Torsion-free groups with all subgroups 4-subnormal

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Let G be a torsion-free group with all subgroups subnormal of defect at most 4. We show that G is nilpotent of class at most 4.

1 Introduction

A subgroup H of a group G is said to be subnormal of defect at most n, or n-subnormal, if there exists a chain of subgroups of the form

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G.$$

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The property of having all subgroups subnormal is a generalized nilpotence property and indeed it is not difficult to see that any group that is nilpotent of class at most n has the property that all subgroups are n-subnormal. A group with all subgroups subnormal need not be nilpotent, as the well known examples of Heineken and Mohammed [4] show. If the subnormal defect of the subgroups of G is bounded the situation is different, and by the celebrated theorem of Roseblade [6], any group which has all subgroups subnormal of defect at most n is nilpotent of class at most $f(n)$ where f is some function depending only on n. This function is not well understood and although the bound given by Roseblade's proof is probably not close to being the best possible there is a lack of examples to form any worthwhile conjecture. For small values of n we have though some detailed information. For $n = 1$ we have the class of Dedekind groups whose structure has been well known for a long time [1,2] and groups with all subgroups 2-subnormal are also quite well understood [8]. From this work we know that $f(1) = 2$ and that $f(2) = 3$. As far as we are aware the exact value of $f(3)$ is still unknown. Restricting oneself to groups of prime exponent one gets the best possible value 3 for almost all primes but curiously enough there is one exception, namely the prime 7, in which case the class goes up to 4 [9]. The best upper bound in general is certainly higher than this but as we have said the value still seems to be unknown.

Whereas the structure of an arbitrary group with all subgroups n-subnormal seems hopelessly complicated with regard to obtaining reasonable general bounds for the nilpotency class, the situation appears to be very different when one adds the extra property that the group is torsion-free. To start with, here there are no counterexamples like the Heineken-Mohammed groups, as all torsion-free groups with all subgroups subnormal are nilpotent [7]. For torsion-free groups with all subgroups n-subnormal there also seems to be some good hope for a reasonable best upper bound for the class. For $n = 1, 2$ and 3 we know indeed that we get the best possible value, n, for the class [9]. It therefore seems natural to conjecture that the same is true for all n. In this paper we verify this for $n = 4$. The case $n = 4$ turns out to be far more difficult than when dealing with smaller values of n. Whereas the proofs for $n \leq 3$ are short and easy, our proof for $n = 4$ is quite long and technical and does not appear to hold much promise for dealing with the general case.

The conjecture that the structure in the torsion-free case is much simpler is also supported by a related result. Let us consider for a while groups with the weaker property that all cyclic subgroups are n-subnormal. It is not difficult to see that these groups satisfy the $(n+1)$-Engel law $[y,_{n+1} x] = 1$ and we know that this Engel degree is the best possible in general. However, if one furthermore assumes that the group has at least one element of infinite order then the situation is different and we get the best possible result: every non-torsion group with all cyclic subgroups n-subnormal is a n-Engel group. This was proved by Heineken [3] for $n = 3$ and in general by Kappe and Traustason [5].

2 A general lemma

Before moving to the case $n = 4$, we will first establish a lemma that works in a more general setting. Let G be a group with all subgroups n-subnormal where n is any integer greater than or equal to 4. For any $x, y \in G$ we have that

$$[y,_n x] = x^m \tag{1}$$

where $m = m(x, y)$ is some integer. It follows that G is $(n+1)$-Engel. In fact it is not difficult to see that under the hypothesis that G is torsion-free one has the stronger property that G is n-Engel. This can be shown as follows. Suppose that $x \neq 1$. By (1) we have that $x^m \in \gamma_{n+1}(\langle x, y\rangle)$. Then also

$$x^{m^2} = [y,_{n-1} x, x^m] \in \gamma_{2n+1}(\langle x, y\rangle)$$

and an easy induction shows that

$$x^{m^r} \in \gamma_{rn+1}(\langle x, y\rangle).$$

As $\langle x, y\rangle$ is nilpotent, it follows that $x^{m^r} = 1$ for some positive integer r. But as G is torsion-free we have that $m = 0$ and thus $[y,_n x] = x^0 = 1$.

Before going further we introduce some notation. Let G be a torsion-free nilpotent group, m a positive integer and a, b elements of G. We will write

$$a \equiv_m b$$

if $ab^{-1} \in G^m$. The following properties are all well known. If p is some prime greater than the class of G then, for each positive integer r, $G^{p^r} = \{g^{p^r} : g \in G\}$. Thus if $g^{p^r} = h^{p^{r+1}} \in G^{p^{r+1}}$ then $g = h^p \in G^p$. Or in the notation above, if $a^{p^r} \equiv_{p^{r+1}} 1$ then $a \equiv_p 1$. Furthermore, we have that

$$[G^{p^r}, G^{p^s}] \leq G^{p^{r+s}}.$$

Also, if $\mathcal{P}$ is an infinite set of primes then $\cap_{p \in \mathcal{P}} G^p = 1$. Thus if $a \equiv_p 1$ for infinitely many primes then $a = 1$.

Now suppose that G is torsion-free with all subgroups n-subnormal and that we want to show that G is nilpotent of class at most n. Without loss of generality we can assume that G has class at most $n+1$. In this context one might use the following lemma.

Lemma 2.1 *Let G be a torsion-free group with all subgroups n-subnormal and suppose that G is nilpotent of class at most $n+1$. Then G is centre by $(n-1)$-Engel.*

Proof We argue by contradiction and assume that there are elements $x, y, z \in G$ such that

$$a = [x,_{n-1} y, z] \neq 1.$$

Let $\mathcal{P}$ be some infinite set of primes greater than the class of G. Then $G^p = \{g^p : g \in G\}$ for all $p \in \mathcal{P}$. Notice that, as the class of G is at most n, for any $g, h_1, \ldots, h_n \in G$ we have that $[g, [h_1, \ldots, h_n]]$ is a product of commutators of the form $[g, h_{\sigma(1)}, \ldots, h_{\sigma(n)}]$ with $\sigma \in S_n$. Using this property we see that for each $p \in \mathcal{P}$ we have that

$$a^p = [x^p,_{n-1} y, z] = [z, [x^p,_{n-1} y]]^{-1} \in \langle x^p, y \rangle$$

as $\langle x^p, y \rangle$ is n-subnormal in G. Notice that any commutator in x^p and y with two or more occurrences of x^p is in G^{p^2}. This implies that for each $p \in \mathcal{P}$ there is an equation of the form

$$a^p \equiv_{p^2} y^{\alpha_p} (x^p)^{\alpha(0,p)} [x^p, y]^{\alpha(1,p)} \cdots [x^p,_{n-1} y]^{\alpha(n-1,p)}$$

where we have used the fact that G is n-Engel.

