On right n-Engel subgroups II

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In this sequel to “On right n-Engel subgroups” we add a new general structure result on right n-Engel subgroups. We also use one of the structure results to prove some results about right n-Engel subgroups in finite p-groups.

1 Introduction

Let n be a non-negative integer. The n-Engel word $[y,_n x]$ is defined recursively by $[y,_0 x] = y$ and $[y,_{n+1} x] = [[y,_n x], x]$. Recall that an element a in a group G is said to be right n-Engel if $[a,_n g] = 1$ for all $g \in G$ and that the group G is said to be n-Engel if every element $a \in G$ is right n-Engel. Clearly for every element a in the $(n+1)$th term of the upper central series, $Z_n(G)$, we have that all the elements in $\langle a \rangle^G$ are right n-Engel. It is conversely not true in general that right n-Engel elements need to be in the hypercentre. Take for example the standard wreath product $C_2 \,\mathrm{wr}\, C_2^{\infty}$, of the cyclic group of order 2 with the infinite countable direct product of groups of order 2. This group is a 3-Engel group with a trivial centre. For a number of classes of groups we do however have that the right Engel elements belong to the hyper centre. For example this is true for finite groups [2] and finitely generated solvable groups [3].

This paper is a sequel to [6]. Our work in that paper and the present one is motivated by the following results on the structure of n-Engel groups. For all four results we let G be a nilpotent n-Engel group.

Structure Theorem 1 *(Wilson [11]) If G is a d-generator group, then the nilpotence class of G is (d, n)-bounded.*

Structure Theorem 2 *(Zel'manov [12]) If G is torsion free, then the nilpotence class of G is n-bounded class.*

Structure Theorem 3 *(Crosby and Traustason [5]) There exist integers $f(n)$ and $c(n)$ such that $G^{f(n)} \leq Z_{c(n)}(G)$.*

Structure Theorem 4 *(Burns and Medvedev [4]) There exist integers $f(n)$ and $c(n)$ such that $\gamma_{c(n)}(G)^{f(n)} = \{1\}$.*

Notice that both the third and the fourth theorem imply the 2nd. It should also be mentioned that the third theorem was preceded by a result of Burns and Medvedev [4], who proved under the same assumptions that there exist integers $f(n), c(n)$ such that $G^{f(n)}$ is nilpotent of class $c(n)$.

We now move to the analogous statements for right Engel subgroups. First a definition.

Definition. Let H be a subgroup of a group G. Then H is said to be a *right n-Engel subgroup* if all the elements of H are right n-Engel elements of G.

As we said above, for any group G, the normal subgroup $Z_n(G)$ consists of right n-Engel elements. We are interested in the reverse problem. Suppose H is a normal right n-Engel subgroup of G and suppose that H belongs to some term of the upper central series. We refer to the smallest integer c such that $H \leq Z_c(G)$ as the upper central degree of H. Our analogous results for this situation are.

Theorem 1 *([6]) If G is a d-generator group, then the upper central degree of H is (d, n)-bounded.*

Theorem 2 *([6]) If H is torsion-free, then the upper central degree of H is n-bounded.*

Theorem 3 *([6]) There exist integers $c(n), f(n)$, only depending on n, such that $H^{f(n)} \leq Z_{c(n)}(G)$.*

Theorem 4 *There exist integers $c(n), f(n)$, only depending on n, such that $[H,_{c(n)} G]^{f(n)} = \{1\}$.*

It is not difficult to see that these theorems imply the structure theorems on Engel groups discussed above and that both Theorem 3 and Theorem 4 imply Theorem 2. The way the proof works is however that one uses Theorems 1 and 2 to prove Theorem 3 and Theorem 4.

Theorems 1,2 and 3 were proved in [6]. In Section 2 we will prove Theorem 4 and we will also use Theorem 3 to obtain some results concerning right n-Engel subgroups of finite p-groups. These results are analogues to the following results on the structure of n-Engel p-groups [1]. Let p be a prime and let $r = r(n,p)$ be the integer saisfying $p^{r-1} < n \leq p^r$.

