On right n-Engel subgroups

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Motivated by three strong structure results on n-Engel groups we state and prove three analogous results on normal subgroups consisting of right n-Engel elements.

1 Introduction

Let n be a non-negative integer. The n-Engel word $[y,_n x]$ is defined recursively by $[y,_0 x] = y$ and $[y,_{n+1} x] = [[y,_n x], x]$. Recall that an element a in a group G is said to be right n-Engel if $[a,_n g] = 1$ for all $g \in G$ and that the group G is said to be n-Engel if every element $a \in G$ is right n-Engel. Clearly for every element a in the $(n+1)$th term of the upper central series, $Z_n(G)$, we have that all the elements in $\langle a \rangle^G$ are right n-Engel. It is conversely not true in general that right n-Engel elements need to be in the hyper centre. Take for example the standard wreath product $C_2 \,\mathrm{wr}\, C_2^\infty$, of the cyclic group of order 2 with the infinite countable direct product of groups of order 2. This group is a 3-Engel group with a trivial centre. For a number of classes of groups we do however have that the right Engel elements belong to the hyper centre. For example this is true for finite groups [2] and finitely generated solvable groups [3].

Our work in this paper is motivated by the following three results on the structure of n-Engel groups.

Structure Theorem 1 *(Wilson [10]) Let G be a d-generator residually nilpotent n-Engel group. Then G is nilpotent of (d, n)-bounded class.*

Structure Theorem 2 *(Zel'manov [12]) Let G be a torsion free locally nilpotent n-Engel group. Then G is nilpotent of n-bounded class.*

Structure Theorem 3 *(Crosby & Traustason [5]) Let G be a locally nilpotent n-Engel group. There exist integers $f(n)$ and $c(n)$ such that $G^{f(n)} \leq Z_{c(n)}(G)$.*

Notice that the third theorem implies the 2nd as if G is torsion-free then we have that $G/Z_{c(n)}(G)$ is torsion-free and thus G nilpotent of class $c(n)$. It should also be mentioned that the third theorem was preceded by a result of Burns and Medvedev [4], who proved under the same assumptions that there exist integers $f(n), c(n)$ such that $G^{f(n)}$ is nilpotent of class $c(n)$.

As we said above, the normal subgroup $Z_n(G)$ consists of right n-Engel elements. We are interested in the reverse problem. Supposing that H is a normal subgroup of a group G consisting of right n-Engel elements, we are interested in conditions that force H to belong to some term of the upper central series.

Definition. (a) Let H be a subgroup of a group G. Then H is said to be a *right n-Engel subgroup* if all the elements of H are right n-Engel elements of G.

(b) Let H be a subgroup of a group G. We say that H is *residually hypercentral* if

$$\cap_{i=0}^{\infty}[H,_i G] = \{1\}.$$

We prove the following analogs of the structure theorems above.

Theorem 1 *Let G be a d-generator group and let H be a normal right n-Engel subgroup of G that is residually hypercentral. Then there exists an integer $m = m(d, n)$, only depending on d and n, such that $H \leq Z_m(G)$.*

Theorem 2 *Let G be a group with a normal right n-Engel group H that is torsion free and belongs to some term of the upper central series. Then there exists an integer $m = m(n)$, only depending on n, such that $H \leq Z_m(G)$.*

Theorem 3 *Let G be a group with a normal right n-Engel group H that is residually hypercentral. Then there exist integers $c(n), f(n)$, only depending on n, such that $H^{f(n)} \leq Z_{c(n)}(G)$.*

Theorem 3 answers question 3.81 in the survey on Engel groups given in [1]. It is not difficult to see that these theorems imply the structure theorems on Engel groups discussed above and that Theorem 3 implies Theorem 2. The way the proof works is however that we use Theorems 1 and 2 to prove Theorem 3. We also remark that a weaker version of Theorem 1 is due to Shalev [9] where the extra assumption is made that G is nilpotent. His result is stated without proof and as far as we know it is not to be found in the literature.

Our main tools are two well known deep results of Zel'manov on Lie rings [11,13,14]. We use the standard left normed convention for Lie products. Thus $a_1 a_2 \cdots a_{n+1} = (a_1 \cdots a_n) \cdot a_{n+1}$. An element a in a Lie ring L is right n-Engel if $ax^n = 0$ for all $x \in L$ and L is n-Engel if all $a \in L$ are right n-Engel. The theorems that we will use are.

