On simple symplectic alternating algebras and their groups of automorphisms

Orazio Puglisi
Dipartimento di Matematica e Informatica “U. Dini”, Università di Firenze
Viale Morgagni 67A, I-50134 Firenze, Italy
puglisi@math.unifi.it

Gunnar Traustason
Department of Mathematical Sciences, University of Bath
Bath BA2 7AY, United Kingdom
gt223@bath.ac.uk

Abstract

Let N be any perfect symplectic alternating algebra. We show that N can be embedded into a larger simple alternating algebra S of dimension $7 \cdot (\dim N) + 6$ such that $\mathrm{Aut}\,(S) = \{\mathrm{id}\}$. This answers a question raised in [9]. Building on this result we show moreover that for any finite group G and characteristic c there exists a symplectic alternating algebra L over a field $\mathbb{F}$ of characteristic c such that $\mathrm{Aut}\,(L) = G$.

Keywords: non-associative algebras; simplectic; engel; simple; automorphisms.

1 Introduction

A symplectic alternating algebra (SAA) is a symplectic vector space L, whose associated alternating form is nondegenerate, that is furthermore equipped with a binary alternating product $\cdot : L \times L \to L$ with the extra requirement that

$$(x \cdot y, z) = (y \cdot z, x)$$

for all $x, y, z \in L$. This condition can be expressed equivalently by saying that $(u{\cdot}x, v) = (u, v{\cdot}x)$ for all $u, v, x \in L$ or in other words that multiplication from the right is self-adjoint with respect to the alternating form.

Symplectic alternating algebras originate from a study of powerful 2-Engel groups [4], [8] and there is in a 1-1 correspondence between a certain rich class of powerful 2-Engel 3-groups of exponent 27 and SAAs over the field $\mathrm{GF}(3)$.

Let $2n$ be a given even integer and $\mathbb{F}$ a fixed field. Let V be the symplectic vector space over the field $\mathbb{F}$ with a nondegenerate alternating form. Fix some basis $u_1, u_2, \ldots, u_{2n}$ for

V . An alternating product $\cdot$ that turns V into a symplectic alternating algebra is uniquely determined by the values

$$\mathcal{P}: \quad (u_i \cdot u_j, u_k), \quad 1 \leq i < j < k \leq 2n.$$

Let L be the resulting symplectic alternating algebra. We refer to the data above as a presentation for L with respect to the basis $u_1, \ldots, u_{2n}$.

Consider the symplectic group $\mathrm{Sp}(V)$. The map $V^3 \to \mathbb{F}, (u, v, w) \mapsto (u \cdot v, w)$ is an alternating ternary form and a moment's reflection should convince the reader that there is a 1-1 correspondence between symplectic alternating algebras of dimension $2n$ over $\mathbb{F}$ and orbits in $(\wedge^3 V)^*$ under the natural action of $\mathrm{Sp}(V)$. In particular a symplectic alternating algebra L has a trivial automorphism group if and only if the corresponding orbit in $(\wedge^3 V)^*$ is regular. From this it is not difficult to determine the growth of symplectic alternating algebras. If $m(n)$ is the number of symplectic alternating algebras over a finite field $\mathbb{F}$ then $m(n) = |\mathbb{F}|^{\frac{4n^3}{3}+O(n^2)}$ [7]. Because of the sheer growth, a general classification does not seem to be within reach although this has been done for small values of n . Thus it is not difficult to see that $m(0) = m(1) = 1$ and $m(2) = 2$. For higher dimensions the classification is already difficult. It is though known that when $\mathbb{F} = \mathrm{GF}(3)$ we have $m(3) = 31$ [9]. Some general structure theory is developed in [9] and [10]. In particular there is a dichotomy result that is an analog to a corresponding theorem for Lie algebras, namely that L either contains an abelian ideal or is a direct sum of simple symplectic alternating algebras. We also have that any symplectic algebra that is abelian-by-nilpotent must be nilpotent while this is not the case in general for solvable algebras. We should also mention here that the study of orbits in $\wedge^3 V$ is a classical problem that has been considered by a number of people (see for example [2], [5] and [6]).