We first show that for almost all primes in $\mathcal{P}$ all the indices $\alpha_{(0,p)}, \dots, \alpha_{(n-1,p)}$ are divisible by p. Assuming this false, there is a least integer i with $0 \leq i \leq n-1$ such that p does not divide $\alpha_{(i,p)}$ for infinitely many primes from $\mathcal{P}$. Omitting those finitely many p that do not divide some of the $\alpha_{(0,p)}, \dots, \alpha_{(i-1,p)}$, we obtain an infinite subset $\mathcal{P}_1$ of $\mathcal{P}$ such that, for each $p \in \mathcal{P}_1$, we have an equation

$$a^p \equiv_{p^2} y^{\alpha_p} [x^p,_i y]^{\alpha_{(i,p)}} \cdots [x^p,_{n-1} y]^{\alpha_{(n-1,p)}}.$$

It follows that for all $p \in \mathcal{P}_1$ we have

$$1 = [a^p,_{n-1-i} y, z] \equiv_{p^2} [x^p,_{n-1} y, z] \equiv_{p^2} a^{p\alpha_{(i,p)}}.$$

As G is torsion-free and all the primes in $\mathcal{P}_1$ are greater than the class of G, we have

$$a \equiv_p 1$$

for all $p \in \mathcal{P}_1$, which gives the contradiction that $a = 1$. So for almost all primes in $\mathcal{P}$, all the indices $\alpha_{(0,p)}, \dots, \alpha_{(n-1,p)}$ are divisible by p. By removing the finitely many exceptions we can thus assume that this is true for all the primes in $\mathcal{P}$.

This means that for all $p \in \mathcal{P}$, we have

$$a^p \equiv_{p^2} y^{\alpha_p}.$$

We claim that for almost all $p \in \mathcal{P}$, α_p is divisible by p^2. Supposing this false, we have an infinite subset $\mathcal{P}_1 \subset \mathcal{P}$ with the property that p^2 does not divide α_p for all $p \in \mathcal{P}_1$. It follows that

$$1 = [x,_{n-2} y, a^p, z] \equiv_{p^2} [x,_{n-1} y, z]^{\alpha_p} \equiv_{p^2} a^{\alpha_p}$$

and again we get the contradiction that

$$a \equiv_p 1$$

for all $p \in \mathcal{P}_1$ and hence $a = 1$. So without loss of generality we can suppose that $a^p \in G^{p^2}$ for all $p \in \mathcal{P}$, and as before

$$a \equiv_p 1$$

for all $p \in \mathcal{P}$ and thus again a must be trivial. This final contradiction completes the proof of the lemma. $\Box$

We remark that this might provide a starting point for an argument to prove that every torsion-free group with all subgroups n-subnormal is nilpotent of class at most n. Unfortunately we do not know how to continue with this general case, and in the rest of the paper we restrict ourselves to groups with all subgroups 4-subnormal.

3 The groups generated by two or three elements

For the rest of this paper we work with a torsion-free group G of nilpotency class at most 5 with all subgroups 4-subnormal. Our aim is to show that they are nilpotent of class at most 4. In this section we deal with two- and three-generator groups. The two-generator groups are easily dealt with using Lemma 2.1.

Lemma 3.1 *Let $x, y \in G$. Then $\gamma_4(\langle x, y\rangle) \leq Z(G)$. In particular, any torsion-free two-generator group with all subgroups 4-subnormal is nilpotent of class at most 4.*

Proof. By Lemma 2.1 we have that $G/Z(G)$ is 3-Engel. Thus modulo $Z(G)$ we have

$$1 \equiv [y, yx, yx, yx] \equiv [y, x, x, y]^2$$

and as $G/Z(G)$ is torsion-free it follows that $[y, x, x, y] \in Z(G)$. It is now clear that all commutators in x, y of weight 4 are in $Z(G)$. $\Box$

Remark. The group G is 4-Engel. Expanding $[y,_4 x_1x_2x_3x_4] = 1$ leads to the linearized 4-Engel identity

$$\prod_{\sigma \in S_4} [y, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}] = 1.$$

In fact any multihomogenous identity of weight 5 satisfied by G implies the corresponding linearized version. As G is torsion-free these two are equivalent.

We next deal with a 3-generator group G. We can assume that G is a counterexample of minimal Hirch length; it follows in particular that $Z(G)$ is cyclic. First we tackle commutators of weight 5 with an entry repeated three times.

Lemma 3.2 *Let $x, y, z \in G$. All commutators of multiweight $(3, 1, 1)$ in x, y, z are trivial.*

Proof This is also an easy corollary of Lemma 2.1. Let $e_1 = [z, y, x, x, x]$, $e_2 = [z, x, y, x, x]$, $e_3 = [z, x, x, y, x]$ and $e_4 = [z, x, x, x, y]$. It suffices to show that these four commutators are trivial. That $e_4 = 1$ is an immediate corollary of Lemma 2.1. That lemma also implies that

$$\begin{aligned} 1 &= [z, [y, x, x, x]] \\ &= [z, y, x, x, x][z, x, y, x, x]^{-3}[z, x, x, y, x]^3[z, x, x, x, y] \\ &= e_1 e_2^{-3} e_3^3. \end{aligned}$$

As G is 4-Engel we also have

$$\begin{aligned} 1 &= [z, y, x, x, x][z, x, y, x, x][z, x, x, y, x][z, x, x, x, y] \\ &= e_1 e_2 e_3 \end{aligned}$$

and

$$\begin{aligned} 1 &= [y, z, x, x, x]^{-1}[y, x, z, x, x]^{-1}[y, x, x, z, x]^{-1} \\ &= [z, y, x, x, x]^3[z, x, y, x, x]^{-3}[z, x, x, y, x] \\ &= e_1^3 e_2^{-3} e_3. \end{aligned}$$

As

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -3 & 3 \\ 3 & -3 & 1 \end{vmatrix} = 20 \neq 0$$

and G is torsion-free, it follows that $e_1 = e_2 = e_3 = 1$. $\Box$

To prove that all commutators of weight 5 in x, y, z are trivial it remains to show that all commutators of multi-weight $(2, 2, 1)$ in x, y, z are trivial. Let $e_1 = [z, x, x, y, y]$, $e_2 = [z, x, y, x, y]$, $e_3 = [z, x, y, y, x]$, $e_4 = [z, y, y, x, x]$, $e_5 = [z, y, x, y, x]$ and $e_6 = [z, y, x, x, y]$.

Lemma 3.3 *The commutators of multiweight* $(2,2,1)$ *in* x, y, z *are all powers of* $e = [z, y, x, y, x]$. *Furthermore*

$$\begin{array}{lll} [z,x,x,y,y] = e^{-3} & [z,x,y,x,y] = e & [z,x,y,y,x] = e^2 \\ [z,y,y,x,x] = e^{-3} & [z,y,x,y,x] = e & [z,y,x,x,y] = e^2. \end{array}$$

Proof From Lemma 3.2 (the linearized version) we have

$$\begin{aligned} 1 &= [\underline{z},\underline{x},x,y,y][\underline{z},\underline{x},y,x,y][\underline{z},\underline{x},y,y,x] \\ &= e_1e_2e_3, \end{aligned}$$

$$\begin{aligned} 1 &= [\underline{z},\underline{y},y,x,x][\underline{z},\underline{y},x,y,x][\underline{z},\underline{y},x,x,y] \\ &= e_4e_5e_6 \end{aligned}$$

$$\begin{aligned} 1 &= [\underline{z},y,\underline{y},x,x][\underline{z},x,\underline{y},y,x][\underline{z},x,\underline{y},x,y] \\ &= e_4e_3e_2 \end{aligned}$$

$$\begin{aligned} 1 &= [\underline{z},x,y,\underline{x},y][\underline{z},y,x,\underline{x},y][\underline{z},y,y,\underline{x},x] \\ &= e_2e_6e_4 \end{aligned}$$

$$\begin{aligned} 1 &= [\underline{x},z,y,y,\underline{x}][\underline{x},y,z,y,\underline{x}][\underline{x},y,y,z,\underline{x}] \\ &= [z,x,y,y,x]^{-3}[z,y,x,y,x]^3[z,y,y,x,x]^{-1} \\ &= e_3^{-3}e_5^3e_4^{-1}. \end{aligned}$$

By trivial linear algebra calculations one sees that the solution to these equations is as described in the lemma. □

Lemma 3.4 *There exist some* $x, y, z \in G$ *such that* $[z, y, x, y, x] \neq 1$ *but* $[y, x, x, x] = 1$.