Theorem Z 1 *Let $L = \langle x_1, \ldots, x_d \rangle$ be a d-generator Lie ring and let n, m be positive integers such that the following two properties hold:*

(1) $\sum_{\sigma \in S_n} uv_{\sigma(1)} \cdots v_{\sigma(n)} = 0$ for all $u, v_1, \ldots, v_n \in L$;
(2) $uv^m = 0$ for all $u \in L$ and all Lie products v in the generators.

Then L is nilpotent of (d, n, m)-bounded class.

Theorem Z 2 *Let L be a torsion free n-Engel Lie ring. Then L is nilpotent of n-bounded class.*

2 Proof of Theorem 1

Let $G = \langle f_1, f_2, \ldots, f_d \rangle$ be any d-generator group and let H be a normal right n-Engel subgroup that is residually hypercentral. We will apply Theorem Z1 and for this we will associate to the pair (H, G) a certain Lie ring L. We start by looking at the chains $(H_i)_{i=0}^{\infty}$ and $(G_i)_{i=1}^{\infty}$ where the latter is the lower central series, $G_i = \gamma_i(G)$, and where $H_i = [H,_i G]$, defined recursively by $[H,_0 G] = H$ and $[H,_{i+1} G] = [[H,_i G], G]$.

Let $A_i = H_i/H_{i+1}$ and $L_i = G_i/G_{i+1}$. We use these to form the abelian groups

$$A = A_0 \oplus A_1 \oplus \cdots, \qquad L_G = L_1 \oplus L_2 \oplus \cdots.$$

We consider A as an abelian Lie ring and L_G is the usual associated Lie ring of G. Thus if $x = aG_{i+1} \in L_i$ and $y = bG_{j+1} \in L_j$ then their Lie product is $xy = [a,b]G_{i+j+1} \in L_{i+j}$. This product is then extended linearly onto L_G.

There is a natural action from L_G on A that is defined as follows. Firstly if $u = hH_{i+1} \in A_i$ and $x = aG_{j+1} \in L_j$ then we let $ux = [h,a]H_{i+j+1} \in A_{i+j}$. Using the three subgroup lemma one can easily show by induction that $[H_i, G_j] \leq H_{i+j}$ and standard commutator arguments show that this implies that ux is well defined. We then extend this linearly to get an action from L_G on A. As A is abelian, we have for all $a, b \in A$ and $x \in L$ that $abx = 0 = (ax)b + a(bx)$ and thus the right multiplication by x is a derivation. Also straightforward calculations show that if $a \in A$ and $x, y \in L_G$ then $a(xy) = axy - ayx$. This action from L_G on A thus gives rise to a semidirect product $M = A \rtimes L_G$ and also $L = A \rtimes L_G/C_{L_G}(A)$. The aim is to show that $AL_G^m = \{0\}$ for some (d,n)-bounded integer m. This would imply that $A_m = A_0 L_1^m = 0$ and thus that $H_m = H_{m+1}$. As H is residually hypercentral it would then follow that $H_m = [H,_m G] = \{1\}$. So it remains to show that in M we have $AL_G^m = \{0\} = 0$ or equivalently that in L we have $A(L_G/C_{L_G}(A))^m = 0$ for some (d,n)-bounded integer m. We want to apply Theorem Z1 and so we first check that the two conditions there hold for L. We first turn to the first condition.

We need to introduce some notation. For any positive integer m we let $C_m = \{1, 2, \ldots, m\}$, $\mathcal{P}(C_m)$ be the powerset of C_m, and

$$\mathcal{R}_m = \{(S,T) \in \mathcal{P}(C_m) \times \mathcal{P}(C_m) : S \cup T = \{1, \ldots, m\} \text{ and } S \cap T = \emptyset\}.$$

We will use the following well known formula that holds in any Lie ring and is easily proved by induction.