For nilpotent symplectic alternating algebras there is a particularly rich structure theory with a number of beautiful results [7]. We can pick a basis $x_1, y_1, \ldots, x_n, y_n$ with the property that $(x_i, x_j) = (y_i, y_j) = 0$ and $(x_i, y_j) = \delta_{ij}$ for $1 \leq i \leq j \leq n$. We refer to a basis of this type as a standard basis. It turns out that for any nilpotent symplectic alternating algebra one can always choose a suitable standard basis such that the chain of subspaces

$$0 = I_0 < I_1 < \ldots < I_n < I_{n-1}^{\perp} < \cdots < I_0^{\perp} = L,$$

with $I_k = \mathbb{F}x_n + \cdots + \mathbb{F}x_{n-k+1}$ for $k \geq 1$, is a central chain of ideals. One can furthermore see from this that $x_i y_j = 0$ if $j \leq i$ and that $I_{n-1}^{\perp}$ is abelian. It follows that a number of the triple values (uv, w) are trivial. Listing only the values that are possibly non-zero it suffices to consider

$$\mathcal{P}: \quad (x_i y_j, y_k) = \alpha_{ijk}, \quad (y_i y_j, y_k) = \beta_{ijk}, \quad 1 \leq i < j < k \leq n$$

for some $\alpha_{ijk}, \beta_{ijk} \in \mathbb{F}$. Such a presentation is called a *nilpotent presentation.* Conversely any such presentation describes a nilpotent SAA. The algebras that are of maximal class turn out to have a rigid ideal structure. In particular when $2n \geq 10$ we can choose our chain of ideals above such that they are all characteristic and it turns out that $I_0, I_2, I_3, \ldots, I_{n-1}, I_{n-1}^{\perp}, I_{n-2}^{\perp}, \ldots, I_0^{\perp}$ are unique and equal to both the terms of the lower and upper central series (see [7] Theorems 3.1 and 3.2). This implies also that the automorphism group in this case is nilpotent-by-abelian, since it can be represented as a group of upper triangular matrices over $\mathbb{F}$. The algebras of maximal class can be identified easily from their nilpotent presentations. In fact, if $\mathcal{P}$ is any nilpotent presentation of L with respect to a standard basis $\{x_1, y_1, \ldots, x_n, y_n\}$, and $2n \geq 8$, we have that L is of maximal class if and only if $x_i y_{i+1} \neq 0$ for all $i = 2, \ldots, n-2$, and $x_1 y_2, y_1 y_2$ are linearly independent (see [7] Theorem 3.4).

From the general theory of nilpotent SAAs one can also determine their growth. Thus if $k(n)$ is the number of nilpotent SAAs of dimension $2n$ over a finite field $\mathbb{F}$ then $k(n) = |\mathbb{F}|^{n^3/3+O(n^2)}$. Again the growth is too large for a general classification to be feasable. The algebras of dimension $2n$ for $n \leq 4$ are classified in [7] over any field $\mathbb{F}$. The challenging classification of algebras of dimension 10 is dealth with in Sorkatti's PhD thesis and is again over any field $\mathbb{F}$.

The structure of symplectic alternating algebras is quite asymmetric in general and one of the questions raised in [9] was whether there exists SAAs with trivial automorphism group. In this paper we will answer this question positively. We will in fact do much more. For any perfect SAA N (i.e. where $N^2 = N$), we will construct a larger algebra S of dimension $7 \cdot (\dim N) + 6$ that is simple and whose automorphism group is trivial. Building on this we will then show that for any finite group G and characteristic c, there exists a symplectic alternating algebra L over a field $\mathbb{F}$ of characteristic c where $\mathrm{Aut}\,(L) = G$. Some preparation work regarding SAAs of maximal class is done in Section 2 and then in Section 3 we finish the construction of a SAA with a trivial automorphism group. Finally, in Section 4 we extend our work to show that any finite group can be realised as the automorphism group of a SAA.