Proof As G has class 5, we know from Lemma 3.3 that there exist $x, y, z \in G$ such that $e = [z, y, x, y, x] \neq 1$. Suppose that $[y, x, x, x] \neq 1$ and let r, s be arbitrary integers. Then using Lemma 3.3, we get

$$\begin{aligned} [y,_3 x^r[z,y]^s] &= [y,x,x,x]^{r^3}([y,[z,y],x,x][y,x,[z,y],x][y,x,x,[z,y]])^{r^2s} \\ &= ([y,x,x,x]^r[z,y,y,x,x]^{-3s}[z,y,x,y,x]^{3s}[z,y,x,x,y]^{-s})^{r^2} \\ &= ([y,x,x,x]^re^{10s})^{r^2}. \end{aligned}$$

As $Z(G)$ is cyclic, there exist some non-zero integers r, s such that $[y, x, x, x]^r e^{10s} = 1$. Replacing x by $x^r[z, y]^s$ we have the elements x, y, z required. □

Up till now we have used the 4-subnormal property only in the proof of Lemma 2.1, everything else being a consequence of that lemma. To finish the proof that G is nilpotent of class at most 4, we need to use the 4-subnormal property again.

Lemma 3.5 *If* $[z, y, x, y, x] \neq 1$ *and* $[y, x, x, x] = 1$ *then* $[y, x, x, y] \neq 1$.

Proof Let $\mathcal{P}$ be the set of all primes greater than the class of G and let $p \in \mathcal{P}$. As $[y, x, x, x] = 1$ we have an equation of the form

$$\begin{aligned} [z, y^p, x, y^p, x] &= x^{\alpha(p)}(y^p)^{\beta(p)}[y^p, x]^{\gamma(p)}[y^p, x, x]^{\eta(p)}[x, y^p, y^p]^{\rho(p)} \\ &\quad [y^p, x, x, y^p]^{\theta(p)}[x, y^p, y^p, y^p]^{\phi(p)}. \end{aligned}$$

Taking commutators on both sides with $[z, y, x, y]$ and using Lemma 3.2 gives $[z, y, x, y, x]^{\alpha(p)} = 1$ and thus $\alpha(p) = 0$. Similarly, the equation $1 = [z, y, x, e^{p^2}, x] = [z, y, x, y^p, x]^{\beta(p)} = 1$ gives $\beta(p) = 0$. We continue in this manner. Next

$$\begin{aligned} 1 &= [z, y, x, e^{p^2}] \\ &= [z, y, x, [y, x]^p]^{\gamma(p)} \\ &= ([z, y, x, y, x][z, y, x, x, y]^{-1})^{p\gamma(p)} \\ &= e^{-p\gamma(p)} \end{aligned}$$

which shows that $\gamma(p) = 0$. Then

$$\begin{aligned} 1 &= [z, y, e^{p^2}] \\ &= [z, y, [y, x, x]^p]^{\eta(p)} \\ &= ([z, y, y, x, x][y, x, y, x, y]^{-2}[z, y, x, x, y])^{p\eta(p)} \\ &= e^{-3p\eta(p)} \end{aligned}$$

which gives $\eta(p) = 0$ and

$$\begin{aligned} 1 &= [z, x, e^{p^2}] \\ &= [z, x, [x, y, y]^{p^2}]^{\rho(p)} \\ &= e^{-3p^2\rho(p)} \end{aligned}$$

giving $\rho(p) = 0$. So we have shown that for each $p \in \mathcal{P}$, there is an equation

$$e^{p^2} \equiv_{p^3} [y, x, x, y]^{p^2\theta(p)}$$

This implies that $[y, x, x, y] \neq 1$, since otherwise

$$e \equiv_p 1$$

for all $p \in \mathcal{P}$ that gives the contradiction that $e = 1$. This finishes the proof of the lemma. □

Proposition 3.6 *Every torsion-free* 3*-generator group with all subgroups* 4*-subnormal is nilpotent of class at most* 4*.*

Proof We argue by contradiction and suppose that this is not the case. Working with a minimal counterexample G as before we know by Lemma 3.4 that there exist $x, y, z \in G$ such that $[z, y, x, y, x] \neq 1$ but $[y, x, x, x] = 1$. From Lemma 3.5 we then have that $[y, x, x, y] \neq 1$. But we also have for any non-zero integers r, s that

$$[z, y^r[z, y]^s, x, y^r[y, z]^s, x] = [z, y, x, y, x]^{r^2} \neq 1$$

and (using Lemma 3.2)

$$[y^r[z, y]^s, x, x, x] = [y, x, x, x]^r[z, y, x, x, x]^s = 1.$$

Thus Lemma 3.5 also gives that $[y^r[z, y]^s, x, x, y^r[z, y]^s] \neq 1$. We obtain a contraction from this. Expanding gives

$$\begin{aligned} 1 &\neq [y, x, x, y]^{r^2}([z, y, x, x, y][y, x, x, [z, y]])^{rs} \\ &= [y, x, x, y]^{r^2}([z, y, x, x, y][y, x, x, z, y])^{rs} \\ &= [y, x, x, y]^{r^2}([z, y, x, y, x]^2[z, x, x, y, y]^{-1})^{rs} \\ &= ([y, x, x, y]^r e^{5s})^r. \end{aligned}$$

By Lemma 2.2 we know that $[y, x, x, y]$ is a nontrivial element of $Z(G)$. But the centre is cyclic so we can choose nonzero integers r, s such that $[y, x, x, y]^r e^{5s} = 1$. By this final contradiction we know that the proposition holds. □

4 The general case

We wish to extend the result of Proposition 3.6 to all torsion-free groups with all subgroups 4-subnormal. This will be deduced easily from the 4-generator case.

Let G be a 4-generator group that is torsion-free and with all subgroups 4-subnormal. Our aim is to show that G is nilpotent of class at most 4. First we will see how much information we can get using only the result from the last section that all 3-generator subgroups are nilpotent of class at most 4. From now on until the final paragraph of this section, we shall assume that $G = \langle y, x_1, x_2, x_3 \rangle$. Let $e_1 = [x_1, y, y, x_2, x_3]$, $e_2 = [x_1, y, x_2, y, x_3]$, $e_3 = [x_1, y, x_2, x_3, y]$, $e_4 = [x_1, x_2, y, y, x_3]$, $e_5 = [x_1, x_2, y, x_3, y]$ and $e_6 = [x_1, x_2, x_3, y, y]$. From Proposition 3.6 it follows that any commutator of multiweight $(1, 1, 1, 2)$ in x_1, x_2, x_3, y is antisymmetric in x_1, x_2, x_3. Notice also that the Hall-Witt identity gives

$$\begin{aligned} 1 &= [x_1, x_2, x_3, y, y][x_2, x_3, x_1, y, y][x_3, x_1, x_2, y, y] \\ &= e_6^3 \end{aligned}$$

which gives that $e_6 = 1$ as G is torsion-free.

