group $T = \langle a, b, c\rangle$ is nilpotent. Without loss of generality we can assume that T has trivial Hirsch-Plotkin radical. Thus in particular T has a trivial centre and no elements of order 2 or 3. As well as the lemmas above the following lemma is going to play a key role in the proof.

Lemma 2.3 *Let u be an element in T. If*

$$H = \langle u\rangle^{\langle [c,a]\rangle} \leq C_T(\langle b, c\rangle)$$

then $u = 1$.

Proof First consider any $h \in H$ that commutes with $[c, a]$. Then h commutes with c, $[c, a]$ and b. We next show that the same holds for the elements $[h, a], [h, a, a]$ and $[h, a, a, a]$. Firstly, as a commutes with b and $[c, a]$ it is clear that these elements commute with b and $[c, a]$. We show by an inductive argument that they also commute with c. But

$$\begin{aligned} [h, a]^c &= [h, a[a, c]] \\ &= [h, [a, c]][h, a]^{[a,c]} \\ &= [h, a], \end{aligned}$$

and thus $[h,a]$ commutes with c. In fact this argument shows that if v in T commutes with c and $[c,a]$ then the same is true for $[v,a]$. Thus the elements $h,[h,a],[h,a,a]$ and $[h,a,a,a]$ all commute with c and b. As T is 4-Engel it follows that $[h,a,a,a]\in Z(T)$ and thus trivial. But then $[h,a,a]$ is in the centre and also trivial. Continuing like this, we see that $h=1$.

We use this now to deduce that $u=1$. As $[u,[c,a],[c,a],[c,a]]$ commutes with $[c,a]$ it is trivial by the previous paragraph. But then $[u,[c,a],[c,a]]$ commutes with $[c,a]$ and is also trivial. Continuing like this we see that $u=1$. $\Box$

We divide the rest of the proof into few steps.

Step 1. We show that $[c,b,b]=1$.

The 2-generator subgroup $\langle ac,b\rangle$ is nilpotent and

$$[ac,b,b,b]=[c,b,b,b]=1.$$

It follows from Lemma 2.1 that $\langle b\rangle^{\langle ac\rangle}$ is nilpotent of class at most two. In particular $[ac,b,b,ac]$ commutes with b. Then

$$[ac,b,b,ac]=[c,b,b,ac]=[c,b,b,a][c,b,b,a,c],$$

and as $[c,b,b,a]$ commutes with b (as $[c,b,b]$ and a do) we conclude that

$$[c,b,b,a,c,b]=1. \tag{1}$$

The Hall-Witt identity gives us

$$\begin{aligned} 1 &= [[c,b,b],c^{-1},a]^c[c,a^{-1},[c,b,b]]^a[a,[c,b,b]^{-1},c]^{[c,b,b]} \\ &= [[c,a]^{-1},[c,b,b]]^a[c,b,b,a,c]. \end{aligned}$$

By (1) the latter commutator commutes with b. Thus $[[c,b,b],[c,a]^{-1}]$ commutes with b and replacing a by a^{-1} we see that $[[c,b,b],[c,a]]$ also commutes with b. We have now seen that

$$[c,b,b],[c,b,b]^{[c,a]},[c,b,b]^{[c,a]^{-1}}$$

all commute with b and by Lemma 2.2 we can then deduce that all the elements in $H = \langle [c,b,b] \rangle^{\langle [c,a] \rangle}$ commute with b. These elements also clearly commute with c. Now Lemma 2.3 gives that $[c,b,b] = 1$.

Step 2. We show that $[b,c,c] = 1$.

By Step 1 we have that

$$[ac,b,b] = [c,b,b] = 1.$$

It follows from Lemma 2.1 that $\langle b \rangle^{\langle ac \rangle}$ is abelian. In particular it follows that the element

$$[ac,b,ac] = [c,b,ac] = [c,b,c][c,b,a][c,b,a,c]$$

commutes with b and as $[c,b,a],[c,b,c]$ commute now with b, we conclude that

$$[c,b,a,c,b] = 1. \tag{2}$$

We again use the Hall-Witt identity. This time we have

$$1 \;=\; [[c,b],a,c]^{a^{-1}}[a^{-1},c^{-1},[c,b]]^{c}[c,[c,b]^{-1},a^{-1}]^{[c,b]}.$$

By (2) the first commutator commutes with b and it is clear that the last one also commutes with b as $[b,c,c], a$ and $[c,b]$ all commute with b. It follows that the second commutator also commutes with b. That is

$$[[c,b]^c,[c,a]^{-1}] \text{ commutes with } b. \tag{3}$$

And replacing a with a^{-1} we see that $[[c,b]^c,[c,a]]$ also commutes with b. This means that for $u = [c,b][c,b,c]$ we have that $[u,[c,a]]$ commutes with b. Replacing c,a by c^{-1},a^{-1} we see that for $v = [c,b]^{-1}[c,b,c]^2$, $[v,[c,a]]$ commutes with b. Then

$$[uv,[c,a]] = [u,[c,a]]^v[v,[c,a]]$$

commutes with b as v also commutes with b by Step 1. But $uv = [b,c,c]^3$. So $[b,c,c]^3, [[b,c,c]^3,[c,a]]$ and $[[b,c,c]^3,[c,a]^{-1}]$ all commute with b. Lemma 2.2 and Lemma 2.3 give just as in the proof of Step 1 that $[b,c,c]^3 = 1$. As T has no elements of order 3 it follows that $[b,c,c] = 1$.