Structure Theorem 5 *([1]) There exists a positive integer $s = s(n)$ such that any finite powerful n-Engel p-group is nilpotent of class at most s.*

Structure Theorem 6 *([1]) Let G be a finite n-Engel p-group.*

(a) If p is odd, then G^{p^r} is powerful.
(b) If $p = 2$, then $(G^{2^r})^2$ is powerful.

As corollary of Theorems 5 and 6 we then have

Structure Theorem 7 *([1]) Let G be a locally finite n-Engel p-group.*

(a) If p is odd, then G^{p^r} is nilpotent of n-bounded class.
(b) If $p = 2$ then $(G^{2^r})^2$ is nilpotent of n-bounded class.

The analogous results for right n-Engel subgroups are

Theorem 5 *There exists a positive integer $s(n)$ such that, for any finite p-group G and right n-Engel subgroup H which is powerfully embedded in G, $[H,_{s(n)} G] = 1$.*

Theorem 6 *Let G be a finite p-group and H be a normal right n-Engel subgroup of G.*

(a) If p is odd, H^{p^r} is powerfully embedded in G^{p^r}.
(b) If $p = 2$, $(H^{2^r})^2$ is powerfully embedded in $(G^{2^r})^2$.

Theorem 7 *Let G be a locally finite p-group and H be a normal right n-Engel subgroup of G. There exists an integer $s = s(n)$ such that the following hold.*

(a) If p is odd, $[H^{p^r},_s G^{p^r}] = 1$.
(b) If $p = 2$, $[(H^{2^r})^2,_s (G^{2^r})^2] = 1$.

Remark. The r given in Theorem 7 is close to being the best bound. Let t be the smallest positive integer such that H^{p^t} is upper central of n-bounded degree. Then $r \in \{r-1, r\}$ if p is odd and $t \in \{r-1, r, r+1\}$ if $p = 2$ [1].

2 Proofs

In this section we prove Theorems 4,5,6 and 7. We start with Theorem 4.

Proof of Theorem 4. By Lemma 3 [6], we know that there exist positive integers $m = m(n)$ and $l = l(n)$ such that, for any $h \in H$ and $g_1, \ldots, g_m \in G$, $[h, g_1, \ldots, g_m]^l = 1$. Fix $h \in H$ and $g_1, \ldots, g_{m+1} \in G$ and let $K = \langle [h, g_1, \ldots, g_m], g_{m+1} \rangle$. Then $K'/[K', K']$ is abelian of exponent dividing l. Let $k = [h, g_1, \ldots, g_m]$. By Theorem 1, there exists a positive integer $s = s(n)$ such that $[< k >^K,_s K] = \{1\}$. It follows in particular that K' is nilpotent of class at most s and thus K' has exponent dividing l^s. Let e be an integer such that $\binom{l^e}{k}$ is divisble by l^s for $k = 1, \ldots, s$ and set $f = l^e$. Let $g = a_1 \cdots a_t$ be any product of commutators of the form $[y, x_1, \ldots, x_m]$, with $y \in H$ and $x_1, \ldots, x_m \in G$. We prove, by induction on t, that $g^f = 1$. if $t = 1$ this is trivial. Now suppose $t \geq 2$ and that the inductive hypothesis holds for smaller values of t. Let $z = a_2 \cdots a_t$ and $a = a_1$. Applying the well known Hall-Petrescu identity, we have that

$$a^f z^f = (az)^f w_2^{\binom{f}{2}} w_3^{\binom{f}{3}} \cdots w_s^{\binom{f}{s}}$$

with $w_i \in \gamma_i(\langle a, z \rangle)$. By inductive hypothesis the left hand side is trivial and by definition of f every $w_i^{\binom{f}{i}}$ is also trivial, since each w_i is in $\langle a_1, z \rangle'$. Hence

$(a_1 \cdots a_t)^f = (az)^f = 1$. This finishes the inductive proof and we conclude that $g^f = 1$ for all $g \in [H,_m G]$. $\Box$

We now move to Theorem 5. Let G be a finite p-group. Recall that a group H is powerfully embedded in G if $[H, G] \leq H^p$ provided that p is odd. If $p = 2$, we require that $[H, G] \leq H^4$. For the proof we need to apply few well known properties of powerfully embedded subgroups. Firstly if H is powerfully embedded in G, then H^p is also powerfully embedded in G. Secondly, if H is powerfully embedded in G, then for each positive integer m we have that $H^m = \{h^m : h \in H\}$. The details can be found in [7] for example.