Lemma 4.1 *Let* $e = [x_1, y, x_2, y, x_3]$. *Then*

$$\begin{aligned} [x_1, y, y, x_2, x_3] &= e^{-3} \\ [x_1, y, x_2, y, x_3] &= e \\ [x_1, y, x_2, x_3, y] &= e^{-1} \\ [x_1, x_2, y, y, x_3] &= e^2 \\ [x_1, x_2, y, x_3, y] &= e^{-2}. \end{aligned}$$

Proof From Proposition 3.6 we have the following identities.

$$\begin{aligned} 1 &= [\underline{x_1}, y, y, x_2, \underline{x_3}][\underline{x_1}, y, x_2, y, \underline{x_3}][\underline{x_1}, x_2, y, y, \underline{x_3}] \\ &= e_1 e_2 e_4 \end{aligned}$$

$$\begin{aligned} 1 &= [\underline{x_1}, \underline{x_2}, x_3, y, y][\underline{x_1}, \underline{x_2}, y, x_3, y][\underline{x_1}, \underline{x_2}, y, y, x_3] \\ &= e_5 e_4 \end{aligned}$$

$$\begin{aligned} 1 &= [\underline{x_1}, x_2, \underline{x_3}, y, y][\underline{x_1}, y, \underline{x_3}, x_2, y][\underline{x_1}, y, \underline{x_3}, y, x_2] \\ &= e_3^{-1}e_2^{-1} \end{aligned}$$

Solving these equations together gives

$$e_5 = e_4^{-1}, \quad e_3 = e_2^{-1}, \quad e_1 = e_2^{-1}e_4^{-1}.$$

Then using the antisymmetry, we also have

$$\begin{aligned} [x_1, y, x_2, y, x_3] &= [x_2, y, x_1, y, x_3]^{-1} \\ &= [x_1, [x_2, y], y, x_3] \\ &= [x_1, x_2, y, y, x_3][x_1, y, x_2, y, x_3]^{-1}, \end{aligned}$$

which gives $e_4 = e_2^2$ and thus $e_5 = e_3^2$ and

$$e_1 = e_2^{-3}, e_3 = e_2^{-1}, e_4 = e_2^2, e_5 = e_2^{-2}.$$

This proves the lemma. □

To go further we need to use the subnormality condition. We want to show that the class of G is at most 4. We argue by contradiction and take a counterexample with minimal Hirsch length. In particular it follows that $Z(G)$ is cyclic. Let

$$H = \{x \in G : \ [g, x, x] \in Z_2(G) \ \forall g \in G\}.$$

Lemma 4.2 *Without loss of generality we can assume that H is a characteristic subgroup of G such that G/H is infinite cyclic.*

Proof G has class 5 and we can thus apply Lemma 4.1 to deduce that $[x_1, y, x_2, y, x_3] \neq 1$ for some generators x_1, x_2, x_3, y for G. Now $Z(G)$ is cyclic. Suppose that

$$|Z(G)/\langle [x_1, y, x_2, y, x_3]\rangle| = r.$$

Now the commutators $[y, x_1, x_2, x_1, x_3]$, $[y, x_2, x_3, x_2, x_1]$, $[y, x_3, x_1, x_3, x_2]$ are in the centre of G. Thus the r-th powers are in $\langle [x_1, y, x_2, y, x_3]\rangle$. Suppose

$$\begin{aligned} [y, x_1, x_2, x_1, x_3]^r &= [x_1, y, x_2, y, x_3]^\alpha \\ [y, x_2, x_3, x_2, x_1]^r &= [x_2, y, x_3, y, x_1]^\beta \\ [y, x_3, x_1, x_3, x_2]^r &= [x_3, y, x_1, y, x_2]^\gamma. \end{aligned}$$

Notice that it follows from the antisymmetry in x_1, x_2, x_3 that $[x_1, y, x_2, y, x_3] = [x_2, y, x_3, y, x_2] = [x_3, y, x_1, y, x_2]$. It is convenient to write this same element differently as we want to apply some symmetry arguments.

Let $\bar{x_1} = x_1^r y^\alpha$, $\bar{x_2} = x_2^r y^\beta$ and $\bar{x_3} = x_3^r y^\gamma$. Then, using the fact that all 3-generator subgroups have class at most 4, we have

$$\begin{aligned} [y, \bar{x_1}, \bar{x_2}, \bar{x_1}, \bar{x_3}] &= [y, x_1^r, x_2^r y^\beta, x_1^r y^\alpha, x_3^r y^\gamma] \\ &= [y, x_1^r, x_2^r, x_1^r y^\alpha, x_3^r] \\ &= [y, x_1, x_2, x_1, x_3]^{r^4} [x_1, y, x_2, y, x_3]^{-r^3\alpha} \\ &= 1. \end{aligned}$$

By symmetry we also have that $[y, \bar{x_2}, \bar{x_3}, \bar{x_2}, \bar{x_1}] = 1$ and $[y, \bar{x_3}, \bar{x_1}, \bar{x_3}, \bar{x_2}] = 1$. As $\langle \bar{x_1}, \bar{x_2}, \bar{x_3}, y\rangle$ has nilpotency class 5, we can without loss of generality assume that $\bar{x_i} = x_i$. From what we have done we know that all commutators of weight 5 in x_1, x_2, x_3, y that are of multiweight different from $(1, 1, 1, 2)$ are trivial, and that $\gamma_5(G) = \langle [x_1, y, x_2, y, x_3]\rangle$.

We next show that $H = \langle x_1, x_2, x_3\rangle\gamma_2(G)$. Firstly it is easy to see from the work above that $[g, x_i, x_i] \in Z_2(G)$ for $i = 1, 2, 3$ and all $g \in G$. Also by the antisymmetry in x_1, x_2, x_3 we have that $[g, x_i, x_j][g, x_j, x_i] \in Z_2(G)$ for all $1 \le i, j \le 3$. It follows that $\langle x_1, x_2, x_3\rangle\gamma_2(G) \subset H$. But if $h \in \langle x_1, x_2, x_3\rangle\gamma_2(G)$, then for each non-zero integer s

$$[x_1, y^s h, y^s h, x_2, x_3] = [x_1, y, y, x_2, x_3]^{s^2}$$

which is non-trivial by Lemma 4.1. Thus no element in $G \setminus \langle x_1, x_2, x_3\rangle\gamma_2(G)$ is in H and it follows that $H = \langle x_1, x_2, x_3\rangle\gamma_2(G)$.