We should also add that the question whether a symplectic alternating algebra can have a trivial automorphism group initially arose from an attempt to answer a question posed by A. Caranti (see problem 11.46 in [3]) whether there exists a finite 2-Engel 3-group G of class 3 such that $\mathrm{Aut}\,(G) = \mathrm{Aut}_c(G) \cdot \mathrm{Inn}\,(G)$ where $\mathrm{Aut}_c(G)$ is the group of central automorphisms of G. As we said above there is in a 1-1 correspondence between a certain rich class of powerful 2-Engel 3-groups of exponent 27 and SAAs over the field $\mathrm{GF}(3)$ and it was pointed out in [9] that a necessary condition for a group from this class to be a counter example to Caranti's question is that the corresponding symplectic alternating algebra would have a trivial automorphism group. Unfortunately further analysis reveals that the condition is not sufficient however and thus our examples to not provide directly such counter example. These examples may though provide a basis for constructing such examples. In any case, a counter example to Cartanti's question has now been found based on the use of GAP and MAGMA [1].

2 Nilpotent algebras

Although we are concerned with simple algebras and their automorphism group, a knowledge of certain nilpotent algebras will be crucial.

Let $2n \geq 20$ and choose a nondegenerate symplectic $\mathbb{F}$-vector space L of dimension $2n$.

Select a standard basis $\{x_1,\dots,x_n,y_1,\dots,y_n\}$ for L. The nilpotent presentation

$$\mathcal{P}:\quad \begin{array}{ll} (x_iy_{i+1},y_{i+3})=1 & \text{for all } i=1,\dots,n-3\\ (x_{n-2}y_{n-1},y_n)=1 & \\ (x_4y_7,y_i)=-1 & \text{for all } i=8,\dots,n\\ (x_5y_8,y_i)=-1 & \text{for all } i=9,\dots,n\\ (x_6y_9,y_i)=-1 & \text{for all } i=10,\dots,n\\ (y_1y_2,y_3)=1 & \end{array}$$

gives L the structure of a nilpotent SAA. As $x_2y_3,x_3y_4,\dots,x_{n-2}y_{n-1}$ are non-zero and as x_1y_2,y_1y_2 are linearly independent we know that L is of maximal class. For $k=1,\dots,n$, let $I_k=\langle x_n,x_{n-1},\cdots,x_{n-k+1}\rangle$. From the general theory of symplectic alternating algebras of maximal class, we know that I_k and $I_k^{\perp}$ are characteristic ideals for $k=2,\dots,n-1$. If I,J are two characteristic ideals of L, their product, although not always an ideal, is a characteristic subspace of L. Choose any k and select $a\in I_k, b\in I_{k-1}^{\perp}$ and $x\in L$. As $I_kL\le I_{k-1}$ and since $(ab,x)=(xa,b)=0$, it follows that $I_kI_{k-1}^{\perp}=0$. Therefore

$$I_k(I_{k-2}^{\perp})=I_k(I_{k-1}^{\perp}+\langle y_{n-k+2}\rangle)=I_k\langle y_{n-k+2}\rangle=\langle x_{n-k+1}y_{n-k+2}\rangle.$$

Notice that $I_4I_2^{\perp}=\langle x_{n-3}y_{n-2}\rangle=\langle x_n\rangle=I_1$ is thus a characteristic subspace contained in $Z(L)$ and thus a characteristic ideal. Also

$$I_n=\{x\in I_{n-1}^{\perp}: xI_{n-2}^{\perp}\le I_{n-3}\}$$

and thus it is also a characteristic ideal. It follows that the subspaces $\langle x_{n-k+1}y_{n-k+2}\rangle$ are characteristic for $k=3,\dots,n$. Hence $\langle x_k\rangle$ is characteristic for $k=4,\dots,n$. In order to make calculations in L easier, it is better to express the above presentation in terms of the product of L. Some relevant products among members of the basis are the following:

$$\begin{array}{ll} x_iy_{i+1}=x_{i+3} & \text{for all } i=1,\dots,n-3\\ x_{n-2}y_{n-1}=x_n & \\ x_4y_7=-(x_5+\sum_{i=8}^n x_i) & \\ x_5y_8=-(x_6+\sum_{i=9}^n x_i) & \\ x_6y_9=-(x_7+\sum_{i=10}^n x_i) & \\ y_1y_2=x_3. & \end{array}$$