Proof of Theorem 5. As H is powerfully embedded in G we have that H^{p^k} is powerfully embedded for any positive integer k. Furthermore $H^{p^k} = \{h^{p^k} : h \in H\}$. We use these properties to show by induction on $k \geq 1$ that $[H,_k G] \leq H^{p^k}$. The induction basis is given by the assumption that H is powerfully embedded in G. Now suppose that $k \geq 2$ and that the result holds for smaller values of k. Then

$$[H,_k G] = [H,_{k-1} G, G] \leq [H^{p^{k-1}}, G] \leq (H^{p^{k-1}})^p = H^{p^k}.$$

This finishes the inductive proof. Let c and f be as in Theorem 3, and let $v = v(n)$ be the largest power of any prime that occurs in $f(n)$. Then $[H^{p^v},_c G] = \{1\}$ and thus $[H,_{v+c} G] \leq [H^{p^v},_c G] = \{1\}$. $\Box$.

We finally turn to Theorems 6 and 7. Let p be a fixed prime and n be a fixed positive integer. Let r be the integer satisfying $p^{r-1} < n \leq p^r$.

Proof of Theorem 6. (a) We can assume that $(H^{p^r})^p = \{1\}$ and then the aim is to show that $[H^{p^r}, G^{p^r}] = \{1\}$. Let $g \in G$ be arbitrary and set $V = H^{p^r}$. Since H is a finite n-Engel p-group, we have by Structure Theorem 6 that V is powerful and hence elementary abelian. Since H is a right n-Engel subgroup, for each $v \in V$, $[v,_n g] = 1$ and thus $[v,_{p^r} g] = 1$. Hence, in End (V), $0 = (-1 + g)^{p^r} = g^{p^r} - 1$. So $[v, g^{p^r}] = 1$ as required.

(b) Let $K = (H^{2^r})^2$. We may assume that $K^4 = 1$ and the aim is then to show that $[K, (G^{2^r})^2] = \{1\}$. Let $g \in G$ be arbitrary and set $V = K/K^2$. As H is a right n-Engel subgroup, we have by Structure Theorem 6 that K is powerful and hence abelian. It follows that V is an elementary abelian 2-group and

$[v,_{2^r} g] = 1$. We can conclude that in End (V), $0 = (-1+g)^{2^r} = t^{2^r} - 1$. This shows that $[K, G^{2^r}] \leq K^2$. Let $k \in K$, then

$$[k, (g^{2^r})^2] = [k, g^{2^r}]^2 [k, g^{2^r}, g^{2^r}]$$

and since $[k, g^{2^r}] \in K^2$ we have that $[k, g^{2^r}]^2 = 1$. It remains to see that $[k, g^{2^r}, g^{2^r}] = 1$ and for this it suffices to show that $[K^2, G^{2^r}] = \{1\}$. But as K^2 is right n-Engel we have as before that in End (K^2), $0 = (-1+g)^{2^r} = g^{2^r} - 1$, and so $[K^2, G^{2^r}] = \{1\}$. This finishes the proof. □

Proof of Theorem 7. (a) Let $s = s(n)$ be as in Theorem 5. Let $h \in H^{p^r}$ and $g_1, \ldots, g_s \in G^{p^r}$. Then $h, g_1, \ldots, g_s \in K^{p^r}$ for some finitely generated, and hence finite subgroup K of G. By Theorem 6, $(\langle h \rangle^K)^{p^r}$ is powerfully embedded in K^{p^r}. Hence, by Theorem 5, $[h, g_1, \ldots, g_s] = 1$. This finishes the proof of part (a). Part (b) is proved similarly. □.