It is clear from the definition of H that it is a characteristic subset of G. Finally if modulo $Z_2(G)$ we have that

$$1 \equiv [g, x^s, x^s] \equiv [g, x, x]^{s^2}$$

then, as $G/Z_2(G)$ is torsion-free, s must be 0. This shows that G/H is torsion-free and thus infinite cyclic. □

Lemma 4.3 *We have that $[g, x, x, x] = 1$ for all $g \in G$ and all $x \in H$.*

Proof Take generators y, x_1, x_2, x_3 for G such that $G = \langle H, y\rangle$ and $H = \langle x_1, x_2, x_3\rangle\gamma_2(G)$. We will show that

$$[y, x_1, x_1, x_2][y, x_1, x_2, x_1][y, x_2, x_1, x_1] = 1. \qquad (2)$$

From this we will then derive the lemma.

Notice first that if y, x_1, x_2 are replaced by $\bar{y} = y^s v$, $\bar{x_1} = x_1^{r_1}u_1$, $\bar{x_2} = x_2^{r_2}u_2$ with s, r_1, r_2 non-zero integers and $v, u_1, u_2 \in \gamma_2(G)$, then

$$[\bar{y}, \bar{x_1}, \bar{x_1}, \bar{x_2}][\bar{y}, \bar{x_1}, \bar{x_2}, \bar{x_1}][\bar{y}, \bar{x_2}, \bar{x_1}, \bar{x_1}] = [y, \bar{x_1}, \bar{x_1}, x_2]^{r_2 s}[y, \bar{x_1}, x_2, \bar{x_1}]^{r_2 s}[y, x_2, \bar{x_1}, \bar{x_1}]^{r_2 s}.$$

Furthermore, if $\bar{x_1} = x_1^r[y, x_1]^\alpha[y, x_2]^\beta[y, x_3]^\gamma c$, with $c \in H'\gamma_3(G)$, then

$$\begin{aligned}[y, \bar{x_1}, \bar{x_1}, x_2][y, \bar{x_1}, x_2, \bar{x_1}][y, x_2, \bar{x_1}, \bar{x_1}] &= [y, x_1, x_1, x_2]^{r_1^2}[y, x_1, x_2, x_1]^{r_1^2}[y, x_2, x_1, x_1]^{r_1^2} \\ &\quad [y, [y, x_3], x_1, x_2]^{r_1\gamma}[y, x_1, [y, x_3], x_2]^{r_1\gamma} \\ &\quad [y, [y, x_3], x_2, x_1]^{r_1\gamma}[y, x_1, x_2, [y, x_3]]^{r_1\gamma} \\ &\quad [y, x_2, [y, x_3], x_1]^{r_1\gamma}[y, x_2, x_1, [y, x_3]]^{r_1\gamma} \\ &= [y, x_1, x_1, x_2]^{r_1^2}[y, x_1, x_2, x_1]^{r_1^2}[y, x_2, x_1, x_1]^{r_1^2}.\end{aligned}$$

Here the last equality holds because every commutator of multiweight $(2, 1, 1, 1)$ in y, x_1, x_2, x_3 is antisymmetric in x_1, x_2. From these calculations it is clear that (2) holds if and only it holds for x_1, x_2, x_3, y replaced by $\bar{x_1}, \bar{x_2}, \bar{x_3}, \bar{y}$. The idea is now to chose $\bar{x_1}, \bar{x_2}, \bar{x_3}, y$ in such a way that we can apply the subnormality property effectively. Before finishing the proof of Lemma 4.3, we summarise those properties needed in two separate lemmas that we will also make use of later.

Lemma 4.4 *There exists an integer r and elements $u_1, u_2, u_3 \in \gamma_2(G)$ such that for $\bar{x}_i = x_i^r u_i$ we have that*

$$[y, \bar{x}_i, \bar{x}_i]$$

commutes with $\bar{x}_j, y$ when $i \neq j$, and

$$[y, \bar{x}_i, \bar{x}_j][y, \bar{x}_j, \bar{x}_i]$$

commutes with y when i, j are distinct. Furthermore the choice can be made such that

$$[y, \bar{x_1}, \bar{x_2}, \bar{x_3}][y, \bar{x_2}, \bar{x_1}, \bar{x_3}] = 1.$$

Proof Suppose that (i,j,k) is one of $(1,2,3),(2,3,1),(3,1,2)$. Then the elements

$$[y,x_i,x_i,y],\ [y,x_i,x_i,x_j],\ [y,x_i,x_i,x_k],\ [y,x_i,x_j,y][y,x_j,x_i,y]$$

all lie in the centre. Suppose that

$$\begin{aligned} [y,x_i,x_i,y]^r &= [x_j,y,x_k,y,x_i]^{-\alpha_i} \\ ([y,x_i,x_j,y][y,x_j,x_i,y])^r &= [x_i,y,x_j,y,x_k]^{-\beta_k} \\ [y,x_i,x_i,x_j]^r &= [x_i,y,x_j,y,x_k]^{\gamma_i} \\ [y,x_i,x_i,x_k]^r &= [x_i,y,x_j,y,x_k]^{-\sigma_i}. \end{aligned}$$

Let

$$\bar{x}_i = x_i^{28r}[x_j,x_k]^{7\alpha_i}[x_k,x_i]^{7\beta_k}[y,x_k]^{4\gamma_i}[y,x_j]^{4\sigma_i}$$

We then have, using Lemmas 4.1 and 4.2,

$$\begin{aligned} [y,\bar{x}_i,\bar{x}_i,y] &= [y,x_i^{28r}[x_j,x_k]^{7\alpha_i},x_i^{28r}[x_j,x_k]^{7\alpha_i},y] \\ &= [y,x_i,x_i,y]^{7^2\cdot 16r^2}[y,x_i,x_j,x_k,y]^{7^2\cdot 16r\alpha_i} \\ &= 1, \end{aligned}$$

$$\begin{aligned} [y,\bar{x}_i,\bar{x}_j,y][y,\bar{x}_j,\bar{x}_i,y] &= [y,x_i^{28r}[x_k,x_i]^{7\beta_k},x_j^{28r},y][y,x_j^{28r},x_i^{28r}[x_k,x_i]^{7\beta_k},y] \\ &= ([y,x_i,x_j,y][y,x_j,x_i,y])^{7^2\cdot 16r^2}[y,x_k,x_i,x_j,y]^{7^2\cdot 16r\beta_k} \\ &= 1, \end{aligned}$$

$$\begin{aligned} [y,\bar{x}_i,\bar{x}_i,\bar{x}_j] &= [y,x_i^{28r}[y,x_k]^{4\gamma_i},x_i^{28r}[y,x_k]^{4\gamma_i},x_j^{28r}] \\ &= [y,x_i,x_i,x_j]^{4^3\cdot 7^3r^3}([y,[y,x_k],x_i,x_j][y,x_i,[y,x_k],x_j])^{4^3\cdot 7^2r^2\gamma_i} \\ &= [y,x_i,x_i,x_j]^{4^3\cdot 7^3r^3}([x_k,y,y,x_i,x_j]^2[x_k,y,x_i,y,x_j]^{-1})^{4^3\cdot 7^2r^2\gamma_i} \\ &= [y,x_i,x_i,x_j]^{4^3\cdot 7^3r^3}[x_k,y,x_i,y,x_j]^{-4^3\cdot 7^3r^2\gamma_i} \\ &= 1, \end{aligned}$$

$$\begin{aligned} [y,\bar{x}_i,\bar{x}_i,\bar{x}_k] &= [y,x_i^{28r}[y,x_j]^{4\sigma_i},x_i^{28r}[y,x_j]^{4\sigma_i},x_k^{28r}] \\ &= [y,x_i,x_i,x_k]^{4^3\cdot 7^3r^3}([y,[y,x_j],x_i,x_k][y,x_i,[y,x_j],x_k])^{4^3\cdot 7^2r^2\sigma_i} \\ &= [y,x_i,x_i,x_k]^{4^3\cdot 7^3r^3}([x_j,y,y,x_i,x_k]^2[x_j,y,x_i,y,x_k]^{-1})^{4^3\cdot 7^2r^2\sigma_i} \end{aligned}$$

$$\begin{aligned} &= [y, x_i, x_i, x_k]^{4^3 \cdot 7^3 r^3} [x_i, y, x_j, y, x_k]^{4^3 \cdot 7^3 r^2 \gamma_i} \\ &= 1. \end{aligned}$$