Step 3. We show that $T = \{1\}$.

Now by Step 1 and Step 2, $[b, c]$ commutes with b and c. By (3) we have as before that all elements in

$$\langle [b, c] \rangle^{\langle [c,a] \rangle}$$

commute with b and c and we conclude as before, using Lemmas 2.2 and 2.3, that $[c, b] = 1$.

Now b is in the centre of T and thus trivial. Then $T = \langle a, c\rangle$ which was nilpotent by our assumption. Hence T is trivial. □

3 A general criterion for the nilpotence of a 3-generator group

Let $S = \langle a, b, c\rangle$ be any group where

$$\langle a, b\rangle \text{ is abelian} \tag{4}$$

$$\langle a, c\rangle \text{ is nilpotent of class at most 2} \tag{5}$$

$$\langle b, c\rangle \text{ is nilpotent of class at most 3.} \tag{6}$$

In the last section we proved that S is nilpotent if S also satisfies the 4-Engel identity. We now drop this last condition. We still want S to keep the property that

$$R(S/R(S)) = \{1\} \text{ and } S/R(S) \text{ has no elements of order } 2, 3. \tag{7}$$

Notice that this property (7) is satisfied by a large class of groups. This holds for example in any group of exponent p or any n-Engel p-group, $p \neq 2, 3$. To this we add the following extra condition

$$[b, c, c] = 1. \tag{8}$$

We have not been able to prove the nilpotence of S without the extra property (8) and maybe the 4-Engel property is needed to prove the stronger version.

Proposition 3.1 *S is nilpotent.*

Proof By property (7) we can assume without loss of generality that S has trivial Hirsch-Plotkin radical. From the property that $\langle a, c\rangle$ is nilpotent of class at most 2 it follows that $\langle c\rangle^{\langle a\rangle}$ is abelian and $1 = [c, a, a]$ gives

$$c^{a^2} = c^{2a}c^{-1}.$$

We now use this together with the property that $\langle c\rangle^{\langle b\rangle}$ is abelian and $[a, b] = 1$. We have for any $r \in Z$

$$\begin{aligned} 1 &= [c^{a^2b^r}, c^{a^2}] \\ &= [c^{2ab^r}c^{-b^r}, c^{2a}c^{-1}] \\ &= [c^{2ab^r}, c^{2a}c^{-1}]^{c^{-b^r}}[c^{-b^r}, c^{2a}c^{-1}] \\ &= [c^{2ab^r}, c^{-1}]^{c^{-b^r}}[c^{-b^r}, c^{2a}]^{c^{-1}} \\ &= [c^{2ab^r}, c^{-1}][c^{-b^r}, c^{2a}]. \end{aligned}$$

It follows that

$$[c^{-1}, c^{2ab^r}] = [c^{-b^r}, c^{2a}].$$

By our conditions, it is clear that the elements c, c^{ab^r} commute with the elements c^a, c^{b^r}. Therefore the LHS commutes with c^{b^r}, c^a and the RHS commutes with c, c^{ab^r}. Thus both sides (being equal) commute with all these four conjugates. In particular we get

$$[c, c^{ab^r}]^{-2} = [c^{b^r}, c^a]^{-2}$$

and as $[c, c^{ab^r}]$ commutes with $[c^{b^r}, c^a]$ and S has no elements of order 2, it follows that $[c, c^{ab^r}] = [c^{b^r}, c^a]$. In particular we have

$$[c, c^{ab}] = [c^b, c^a] \quad \text{commutes with } c, c^b, c^a, c^{ab} \tag{9}$$

$$[c, c^{ab^2}] = [c^{b^2}, c^a] \quad \text{commutes with } c, c^{b^2}, c^a, c^{ab^2} \tag{10}$$

$$[c^b, c^{ab^2}] = [c^{b^2}, c^{ab}] \quad \text{commutes with } c^b, c^{b^2}, c^{ab}, c^{ab^2} \tag{11}$$

where (11) follows from (9) taking a conjugate with b. The next aim is to show that $H = \langle c\rangle^S$ is abelian. Clearly