Notice that the center of L is $Z(L)=\langle x_n,x_{n-1}\rangle$ and that $Z_2(L)$ is $\langle x_n,x_{n-1},x_{n-2}\rangle$. We have seen above that the subspaces $\langle x_k\rangle$ are characteristic for $k=4,\dots,n$. Thus, for each $\theta\in\mathrm{Aut}(L)$, ${x_k}^{\theta}=\lambda_kx_k$ for suitable $\lambda_k\in\mathbb{F}$ and for all $k\ge 4$. Thus, if we choose any subset $\Omega\subseteq\{i\mid 4\le i\le n\}$, the subspace $\langle x_i\mid i\in\Omega\rangle$ is characteristic.

It follows that the subspaces $U=\langle I_n,y_1,y_2,y_3,y_4,y_7\rangle$, $W=\langle I_n,y_1,y_2,y_3,y_4,y_5,y_8\rangle$ and $T=\langle I_n,y_1,y_2,y_3,y_4,y_5,y_6,y_9\rangle$ are characteristic, since they can be expressed as

$$U=\langle x_5,x_6,x_i\mid i\ge 8\rangle^{\perp}\qquad W=\langle x_6,x_7,x_i\mid i\ge 9\rangle^{\perp}\qquad T=\langle x_7,x_8,x_i\mid i\ge 10\rangle^{\perp}$$

and the product $\langle x_4\rangle U=\langle x_4y_7\rangle$ is characteristic as well. Similarly we get that $\langle x_5y_8\rangle$ and $\langle x_6y_9\rangle$ are characteristic because they can be expressed as products $\langle x_5\rangle W$ and $\langle x_6\rangle T$.

Thus there exists $\mu, \eta, \tau \in \mathbb{F}$ such that $(x_4y_7)^\theta = \mu x_4y_7 = -\mu(x_5 + \sum_{i=8}^n x_i)$, $(x_5y_8)^\theta = \eta x_5y_8 = -\eta(x_6 + \sum_{i=9}^n x_i)$ and $(x_6y_9)^\theta = \tau x_6y_9 = -\tau(x_7 + \sum_{i=10}^n x_i)$. But we also get

$$(x_4y_7)^\theta = (-x_5 - \sum_{i=8}^n x_i)^\theta = -({x_5}^\theta + \sum_{i=8}^n {x_i}^\theta) = -(\lambda_5x_5 + \sum_{i=8}^n \lambda_i x_i).$$

Matching the two expressions of $(x_4y_7)^\theta$, we see that $\mu = \lambda_5 = \lambda_i$ for all $i \geq 8$. The same argument can be applied to the images of x_5y_8 and x_6y_9 in order to get that $\eta = \lambda_6 = \lambda_i$ for all $i \geq 9$ and $\tau = \lambda_7 = \lambda_i$ for all $i \geq 10$. It follows that $\mu = \eta = \tau = \lambda_i$ for all $i \geq 5$. We indicate by λ this element of $\mathbb{F}$. Since y_6 belongs to $I_{n-6}^\perp = \langle y_6 \rangle + I_{n-5}^\perp$, there exist $\rho \in \mathbb{F}$ and $v \in I_{n-5}^\perp$ such that $y_6^\theta = \rho y_6 + v$. The automorphism θ is a symplectic map, hence

$$1 = (x_6, y_6) = (x_6^\theta, y_6^\theta) = (\lambda x_6, \rho y_6 + v) = \lambda\rho + (x_6, v) = \lambda\rho$$

because $x_6 \in I_{n-5}$, showing that $\rho = \lambda^{-1}$. Thus

$$\lambda x_5y_6 = (x_5y_6)^\theta = x_5^\theta y_6^\theta = (\lambda x_5)(\lambda^{-1}y_6) = x_5y_6$$

and hence λ must be 1. We have therefore proved the following fact.