Finally suppose that

$$([y, \bar{x_1}, \bar{x_2}, \bar{x_3}][y, \bar{x_2}, \bar{x_1}, \bar{x_3}])^l = [\bar{x_1}, y, \bar{x_2}, y, \bar{x_3}]^\tau.$$

Let $\tilde{x_1} = \bar{x_1}^{7l}[y, \bar{x_1}]^\tau$. One easily checks that all the previous properties still hold if $\bar{x_1}$ is replaced by $\tilde{x_1}$ and furthermore we have

$$\begin{aligned} [y, \tilde{x_1}, \bar{x_2}, \bar{x_3}][y, \bar{x_2}, \tilde{x_1}, \bar{x_3}] &= [y, \bar{x_1}, \bar{x_2}, \bar{x_3}]^{7l}[y, \bar{x_2}, \bar{x_1}, \bar{x_3}]^{7l} \\ &\quad [y, [y, \bar{x_1}], \bar{x_2}, \bar{x_3}]^\tau [y, \bar{x_2}, [y, \bar{x_1}], \bar{x_3}]^\tau \\ &= [y, \bar{x_1}, \bar{x_2}, \bar{x_3}]^{7l}[y, \bar{x_2}, \bar{x_1}, \bar{x_3}]^{7l} \\ &\quad [\bar{x_1}, y, y, \bar{x_2}, \bar{x_3}]^{2\tau}[\bar{x_1}, y, \bar{x_2}, y, \bar{x_3}]^{-\tau} \\ &= [y, \bar{x_1}, \bar{x_2}, \bar{x_3}]^{7l}[y, \bar{x_2}, \bar{x_1}, \bar{x_3}]^{7l} \\ &\quad [\bar{x_1}, y, \bar{x_2}, y, \bar{x_3}]^{-7\tau} \\ &= 1. \end{aligned}$$

This finishes the proof of the Lemma 4.4. □

Notice that the above lemma still holds if y is replaced by any element from $\langle y\rangle\gamma_2(G)$ and $\bar{x_i}$ by any element from $\langle \bar{x_i}\rangle\gamma_3(G)$. To simplify the notation we assume that we have replaced x_i by $\bar{x_i}$ and thus we use x_i as notation instead of $\bar{x_i}$.

Lemma 4.5 *There exists some non-zero integer r and $v \in \gamma_2(G)$, such that for $\bar{y} = y^r v$, we have*

$$\begin{aligned} [x_i, \bar{y}, \bar{y}, \bar{y}] &= 1 \\ [\bar{y}, x_j, [\bar{y}, x_i]][\bar{y}, x_i, x_j, \bar{y}]^{-4} &= 1 \end{aligned}$$

for all $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$.

Proof We have that the commutators $[x_i, y, y, y]$, $i = 1, 2, 3$, are in the centre of G. Suppose that

$$\begin{aligned} [x_1, y, y, y]^r &= [x_1, y, x_2, y, x_3]^\alpha \\ [x_2, y, y, y]^r &= [x_2, y, x_3, y, x_1]^\beta \\ [x_3, y, y, y]^r &= [x_3, y, x_1, y, x_2]^\gamma. \end{aligned}$$

Using Lemma 4.1, we also have

$$\begin{aligned}
[y, x_2, [y, x_1]][y, x_1, x_2, y]^{-4}, x_3] &= [x_2, y, y, x_1, x_3]^{-1}[x_2, y, x_1, y, x_3][x_1, y, x_2, y, x_3]^4 \\
&= [x_1, y, y, x_2, x_3][x_1, y, x_2, y, x_3]^3 \\
&= 1.
\end{aligned}$$

This shows that $[y, x_2, [y, x_1]][y, x_1, x_2, y]^{-4}$ lies in the centre of G. By symmetry the same is true for $[y, x_3, [y, x_2]][y, x_2, x_3, y]^{-4}$ and $[y, x_1, [y, x_3]][y, x_3, x_1, y]^{-4}$. Suppose

$$\begin{aligned}
([y, x_2, [y, x_1]][y, x_1, x_2, y]^{-4})^r &= [x_1, y, x_2, y, x_3]^\sigma \\
([y, x_3, [y, x_2]][y, x_2, x_3, y]^{-4})^r &= [x_2, y, x_3, y, x_1]^\tau \\
([y, x_1, [y, x_3]][y, x_3, x_1, y]^{-4})^r &= [x_3, y, x_1, y, x_2]^\rho.
\end{aligned}$$

Let

$$\bar{y} = y^{8r}[x_2, x_3]^\alpha [x_3, x_1]^\beta [x_1, x_2]^\gamma [y, x_3]^{-\sigma}[y, x_1]^{-\tau}[y, x_2]^{-\rho}.$$

Calculations show that $\bar{y}$ has the properties required. □

We can now finish the proof of Lemma 4.3. By replacing x_1, x_2, x_3, y by $\bar{x_1}, \bar{x_2}, \bar{x_3}, \bar{y}$, we can assume that all the equations from the last two lemmas hold.

First we notice that we can without loss of generality assume that either $[y, x_1, x_1, x_1] \neq 1$ or $[y, x_1, x_1] = 1$. To see this suppose that $[y, x_1, x_1, x_1] = 1$. This together with previous lemmas then implies that $[y, x_1, x_1]$ is in the centre of G. Suppose that

$$[y, x_1, x_1]^r = [x_1, y, x_2, y, x_3]^{-s}.$$

Let $\bar{x_1} = x_1^{3r}[y, x_2, x_3]^s$. Then

$$\begin{aligned}
[y, \bar{x_1}, \bar{x_1}] &= [y, x_1, x_1]^{9r^2}([y, [y, x_2, x_3], x_1][y, x_1, [y, x_2, x_3]])^{3rs} \\
&= [y, x_1, x_1]^{9r^2}[x_1, y, x_2, y, x_3]^{6rs}[x_1, y, x_2, x_3, y]^{-3rs} \\
&= [y, x_1, x_1]^{9r^2}[x_1, y, x_2, y, x_3]^{9rs} \\
&= 1.
\end{aligned}$$

By replacing x_1 by $\bar{x_1}$ we would thus have the property wanted. Notice that, as we have remarked before, the change of x_1 has no influence on all the previous established properties. By replacing x_2 by a suitable element if necessary, we can also assume that either $[y, x_2, x_2, x_2] \neq 1$ or $[y, x_2, x_2] = 1$.