$$b(yu_1 \cdots u_m) = \sum_{(S,T) \in \mathcal{R}_m} (-1)^{|T|} b\bar{u}_T y u_S$$

where, if $S = \{i_1, \ldots, i_r\}$, $i_1 < i_2 < \ldots < i_r$, and $T = \{j_1, \ldots, j_k\}$, $j_1 < j_2 < \ldots < j_k$, then $b\bar{u}_T y u_S = bu_{j_k} u_{j_{k-1}} \cdots u_{j_1} y u_{i_1} \cdots u_{i_r}$. For $\sigma \in S_m$ we also use $(b\bar{u}_T y u_S)^\sigma$ for

$$bu_{\sigma(j_k)} u_{\sigma(j_{k-1})} \cdots u_{\sigma(j_1)} y u_{\sigma(i_1)} \cdots u_{\sigma(i_r)}.$$

We will furthermore use the following useful notation If $u, v_1, \ldots, v_m$ are elements in a Lie algebra then

$$u\{v_1, \ldots, v_m\} = \sum_{\sigma \in S_m} uv_{\sigma(1)} \cdots v_{\sigma(m)}.$$

Lemma 1 *Let $a = hH_{i+1} \in A_i, x = gG_{j+1} \in L_j$ and $x_i = g_iG_{\beta(i)+1} \in L_{\beta(i)}$ for $i = 1, \ldots, 3n-2$. Then*

(a) $a\{x_1, \ldots, x_n\} = 0$;
(b) $x\{x_1, \ldots, x_{2n-1}\} \in C_{L_G}(A)$;
(c) $x\{a, x_1, \ldots, x_{3n-2}\} = 0$.

Proof. As h is a right n-Engel element we have $[h,_n y] = 1$ for all $y \in G$. Take arbitrary elements $y_1, y_2, \ldots, y_n \in G$ we have that $[h,_n y_1y_2 \cdots y_n] = 1$. Expanding this commutator and using Hall's collection process we see that

$$\prod_{\sigma \in S_n} [h, y_{\sigma(1)}, \ldots, y_{\sigma(n)}] = u$$

where u is a product of commutators of weight higher than $(1, \ldots, 1)$ in $h, y_1, \ldots, y_n$. In particular it follows that

$$\prod_{\sigma \in S_n} [h, g_{\sigma(1)}, \ldots, g_{\sigma(n)}] \in H_{i+\beta(1)+\cdots+\beta(n)+1}.$$

Hence we get in M that

$$\sum_{\sigma \in S_n} ax_{\sigma(1)} \cdots x_{\sigma(n)} = 0$$

which proves (a).

We now move to (b). It suffices to show that $a(x\{x_1 \cdots x_{2n-1}\}) = 0$. Using the extension formula above this is equal to

$$\sum_{(S,T) \in \mathcal{R}_{2n-1}} (-1)^{|T|} \sum_{\sigma \in S_{2n-1}} (a\bar{x}_T x x_S)^\sigma.$$

As for each pair (S, T) we either have $|S| \geq n$ or $|T| \geq n$ it now follows from (a) that this sum is equal to 0. This proves (b).

For (c) notice that

$$x\{a, x_1, \ldots, x_{3n-2}\} = \sum_{r=0}^{3n-2} \sum_{\sigma \in S_{3n-2}} x x_{\sigma(1)} \cdots x_{\sigma(r)} a x_{\sigma(r+1)} \cdots x_{\sigma(3n-2)}$$

and as either $r \geq 2n-1$ or $3n-2-r \geq n$ it follows from (a) and (b) that the sum is trivial. □

As A is abelian every product in L with two occurrences of an element from A is trivial. It follows thus from Lemma 1 and multilinearity that L satisfies the identity

$$\sum_{\sigma \in S_{3n-1}} u v_{\sigma(1)} \cdots v_{\sigma(3n-1)} = 0. \tag{1}$$