3 Right 2-Engel subgroups

In this section we consider the simplest non-trivial case of right 2-Engel subgroups. First we determine the integers $f(2)$ and $c(2)$ of Theorem 4. Let G be any group with a normal subgroup right 2-Engel subgroup H. In [8,9] (see also [10] Theorem 7.13), it is shown that $[h, x, y, z]^2 = 1$ for all right 2-Engel elements h in G and all $x, y, z \in G$ and that $\langle h \rangle^G$ is an abelian right 2-Engel subgroup. We also have

$$1 = [h, x, xy, xy] = [h, x, y, xy] = [h, x, y, x]^y$$

and $[h, x^{-1}] = [h, x]^{-1}$. From this it is clear that any commutator $[h, u_1, \ldots, u_m]$ with $u_1, \ldots, u_m \in \{x, y\}$ and with a repeated entry of either x or y is trivial. In particular, such a commutator is trivial if $m \geq 3$. It follows that

$$1 = [h, xy, xy] = [h, x, y][h, y, x]$$

and $[h, y, x] = [h, x, y]^{-1}$. It follows that if $h \in H$ and $x_1, \ldots, x_m \in G$, then any commutator $[h, x_{i_1}, \ldots, x_{i_t}]$, with some x_i repeated, is trivial. Thus for $h \in H$ and $x, y, z \in G$, we have

$$[h, x, [y, z]] = [h, x, y, z][h, x, z, y]^{-1} = [h, x, y, z]^2 = 1.$$

It follows that $[H,G,G,G] = \{[h,x,y,z] : h \in H,\ x,y,z \in G\}$, is abelian and so $[H,G,G,G]^2 = 1$. Examples 1 and 2 in [6] then show that this is the best possible. Thus $c(2) = 3$ and $f(2) = 2$.

We next move to Theorems 5 and 7. Let $s(2,p)$ be the smallest positive integer such that $[H,_{s(2,p)} G] = \{1\}$ for all pairs (H,G), where G is a finite p-group and H is a right n-Engel subgroup that is powerfully embedded in G. Also, let $e(2,p)$ and $f(2,p)$ be integers such that for any pair pairs (H,G), where G is a locally finite p-group and H is a normal right 2-Engel subgroup of G, $[H^{p^{f(2,p)}},_{e(2,p)} G^{p^{f(2,p)}}] = \{1\}$. We want to find the value of $s(2,2)$ and the best possible values for $e(2,p)$ and $f(2,p)$.

First we deal with the case when p is odd. From [6], we know that $[a^2,x,y,z] = 1$ when a is a right 2-Engel element and $x,y,z \in G$. Hence $[a,x,y,z] = 1$ when p is odd. Hence $s(2,p) \leq 3$ and the best possible value for $f(2,p) = 0$. Next example shows that $s(2,p) = 3$ and that the best possible value for $e(2,p)$ is 3.

Example 1. For any given positive integer s we let $\mathbb{Z}_{p^s}$ be the congruence class of the integers modulo p^s. We let $N(s) = \mathbb{Z}_{p^s}^4$ and we let $M(s)$ be the subgroup of $\mathrm{GL}(4,\mathbb{Z}_{p^s}$, generated by

$$X(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ p & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -p & 1 \end{pmatrix}, \quad Y(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ p & 0 & 1 & 0 \\ 0 & p & 0 & 1 \end{pmatrix}.$$

Let $L(s) = N(s) \rtimes M(s)$ where $M(s)$ acts on $N(s)$ by multiplication on the left. Notice that if v_1, v_2, v_3, v_4 is the standard basis for the $\mathbb{Z}_{p^s}$-module $N(s)$ then

$$\begin{array}{llll} [v_1,X] = pv_2 & [v_2,X] = 0 & [v_3,X] = -pv_4 & [v_4,X] = 0 \\ [v_1,Y] = pv_3 & [v_2,Y] = pv_4 & [v_3,Y] = 0 & [v_4,Y] = 0. \end{array}$$

Notice that $L(s)$ is a finite p-group and that $N(s)$ is powerfully embedded right 2-Engel subgroup in $L(s)$. Then

$$[p^t v_1, X^{p^t}, Y^{p^t}] = p^{3t+2} v_4,$$

which is non trivial in $L(3t+3)$. This shows that the best possible value for $e(2,p)$ is 3. Since $[v_1,X,Y] = p^2 v_4$,that is non-trivial in $L(3)$, we also see that $c(2,p)=3$.