Let $\mathcal{P}$ be a infinite set of primes greater than the class of G, and let $p \in \mathcal{P}$. As $H = \langle x_1^{p^2}, x_2^p, y\rangle$ is 4-subnormal in G, we have

$$1 \neq [x_1^{p^2}, y, x_2^p, y, x_3] = [x_3, y, x_1^{p^2}, y, x_2^p] \in H$$

We can thus express $[x_1^{p^2}, y, x_2^p, y, x_3]$ in basic commutators in $x_1^{p^2}, x_2^p, y$ of weight at most 4. We will eventually work modulo G^{p^4} and we will thus not write down those of the commutators which are in G^{p^4}. These are the commutators that have either at least two occurrences of x_1 or include x_1 and have at least two occurrences of x_2. With the order $x_1^{p^2} < x_2^p < y$ and $[x_2^p, x_1^{p^2}] < [y, x_1^{p^2}] < [y, x_2^p]$, the basic commutators are

$$\begin{aligned}&x_1^{p^2}, x_2^p, y, [x_2^p, x_1^{p^2}], [y, x_1^{p^2}], [y, x_2^p], [x_2^p, x_1^{p^2}, y], [y, x_1^{p^2}, x_2^p], [y, x_1^{p^2}, y],\\ &[y, x_2^p, x_2^p], [y, x_2^p, y], [x_2^p, x_1^{p^2}, y, y], [y, x_1^{p^2}, x_2^p, y], [y, x_1^{p^2}, y, y],\\ &[y, x_2^p, x_2^p, x_2^p], [y, x_2^p, x_2^p, y], [y, x_2^p, y, y], [y, x_2^p, [y, x_1^{p^2}]],\end{aligned}$$

and commutators that lie in G^{p^4}, all of which lie in $\gamma_3(H)$. The elements written can clearly be replaced by

$$\begin{aligned}&x_1^{p^2}, x_2^p, y, [x_2^p, x_1^{p^2}], [y, x_1^{p^2}], [y, x_2^p], [y, x_1^{p^2}, x_2^p][y, x_2^p, x_1^{p^2}], [y, x_1^{p^2}, x_2^p],\\ &[y, x_1^{p^2}, y], [y, x_2^p, x_2^p], [y, x_2^p, y], [y, x_1^{p^2}, x_2^p, y][y, x_2^p, x_1^{p^2}, y], [y, x_1^{p^2}, x_2^p, y],\\ &[y, x_1^{p^2}, y, y], [y, x_2^p, x_2^p, x_2^p], [y, x_2^p, x_2^p, y], [y, x_2^p, y, y], [y, x_2^p, [y, x_1^{p^2}]][y, x_1^{p^2}, x_2^p, y]^{-4}.\end{aligned}$$

From the previous lemmas we know that, of those elements listed of weight 4, the only possible non-trivial elements are $[y, x_1^p, x_2^p, y]$ and $[y, x_2^p, x_2^p, x_2^p]$. We thus have an equation of the form

$$\begin{aligned}[x_1^{p^2}, y, x_2^p, y, x_3] \quad = \quad &(x_1^{p^2})^{\alpha_1}(x_2^p)^{\alpha_2}y^{\alpha_3}[x_2^p, x_1^{p^2}]^{\alpha_4}[y, x_1^{p^2}]^{\alpha_5}[y, x_2^p]^{\alpha_6}\\ &[y, x_1^{p^2}, x_2^p]^{\alpha_7}[y, x_1^{p^2}, y]^{\alpha_8}[y, x_2^p, y]^{\alpha_9}[y, x_2^p, x_2^p]^{\alpha}\\ &([y, x_1^{p^2}, x_2^p][y, x_2^p, x_1^{p^2}])^{\beta}[y, x_2^p, x_2^p, x_2^p]^{\gamma}[y, x_1^{p^2}, x_2^p, y]^{\tau}C.\end{aligned}$$

Where C is a product of commutators which all lie in $\gamma_3(H)$ and have two occurrences of either x_1 or x_2 and thus lie in the second centre of G.

Before continuing we introduce some notation. For elements $g, g_1, \ldots, g_n \in G$ we define the commutator of g and $(g_1, \ldots, g_n)$ as the element $[g, g_1, \ldots, g_n]$. Now we work with the equation above. Taking commutators on both sides with (y, x_2, y, x_3), (y, x_1, y, x_3) and (x_1, x_2, y, x_3) gives

$$1 = [x_1, y, x_2, y, x_3]^{p^2\alpha_1} = [x_2, y, x_1, y, x_3]^{p\alpha_2} = [y, x_1, x_2, y, x_3]^{\alpha_3},$$

which implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Similarly, taking commutators on both sides with (y, y, x_3), (x_2, y, x_3) and (y, x_1, x_3) gives that $\alpha_4 = \alpha_5 = \alpha_6 = 0$. We continue like this, taking next commutators on both sides with (y, x_3), (x_2, x_3) and (x_1, x_3) which gives $\alpha_7 = \alpha_8 = \alpha_9 = 0$. We have thus shown that for all $p \in \mathcal{P}$ there is modulo G^{p^4} an equation of the form

$$[x_1^{p^2}, y, x_2^p, y, x_3] \equiv [y, x_2^p, x_2^p]^{\alpha_p}([y, x_1^{p^2}, x_2^p][y, x_2^p, x_1^{p^2}])^{\beta_p}[y, x_2^p, x_2^p, x_2^p]^{\gamma_p}[y, x_1^{p^2}, x_2^p, y]^{\tau_p}.$$

We next show that, for all but finitely many primes p in $\mathcal{P}$, p divides τ_p. Otherwise there would be an infinite subset $\mathcal{P}_1$ of $\mathcal{P}$ such that τ_p is coprime with p for all $p \in \mathcal{P}_1$. From the previous lemmas we know that $[y, x_2^p, x_2^p]$, $[y, x_1^{p^2}, x_2^p][y, x_2^p, x_1^{p^2}]$ and $[y, x_2^p, x_2^p, x_2^p]$ commute with x_3, thus taking commutator on both sides with x_3 gives that $[y, x_1, x_2, y, x_3]^{p^3}$ is in G^{p^4} for all $p \in \mathcal{P}_1$. But then we get the contradiction that

$$[x_1, y, x_2, y, x_3] \in \bigcap_{p \in \mathcal{P}_1} G^p = \{1\}.$$

Without loss of generality we can thus assume that p divides τ_p for all $p \in \mathcal{P}$. So for each $p \in \mathcal{P}$ we have the equation

$$[x_1^{p^2}, y, x_2^p, y, x_3] \equiv_{p^4} [y, x_2^p, x_2^p]^{\alpha_p}([y, x_1^{p^2}, x_2^p][y, x_2^p, x_1^{p^2}])^{\beta_p}[y, x_2^p, x_2^p, x_2^p]^{\gamma_p}. \quad (3)$$