We next take a fixed element $h \in H$ and the generators $f_1, \ldots, f_d$ of G. Consider the elements $b = h[H, G] \in A_0$ and $y_1 = f_1[G, G], \ldots, y_d = f_d[G, G] \in L_1$. Notice that $y_1, \ldots, y_d$ generate L_G. We consider the subalgebra N of L generated by b and $y_1 + C_{L_G}(A), \ldots, y_d + C_{L_G}(A)$. Now N satisfies the linearised Engel identity (1) and in order to apply Theorem Z1 we will show that for any Lie product v in the generators $b, y_1 + C_{L_G}(A), \ldots, y_d + C_{L_G}(A)$, we have that

$$uv^{2n-1} = 0 \tag{2}$$

for all $u \in N$. It follows then from Theorem Z1 that N is nilpotent of (d, n)-bounded class m. In particular we can conclude that

$$bx_1 x_2 \cdots x_m = 0$$

for all $x_1, \ldots, x_m \in L_G/C_{L_G}(A)$ and as $b \in A_0$ is arbitrary it follows that $A_m = A_0(L_G/C_{L_G}(A))^m = 0$. Hence $[H,_m G] = [H,_{m+1} G]$ and as H is residually hypercentral, it then follows that $[H,_m G] = 1$. To finish the proof of Theorem 1, it thus suffices to show that equation (2) holds in N. In fact we prove something stronger, namely that (2) holds for all $u \in L$. As A is an abelian ideal of L it is clear that this holds if v has more than 1 occurrence of b. Now suppose that v has exactly one occurrence of b. Then $uv^n = 0$ when $n \geq 2$ again for the reason that A is abelian. So in this case we can assume that $n = 1$. But then $H \leq Z(G)$ and thus $AL = \{0\}$ which implies that $uv \in LA = \{0\}$. So we can assume that v is a Lie product in the generators

$y_1+C_{L_G}(A), \cdots, y_d+C_{L_G}(A)$. Then $v = w+C_{L_G}(A)$ where $w \in L_j$ for some $j \geq 1$, say $w = gG_{j+1}$. We first prove that

$$av^n = 0 \tag{3}$$

for all $a \in A$. By linearity we can suppose that $a \in A_i$ for some $i \geq 0$, say $a = kH_{i+1}$. Since $k \in H_i$ is a right n-Engel element we have

$$av^n = aw^n = [k,_n g]H_{i+nj+1} = 1H_{i+nj+1} = 0.$$

It now only remains to show that $xv^{2n-1} = 0$ for all $x \in L_G/C_{L_G}(A)$ or equivalently that $xw^{2n-1} \in C_{L_G(A)}$ for all $x \in L_G$. But

$$a(xw^{2n-1}) = \sum_{r=0}^{2n-1}(-1)^r\binom{2n-1}{r}aw^r xw^{2n-1-r}$$

and for each r we have that either $r \geq n$ or $2n-1-r \geq n$. Thus it follows from (3) that $a(xw^{2n-1}) = 0$ for all $a \in A$. This finishes the proof of Theorem 1.

3 Proof of Theorem 2

Let G be a group with a torsion-free normal right n-Engel subgroup H that is contained in the lth term of the upper central series of G. We want to apply Theorem Z2 and in order to do this we will associate to the pair (H, G) a certain torsion-free Lie ring. We need then to modify the construction that we used in section 2 and replace the terms $[H,_i G]$ and $\gamma_j(G)$ by their isolators.

Definition. Let K be a subgroup of a group F the *isolator* of K in F is $\sqrt[F]{K} = \{t \in F :$ there exists a positive integer r such that $t^r \in K\}$.

It is well known that if F is nilpotent then $\sqrt[F]{K}$ is a subgroup of F and it is easy to see that this implies more generally that if $N \trianglelefteq F$ with F/N nilpotent and $N \leq K \leq F$, then $\sqrt[F]{K}$ is a subgroup of F. Let us consider the two series $(H_i)_{i=0}^{\infty}$ and $(G_i)_{i=1}^{\infty}$ where $H_i = \sqrt[H]{[H,_i G]}$ and $G_i = \sqrt[G]{\gamma_i}$. By what we have just said above, all these are subgroups of G. For the Lie ring construction we need the following lemma.