It remains to deal with $p=2$. Notice that Example 1 in fact shows also that the best value for $e(2,2)$ is 3 as we know that $[a^2,x^2,y^2,z^2]=1$ for $a\in H$ and $x,y,z\in G$. This also shows that the best value for $f(2,2)$ is at most 1. The group $N(2)$ is however not powerfully embedded in $L(2)$ so it remains to deal with $s(2,2)$ and $f(2,2)$. The first example shows that the best possible value of $f(2,2)$ is 1.

Example 2. For each $t\in\mathbb{N}$, we let

$$R(t)=C_2\,\mathrm{wr}\,C_2^t=\prod_{g\in C_2^t}\langle a^g\rangle\rtimes C_2^t.$$

The base group $B(t)=\prod_{g\in C_2^t}\langle a^g\rangle$ is then a normal right 2-Engel subgroup. However if $g_1,\ldots,g_t$ is the standard basis for C_2^t then

$$[a,g_1,\ldots,g_t]=a^{(-1+g_1)\cdots(-1+g_t)}\neq 1.$$

This shows that the best possible value of $f(2,2)$ is 1.

It now only remains to deal with $s(2,2)$. As we have remarked before we know that $[H^2,_3G]=\{1\}$ for any pair (H,G) where H is a normal right 2-Engel subgroup of G. Thus, if H is a powerfully embedded subgroup of a finite p-group G, then

$$[H,_4G]\leq[H^4,_3G]=\{1\}.$$

We now show that $s(2,2)=4$, by giving an example that shows that $s(2,2)>3$.

Example 3. The construction is similar to the one in Example 1. This time we let $N(s)=\mathbb{Z}_{2^s}^8$ and we consider the subgroup of $\mathrm{GL}\,(8,\mathbb{Z}_{2^s})$ generated by the following three matrices

$$X(s)=\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 4&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-4&0&1&0&0&0\\ 0&0&0&-4&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&4&1 \end{pmatrix}, Y(s)=\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 4&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&4&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&-4&0&0&1&0\\ 0&0&0&0&0&-4&0&1 \end{pmatrix}$$

and

$$Z(s)=\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 4&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&4&0&0&0&1&0&0\\ 0&0&4&0&0&0&1&0\\ 0&0&0&0&4&0&0&1 \end{pmatrix}.$$

We then let $L(s) = N(s) \rtimes M(s)$, where as before $M(s)$ acts on $N(s)$ by matrix multiplication on the left. The group $L(s)$ is then a finite 2-group where $N(s)$ is powerfully embedded in $L(s)$. If we let $v_1, \ldots, v_8$ be the standard basis for $N(s)$ then

$$\begin{array}{llll} [v_1,X]=4v_2, & [v_2,X]=0, & [v_3,X]=-4v_5, & [v_4,X]=-4v_6,\\ [v_5,X]=0, & [v_6,X]=0, & [v_7,X]=4v_8, & [v_8,X]=0,\\ [v_1,Y]=4v_3, & [v_2,Y]=4v_5, & [v_3,Y]=0, & [v_4,Y]=-4v_7,\\ [v_5,Y]=0, & [v_6,Y]=-4v_8, & [v_7,Y]=0, & [v_8,Y]=0\\ [v_1,Z]=4v_4, & [v_2,Z]=4v_6, & [v_3,Z]=4v_7, & [v_4,Z]=0,\\ [v_5,Z]=4v_8, & [v_6,Z]=0 & [v_7,Z]=0 & [v_8,Z]=0. \end{array}$$

Notice that for all $v \in \{v_1, \ldots, v_8\}$ that

$$[v,X,X]=[v,Y,Y]=[v,Z,Z]=0$$

and

$$[v,X,Y]=-[v,Y,X],\ [v,X,Z]=-[v,Z,X],\ [v,Y,Z]=-[v,Z,Y].$$

Notice also that

$$[v, X, [Y, Z]] = [v, Y, [Z, X]] = [v, Z, [X, Y]] = 2[v, X, Y, Z].$$

It follows that in $L(7)$, we have $[v, X, [Y, Z]] = [v, Y, [Z, X]] = [v, Z, [X, Y]] = 2[v, X, Y, Z] \in 2^7 N(s) = \{0\}$ which implies that $N(7)$ is a right 2-Engel subgroup of $L(7)$. However

$$[v_1, X, Y, Z] = 2^6 v_4 \neq 0.$$

This shows that $s(2, 2) = 4$.

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