We next show that for all but finitely many primes in $\mathcal{P}$, we have that β_p is coprime to p. To see this we argue by contradiction and suppose this is not the case. Then there is a infinite subset $\mathcal{P}_1$ of $\mathcal{P}$ such that we have an equation of the form

$$[x_1^{p^2}, y, x_2^p, y, x_3] \equiv_{p^4} [y, x_2^p, x_2^p]^{\alpha_p}[y, x_2^p, x_2^p, x_2^p]^{\gamma_p}, \quad (4)$$

for all $p \in \mathcal{P}_1$ There are then two possibilities. For infinitely many of the primes $p \in \mathcal{P}_1$, we have that p^2 does not divide α_p and taking the commutator with x_2 on both sides gives that $[y, x_2, x_2, x_2]^{p^3} \equiv_{p^4} 1$ for all those primes. Otherwise we have the equation

$$[x_1^{p^2}, y, x_2^p, y, x_3] \equiv_{p^4} [y, x_2^p, x_2^p, x_2^p]^{\gamma_p} \tag{5}$$

for infinitely many primes. Then for infinitely many primes we must have that γ_p is coprime to p since otherwise there is an infinite subset $\mathcal{P}_2$ of $\mathcal{P}_1$ such that $[x_1, y, x_2, y, x_3]^{p^3} \equiv_{p^4} 1$ for all $p \in \mathcal{P}_2$ and we have the contradiction that $[x_1, y, x_2, y, x_3] = 1$. Thus in both cases we have that $[y, x_2, x_2, x_2]^{p^3} \equiv_{p^4} 1$ for infinitely many primes, which shows as before that $[y, x_2, x_2, x_2] = 1$ and by our choice of x_2 we then have that $[y, x_2, x_2] = 1$. Thus our equation becomes

$$[x_1, y, x_2, y, x_3]^{p^3} \equiv_{p^4} 1$$

for infinitely many primes and as before we get the contradiction that $[x_1, y, x_2, y, x_3] = 1$.

This shows that we can assume without loss of generality that β_p is coprime to p for all p in $\mathcal{P}$. Taking the commutator with x_1 on both sides in (2) and using previous lemmas gives that

$$([y, x_1, x_2, x_1][y, x_2, x_1, x_1])^{p^3} \equiv_{p^4} 1$$

for all $p \in \mathcal{P}$ and this implies that $[y, x_1, x_2][y, x_2, x_1]$ commutes with x_1. But, as $[y, x_1, x_1, x_2] = 1$, it follows that

$$[y, x_1, x_1, x_2][y, x_1, x_2, x_1][y, x_2, x_1, x_1] = 1.$$

So we have established that (2) holds. This is true for any choice of generating set $\{y, x_1, x_2, x_3\}$ with $x_1, x_2, x_3 \in H$. Thus we can replace x_2 by x_1x_2, which gives

$$[y, x_1, x_1, x_1] = 1.$$

Replacing x_1 by x_1x_3 in (2) also gives

$$\prod_{\sigma \in S_3} [y, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = 1.$$

It is now clear that $[y, x, x, x] = 1$ for all $x \in H$ and all $y \in G \setminus H$. Replacing y by yh for some $h \in H$ we get also that

$$[h, x, x, x] = 1$$

for all $h \in H$. This finishes the proof of the lemma. □

Proposition 4.6 *G is nilpotent of class at most* 4.

Proof We argue in a similar manner as in the proof of Lemma 4.3. By Lemma 4.4 and 4.5 we can assume that G is generated by elements x_1, x_2, x_3, y with $x_1, x_2, x_3 \in H$ such that the elements $[y, x_1, x_1]$, $[y, x_2, x_2]$ and $[y, x_1, x_2][y, x_2, x_1]$ commute with y and x_3. By replacing x_2 by $x_2[y, x_3]$ if necessary we can furthermore assume that

$$[y, x_2, x_2, x_1] \neq 1. \tag{6}$$

Notice that as

$$[y, x_2, x_2, x_1][y, x_2, x_1, x_2][y, x_1, x_2, x_2] = 1$$

we also have that

$$[y, x_1, x_2, x_2][y, x_2, x_1, x_2] \neq 1. \tag{7}$$

Let $\mathcal{P}$ be a infinite set of primes greater than the class of G. The subgroup $\langle x_1^{p^2}, x_2^p, y\rangle$ is 4-subnormal in G. As in the proof of Lemma 4.3 we have an equation of the form

$$[x_1^{p^2}, y, x_2^p, y, x_3] \equiv_{p^4} [y, x_2^p, x_2^p]^{\alpha_p}([y, x_1^{p^2}, x_2^p][y, x_2^p, x_1^{p^2}])^{\beta_p}[y, x_2^p, x_2^p, x_2^p]^{\gamma_p}[y, x_1^{p^2}, x_2^p, y]^{\tau_p}.$$

But now we know from Lemma 4.3 that $[y, x_2^p, x_2^p, x_2^p] = 1$. Also, taking the commutator with x_3 on both sides, using the fact that $[y, x_2, x_2]$, and $[y, x_1, x_2][y, x_2, x_1]$ commute with x_3, we see that τ_p must be divisible by p for almost all the primes in $\mathcal{P}$. (Otherwise we would get the contradiction as in the proof of Lemma 4.3 that $[y, x_1, x_2, y, x_3] = 1$.) So we can assume that

$$[x_1^{p^2}, y, x_2^p, y, x_3] \equiv_{p^4} [y, x_2^p, x_2^p]^{\alpha_p}([y, x_1^{p^2}, x_2^p][y, x_2^p, x_1^{p^2}])^{\beta_p} \tag{8}$$

for all $p \in \mathcal{P}$. We use from Lemma 4.3 the fact that $[y, x_2^p, x_2^p, x_2^p] = 1$. Thus taking the commutator with x_2 on both sides in (8), gives

$$([y, x_1, x_2, x_2][y, x_2, x_1, x_2])^{p^3\beta_p} \equiv_{p^4} 1.$$

But then we must have that for all but finitely many β_p, we have that p divides β_p. Otherwise

$$[y, x_1, x_2, x_2][y, x_2, x_1, x_2] = 1,$$

which contradicts (7). Thus we can assume that

$$[x_1^{p^2}, y, x_2^p, y, x_3] \equiv_{p^4} [y, x_2^p, x_2^p]^{\alpha_p} \tag{9}$$

for all $p \in \mathcal{P}$. Taking the commutator with x_1 on both sides in (9) gives that

$$([y, x_2, x_2, x_1]^{p\alpha_p p^2} \equiv_{p^4} 1$$

for all $p \in \mathcal{P}$. Again it follows that for almost all primes p, we have that p^2 divides α_p, since otherwise we would get $[y, x_2, x_2, x_1] = 1$ which contradicts (6). Thus we can assume that

$$[x_1, y, x_2, y, x_3]^{p^3} \equiv_{p^4} 1 \tag{10}$$

for all $p \in \mathcal{P}$. Hence

$$[x_1, y, x_2, y, x_3] \in \bigcap_{p \in \mathcal{P}} G^p = \{1\}.$$

This gives the final contradiction! Hence every torsion-free 4-generator group with all subgroups 4-subnormal is nilpotent of class at most 4. □

We can now prove our theorem.

Theorem 4.7 *Let G be a torsion-free group with all subgroups 4-subnormal. Then G is nilpotent of class at most 4.*

Proof By Proposition 4.6 we have that G satisfies the law

$$[y, x, x, z, t] = 1.$$

Hence $G/Z_2(G)$ is a torsion-free 2-Engel group. But it is well known that such a group is nilpotent of class at most 2. Hence G is nilpotent of class at most 4. □

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