Lemma 2 *For all integers $i \geq 0$ and $j \geq 1$, we have $[H_i, G_j] \leq H_{i+j}$.*

Proof Let $h \in H_i$ and $g \in G_j$ and consider $a = hH_{i+j}, b = gH_{i+j}$ in G/H_{i+j}. It suffices to show that $R = \langle a, b\rangle$ is abelian. Let $S = \langle a\rangle^R$. Let r be a positive integer such that $h^r \in [H,_i G]$ and $g^r \in \gamma_j(G)$. Then $[h^r, g^r] \in [[H,_i G], \gamma_j(G)] \leq H_{i+j}$ and thus $[a^r, b^r] = 1$. It follows that $[a, b]^{r^2} \in [S,_2 R]$ and thus $[S, R]^{r^2} \leq [S,_2 R]$. Inductively it follows that $[S,_k R]^{r^2} \leq [S,_{k+1} R]$. As $[S,_l R] = 1$ we can conclude that

$$[S, R]^{r^{2(l-1)}} = \{1\}$$

As R is torsion-free it follows that $[S, R] = 1$ and thus $[a, b] = 1$. $\square$

As in Section 2, we let $A_i = H_i/H_{i+1}$, for $i \geq 0$, and $L_j = G_j/G_{j+1}$, for $j \geq 1$. We then form the abelian groups

$$A = A_0 \oplus A_1 \oplus \cdots, \ L_G = L_1 \oplus L_2 \oplus \cdots.$$

We consider A as an abelian Lie ring and L_G as the usual associated Lie ring of G with respect to the series $(G_j)_{j=1}^{\infty}$ (using the well known fact that $[G_i, G_j] \leq G_{i+j}$). Because of Lemma 2 we get semidirect products $M = A \rtimes L_G$ and $L = A \rtimes L_G/C_{L_G}(A)$. Notice that in the proof of Lemma 1, the fact that G was finitely generated was never used. Lemma 1 therefore also holds in the present setting and we get therefore as before that L satisfies the linearised n-Engel identity

$$\sum_{\sigma \in S_n} uv_{\sigma(1)}v_{\sigma(2)} \cdots v_{\sigma(n)} = 0.$$

In this setting L is however a torsion-free Lie ring and thus the linearised n-Engel identity is equivalent to the n-Engel identity. By Theorem Z2 it then follows that L is nilpotent of n-bounded class m. Hence, we have in particular that $A_m = A(L_G/C_{L_G}(A))^m = 0$ or $\sqrt[H]{[H,_m G]} = \sqrt[H]{[H,_{m+1} G]}$. As $[H,_l G] = 1$ it follows that $\sqrt[H]{[H,_m G]} = \sqrt[H]{\{1\}}$. As H is torsion-free we that $\sqrt[H]{1} = \{1\}$. It follows then that $[H,_m G] = 1$. This finishes the proof of Theorem 2.

4 Proof of Theorem 3

In this section we will use Theorem 1 and Theorem 2 to prove Theorem 3. It will be easily derived from the following Lemma.

Lemma 3 *Let G be a group with a normal right n-Engel subgroup H that is residually hypercentral. Then there exists a positive integer $l = l(n)$ and a non-negative integer $k = k(n)$ such that*

$$[h, g_1, \ldots, g_k]^l = 1$$

for all $h \in H$ and all $g_1, \ldots, g_k \in G$.

Proof Let $m = m(n)$ be as in Theorem 2. Consider the largest group $F = \langle y, x_1, \ldots, x_m \rangle$, subject to $\langle y \rangle^F$ consisting of right n-Engel elements of F. Let $K = \bigcap_{i=1}^{\infty} [\langle y \rangle^F,_i F]$. By Theorem 1, there is an n-bounded integer r such that $[\langle y \rangle^F,_r F] \leq K$. In particular, $\langle y \rangle^F K/K$ is nilpotent and so the torsion elements in $\langle y \rangle^F K/K$ form a normal subgroup, say T/K. Then $\langle y \rangle^F T/T$ is a torsion-free subgroup of F/T. This allows us to use Theorem 2 to conclude that $[\langle y \rangle^F,_m F] \leq T$ and thus there is some positive integer $l = l(n)$ such that

$$[y, x_1, \ldots, x_m]^l \in K.$$

Consider a homomorphism ϕ from F to G that maps y to h and x_i to g_i. Then $[h, g_1, \ldots, g_m]^l = \phi([y, x_1, \ldots, x_m]^l) \in \phi(K) \leq \bigcap_{i=1}^{\infty} [H,_i G] = \{1\}$. This finishes the proof. $\Box$