$$\langle c\rangle^S = \langle c, c^b, c^{b^2}, c^a, c^{ab}, c^{ab^2}\rangle.$$

As H is normal in S and $R(S) = 1$, it follows that $R(H) = 1$. The only commutators of weight 2 in these generators that are not obviously trivial are those in (9)-(11). Now $\langle c\rangle^{\langle b\rangle}$ is abelian and $1 = [c, b, b, b]$ implies that

$$c^{b^3} = c^{3b^2}c^{-3b}c.$$

Using this together with the property that $\langle c\rangle^{\langle a\rangle}$ is abelian and $[a, b] = 1$, we see that

$$\begin{aligned}
1 &= [c^{b^3}, c^{ab^3}] \\
&= [c^{3b^2}c^{-3b}c, c^{3ab^2}c^{-3ab}c^a] \\
&= [c^{3b^2}c^{-3b}, c^{3ab^2}c^{-3ab}c^a]^c[c, c^{3ab^2}c^{-3ab}c^a] \\
&= [c^{3b^2}c^{-3b}, c^a]^c[c^{3b^2}c^{-3b}, c^{3ab^2}c^{-3ab}]^{c^a c}[c, c^a c^{3ab^2}c^{-3ab}] \\
&= [c^{3b^2}c^{-3b}, c^a][c^{3b^2}c^{-3b}, c^{3ab^2}c^{-3ab}]^{c^a c}[c, c^{3ab^2}c^{-3ab}].
\end{aligned}$$

This implies that

$$[c^{3b^2}c^{-3b}, c^{3ab^2}c^{-3ab}]^{c^a c} = [c^a, c^{3b^2}c^{-3b}][c^{3ab^2}c^{-3ab}, c]. \tag{12}$$

We work now for a moment with the two commutators on the RHS of (12). We have

$$\begin{aligned}
[c^a, c^{3b^2}c^{-3b}] &= [c^a, cc^{3b^2}c^{-3b}] = [c^a, c^{b^3}] \\
[c^{3ab^2}c^{-3ab}, c] &= [c^a c^{3ab^2}c^{-3ab}, c] = [c^{ab^3}, c].
\end{aligned}$$

By the result obtained above we thus have that the two commutators are equal and that they commute with c and c^a. From this and (12) we get

$$[c^a, c^{3b^2}c^{-3b}]^2 = [c^{3b^2}c^{-3b}, c^{3ab^2}c^{-3ab}] \tag{13}$$

and that both sides of (13) commute with c and c^a. Next we consider the RHS of (13). We have

$$\begin{aligned}
[c^{3b^2}c^{-3b}, c^{3ab^2}c^{-3ab}] &= [c^{3b^2}, c^{3ab^2}c^{-3ab}]^{c^{-3b}}[c^{-3b}, c^{-3ab}c^{3ab^2}] \\
&= [c^{3b^2}, c^{-3ab}]^{c^{-3b}}[c^{-3b}, c^{3ab^2}] \\
&= [c^{3b^2}, c^{-3ab}][c^{-3b}, c^{3ab^2}] \\
&\overset{(11)}{=} [c^{b^2}, c^{ab}]^{-9}[c^b, c^{ab^2}]^{-9} \\
&\overset{(11)}{=} [c^b, c^{ab^2}]^{-18}.
\end{aligned}$$

By (11) this last element commutes with $c^b, c^{ab^2}, c^{b^2}, c^{ab}$ and as we had already seen that the RHS of (13) commutes with c and c^a, it follows that this element is in the centre of H and thus trivial. As S has no elements of order 2 and 3 it follows that $[c^b, c^{ab^2}] = 1$. As this was the RHS of (13) we also get that the LHS is trivial. It remains now to see that $[c^a, c^{b^2}] = [c^a, c^b] = 1$. We use the fact that the LHS of (13) is trivial to obtain this. We have

$$\begin{aligned} 1 &= [c^a, c^{3b^2} c^{-3b}] \\ &= [c^a, c^{-3b}][c^a, c^{3b^2}]^{c^{-3b}} \end{aligned}$$

and thus

$$\begin{aligned} [c^a, c^{3b^2}] &= [c^a, c^{-3b}]^{-c^{-3b}} \\ &\overset{(9)}{=} [c^a, c^b]^3 \end{aligned}$$

which gives (using (10)) that

$$[c^a, c^{b^2}]^3 = [c^a, c^b]^3. \tag{14}$$

From (10) we have that the LHS commutes with $c^a, c^{b^2}, c, c^{ab^2}$ and from (9) we have that the RHS commutes with c^b and c^{ab}. Thus both sides of (14) are in the centre of H and thus trivial. As S has no elements of order 3 it follows that $[c^a, c^{b^2}] = [c^a, c^b] = 1$. Thus H is abelian and therefore trivial. This implies that $S = \langle a, b \rangle$ is abelian and therefore trivial. $\square$

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