We will now derive Theorem 3 from Lemma 3. Assume that H and G are like in the assumptions of Lemma 3 and let $k = k(n), l(n)$ be as in Lemma 3. Let $h \in H$ and let $g_1, \ldots, g_k \in G$. Then let $E = \langle h, g_1, \ldots, g_k \rangle$ and $K = \langle h \rangle^G$. By Theorem 1 we have that $K \leq Z_c(E)$ for some n-bounded number c. We show by reverse induction on $r \in \{0, 1, \ldots, c-k\}$ that $h^{l^{c-k-r}} \in Z_{k+r}(E)$. As $h \in Z_c(E)$ this is clearly true for $r = c - k$. Now suppose that $0 \leq r < c - k$ and that the result holds for larger values of $r \in \{1, \ldots, c-k\}$. By induction hypothesis we then have that $h^{l^{c-k-r-1}} \in Z_{k+r+1}(E)$. But then we have for any $e_1, \ldots, e_{k+r} \in E$ that

$$\begin{aligned} [h^{l^{c-k-r}}, e_1, \ldots, e_{k+r}] &= [h^{l^{c-k-r-1}l}, e_1, \ldots, e_{k+r}] \\ &= [h^{l^{c-k-r-1}}, e_1, \ldots, e_{k+r}]^l \end{aligned}$$

which is 1 by Lemma 3. This shows that $h^{l^{c-k-r}} \in Z_{k+r}(E)$ that finishes the induction step. Thus the claim holds and in particular for $r = 0$ we get that

$$[h^{l^{c-k}}, g_1, \ldots, g_k] = 1.$$

As $h \in H$ and $g_1, \ldots, g_k \in G$ were arbitrary, this finishes the proof of Theorem 3.

5 Right 2-Engel subgroups

In this section we consider the simplest non-trivial case of right 2-Engel subgroups. First we determine the integers $f(2)$ and $c(2)$ of Theorem 3. Let G be any group with a normal subgroup right 2-Engel subgroup H. In [6,7] (see also [8] Theorem 7.13), it is shown that $[h, x, y, z]^2 = 1$ for all right 2-Engel elements h in G and all $x, y, z \in G$ and that $\langle h \rangle^G$ is an abelian right 2-Engel subgroup. Hence

$$[h^2, x, y, z] = 1$$

for all $h \in H$ and all $x, y, z \in G$. We show that this is the best result one can obtain and that $f(2) = 2$ and $c(2) = 3$. First we give an example that shows that the value of $c(2)$ cannot be smaller than 3.

Example 1. Let M be the multiplicative group generated by the real matrices X and Y, where,

$$X = \begin{pmatrix} 0 & -1 & -2 & -2 \\ 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix},$$

and let $N = \mathbb{R}^4$ considered as a group with respect to addition. Consider the semidirect product $G = N \rtimes M$, where M acts on N by matrix multiplication. Let $v = (v_1, v_2, v_3, v_4) \in N$ and $w = v_1 + v_2 + v_3 + v_3$. Then,

$$[v, X, Y] = (w, -w, -w, w) = -[v, Y, X].$$

Direct calculations show also that $[v, X, X] = [v, Y, Y] = 0$. Hence N is a right 2-Engel subgroup of G. However if v is chosen such that $w \neq 0$ then $[nv, X, Y] \neq 0$ and thus $nv \notin Z_2(G)$ for any integer $n > 0$.

Example 2. Consider the standard wreath product of the cyclic group of order 2 with the countable infinite direct product of groups of order 2, $G = C_2 \operatorname{wr} C_2^\infty = \prod_{g \in C_2^\infty} C_2^g \rtimes C_2^\infty$. Let $A = \prod_{g \in C_2^\infty} C_2^g$ be the base group and $M = C_2^\infty$ be the group acting on it. For each $g, h \in A$ and $x \in M$, we have

$$[g, hx, hx] = [g, x, x] = g^{(-1+x)^2} = g^{1-2x+x^2} = 1.$$

This shows that A is a right 2-Engel subgroup of G. However, if we let $e_i \in M$ be the element with (-1) in coordinate i but 1 in all the other coordinates and we consider $(-1)^1 \in A$ then $[(-1)^1, e_1, e_2, \ldots, e_n] \neq 1$ for all positive integers n. Hence A is not contained in the hypercentre of G.

Having sorted out the integers $c(2)$ and $f(2)$ in Theorem 3, we determine the integer $m(d, 2)$ in Theorem 1. Of course $m(1, n) = 1$ for all $n \geq 1$. We have already mentioned above that if a is a right 2-Engel element of G then the $\langle a \rangle^G$ is abelian. As $[a, u, u] = 1$ and $[a, u, v] = [a, v, u]^{-1}$ for all $u, v \in G$, it follows that if $G = \langle g_1, \ldots, g_d \rangle$ then any commutator $[a, g_{i_1}, \ldots, g_{i_{d+1}}] = 1$. So $a \in Z_{d+1}(G)$. The following example shows that $m(d, 2) = d + 1$ when $d \geq 2$.

Example 3. Let $d \geq 2$. Let F be the free associative algebra over GF(2) with unity generated by $x_1, x_2, \ldots, x_d$. Let I be the ideal generated by all monomials of multiweight $(w_1, w_2, w_3, \ldots, w_d)$ in $x_1, \ldots, x_d$ where either one of w_1, w_2 is at least three or one of $w_3, w_4, \ldots, w_d$ is greater than one. Let J be the ideal generated by all elements of the form

$$x_i x_j x_{i_1} \cdots x_{i_r} - x_i x_j x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(r)}}$$

with $r \geq 2$, $i, j, i_1, \ldots, i_r \geq 1$ and $\sigma \in S_r$. Consider the ring $R = F/(I + J)$. Let $y_i = x_i + (I + J)$. Notice that any $y_i y_j y_{i_1} \cdots y_{i_r}$ is symmetric in the last r factors. Let $R_{(w_1, w_2, \ldots, w_d)}$ be the subspace generated by all monomials of multiweight $(w_1, w_2, \ldots w_d)$ in $y_1, y_2, \ldots, y_d$. Then

$$R = \sum_{(w_1, \ldots, w_d) \in \mathbb{N}^d} R_{(w_1, \ldots, w_d)}$$

where $R_{(w_1, \ldots, w_d)} = \{0\}$ whenever either one of w_1, w_2 is greater than 2 or one of $w_3, \ldots, w_d$ is greater than one. Let S be the ideal generated by all monomials of multiweight greater than $(0, \ldots, 0)$. Then $E = 1 + S$ is a finite 2-group with respect to multiplication. We can consider S as a Lie ring with respect to the bracket product $(s_1, s_2) = s_1 s_2 - s_2 s_1 = s_1 s_2 + s_2 s_1$. We use the well known fact that if s_1, s_2 are multihomogenous elements in $R_{(\alpha_1, \ldots, \alpha_d)}$ and $R_{(\beta_1, \ldots, \beta_d)}$ respectively then $[1 + s_1, 1 + s_2] = 1 + (s_1, s_2) + t$ where t is a sum of monomials of higher multiweight than $(\alpha_1 + \beta_1, \ldots, \alpha_d + \beta_d)$. Let $b_i = 1 + y_i$ and let G be the subgroup of E generated by $b_1, \ldots, b_d$. We consider the element $a = [b_1, b_2] = 1 + u$. Then calculations give that

$$u = (y_1, y_2) + (y_1^2 y_2 + y_1 y_2 y_1) + (y_2^2 y_1 + y_2 y_1 y_2) + (y_1 y_2^2 y_1 + y_1 y_2 y_1 y_2).$$

Notice that as we have quotiented out by the ideal J we have for all $s_1, s_2 \in S$

$$\begin{aligned} u(s_1s_2 + s_2s_1) &= 2us_1s_2 = 0 \\ s_1s_2u &= 2s_1s_2(y_1y_2 + y_1^2y_2 + y_1y_2^2 + y_1^2y_2^2) = 0. \end{aligned} \tag{4}$$

We show next that a is a right 2-Engel element in G. Notice first that as we have quotiented out by the ideal I we have

$$[a, b_{i_1}, \ldots, b_{i_r}] = 1 + (u, y_{i_1}, \ldots, y_{i_r}) \tag{5}$$

as any monomial in S of weight higher than $(1, 1, \ldots, 1)$ in $u, y_{i_1}, \ldots, y_{i_r}$ is zero. For the same reason we have that (5) is trivial if there is a repetition among $y_{i_1}, \ldots, y_{i_r}$. Using the equations (4) we see that

$$(u, y_{i_1}, \ldots, y_{i_r}) = u \prod_{j=1}^{r} y_{i_j} + \sum_{j=1}^{r} y_{i_j} u \prod_{k \neq j} y_{i_k}.$$

This last expression is symmetrical in $y_{i_1}, \ldots, y_{i_r}$. For a subset $I = \{j_1, \ldots, j_s\}$ of $\{1, \ldots, r\}$ and $b = 1 + v \in \langle a \rangle^G$, we will use the notation (v, y_I) for $(v, y_{i_{j_1}}, \ldots, y_{i_{j_s}})$. And similarly $[b, b_I]$ for the commutator expression $[b, b_{i_{j_1}}, \ldots, b_{i_{j_s}}]$.

Now take any $g = b_{i_1} \cdots b_{i_r} \in G$. From the discussion above we see that

$$\begin{aligned} [a, g, g] &= \prod_{\emptyset \neq I, J \subseteq \{1, \ldots, r\}} [a, b_I, b_J] \\ &= \prod_{\emptyset \neq I, J \subseteq \{1, \ldots, r\}} 1 + (u, y_I, y_J) \\ &\overset{(4)}{=} 1 + \sum_{\emptyset \neq I, J \subseteq \{1, \ldots, r\}} (u, y_I, y_J). \end{aligned}$$

Now $(u, y_I, y_J) = 0$ if $I \cap J \neq \emptyset$. As $(u, y_I, y_J) = (u, y_J, y_I)$, the remaining terms come in pairs. Since the characteristic of R is 2, it follows that $[a, g, g] = 1$. Thus a is a right 2-Engel element in G and thus $H = \langle a \rangle^G$ is a right 2-Engel subgroup. However

$$[a, b_1, b_2, b_3, \ldots, b_d] = 1 + (y_1, y_2, y_1, y_2, y_3, \ldots, y_d) \neq 1.$$

The reason why this commutator is non-trivial is that (for example) after expansion the coefficient of the basis element $y_1^2 y_2^2 y_3 \ldots y_d$ is 1. Thus H is not in $Z_d(G)$.

References

[1] A. Abdollahi, Engel elements in groups, Groups St Andrews 2009 in Bath, to appear.

[2] R. Baer, Engelsche Elemente Noetherscher Gruppen, *Math. Ann.* **133** (1957), 256-270.

[3] C. Brookes, Engel elements of soluble groups, *Bull. London Math. Soc.* **18** (1986), 7-10.

[4] R. G. Burns and Y. Medvedev, A note on Engel groups and local nilpotence, *J. Austral Math. Soc. (Series A)* **64** (1998), 92-100.

[5] P. G. Crosby and G. Traustason, A remark on the structure of n-Engel groups, Comm. Alg., to appear.

[6] W. P. Kappe, Die A-Norm einer Gruppe, *Illinois J. Math.* **5** (1961), 187-197.

[7] F. W. Levi, Groups in which the commutator operation satisfies certain algebraic conditions, J. Indian Math. Soc. **6** (1942), 87-97.

[8] D. J. Robinson, Finiteness conditions and generalized soluble groups, Parts II, Ergebinsse der Mathematik und ihrer Grenzgebiete, **62** (Springer-Verlag, New York 1972).

[9] A. Shalev, Combinatorial conditions in residually finite groups II, *J. Algebra* **157** (1993), 51-62.

[10] J. S. Wilson, Two-generator conditions for residually finite groups, *Bull. Lond. Math. Soc.* **23** (1991), 239-248.

[11] E. I. Zel'manov, Engel Lie-algebras, *Dokl. Akad. Nauk SSSR* **292** (1987), 265-268.

[12] E. I. Zel'manov, Some problems in the theory of groups and Lie algebras, *Math. Sb.* **180** (1989), 159-167.

[13] E. I. Zel'manov, The solution of the restricted Burnside problem for groups of odd exponent, *Math. USSR Izvestia* **36** (1991), 41-60.

[14] E. I. Zel'manov, The solution of the restricted Burnside problem for 2-groups, *Mat. Sb.* **182** (1991), 568-592.