Theorem 2.1 *Let $\mathbb{F}$ be any field, and $n \geq 10$ a natural number. There exists a nilpotent SAA over $\mathbb{F}$ of maximal class, with a standard basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ such that, for every $\theta \in \mathrm{Aut}(L)$ we have $x_i^\theta = x_i$ for all $i = 5, 6, \ldots, n$.*

3 Simple algebras

In this section we shall construct simple symplectic alternating algebras whose automorphism groups are trivial. In [9] examples of simple SAAs are discussed and we will use, in our construction, one of them. Fix a field $\mathbb{F}$ and let M be a symplectic $\mathbb{F}$-vector space of dimension 6 with a standard basis $\{x_i, y_i \mid i = 1, 2, 3\}$. We turn M into a SAA by introducing the non-trivial relations $(x_1x_2, x_3) = -1$, $(y_1y_2, y_3) = 1$. Notice that as a result we have the following non-trivial products in the basis elements:

$$\begin{array}{ll} x_2x_3 = y_1 & y_2y_3 = x_1 \\ x_3x_1 = y_2 & y_3y_1 = x_2 \\ x_1x_2 = y_3 & y_1y_2 = x_3 \end{array}$$

(and we also have $x_iy_j = 0$ for $1 \leq i, j \leq 3$). The algebra M is simple, as shown in [9] Section 5.1. Choose now any perfect SAA N of dimension $2n$ and single out a standard basis $\{a_i, b_i \mid i = 1, \ldots, n\}$. Finally let L be a nilpotent SAA of maximal class, of dimension $12n = 2m$. Notice that $n \geq 3$, since there is no perfect SAA of dimension less than 6 [9]. Therefore $m \geq 18$ and the results of the previous section can be used. It is then possible to choose L in such a way that, for a suitable standard basis $\{v_i, w_i \mid i = 1, \ldots, m\}$, we have $v_i^\theta = v_i$ for all $i = 5, \ldots, m$ and all $\theta \in \mathrm{Aut}\,(L)$. Let $S = L \boxplus M \boxplus N$ be the orthogonal sum of L, M, N as symplectic vector spaces. It will be helpful to set up some notation. Let

$$\begin{aligned}(x_1, x_2, x_3, y_1, y_2, y_3) &= (e_1, e_2, \ldots, e_6)\\ (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n) &= (f_1, f_2, \ldots, f_{2n}).\end{aligned}$$

Notice that we can extend the given products on L, M, N to a product on S such that the elements $e_i f_j$, $1 \leq i \leq 6$ and $1 \leq j \leq 2n$, are the same as the elements $v_1, \ldots, v_m, w_1, \ldots, w_m$ in any order we wish. This is easy to achieve, and to arrange for $e_i f_j = v_k$ we just need to add the relation $(e_i f_j, w_k) = 1$. Similarly if wanted instead the $e_i f_j = w_k$ then one would add the relation $(e_i f_j, v_k) = -1$. We now add the $2m$ neccessary relations where we have chosen the order to get the additional property that

$$\mathcal{R}: \begin{array}{lll} w_m = y_1 a_n = e_4 f_n, & w_{m-1} = y_2 b_n = e_5 f_{2n}, & w_{m-2} = y_3 b_n = e_6 f_{2n} \\ v_{m-i} = e_1 f_{i+1} & \text{for all } i = 0, \ldots, 2n-1 & \\ v_{m-2n-i} = e_{2+i} f_{2n} & \text{for all } i = 0, \ldots, 4. & \end{array}$$

Notice that $m - 2n - 4 = 4n - 4 \geq 8$ because $n \geq 3$. Hence every automorphim θ of L fixes $e_1 f_1, e_1 f_2, \ldots, e_1 f_{2n}, e_2 f_{2n}, \ldots, e_6 f_{2n}$ a fact we will make use of later.

Remark. The product thus defined has the following features: $MN = L,\ NL = M,\ LM = N,\ L^2 \leq L,\ M^2 = M$ and $N^2 = N$. We will make use of these properties later.

Lemma 3.1 *Let* $A = \{a \in S \mid \dim(aS) = 2\}$ *. Then* $\langle A \rangle = Z(L)$ *.*

Proof. The center of L is generated by $v_m = e_1 f_1, v_{m-1} = e_1 f_2$. Given e_i, f_j we have $(v_m e_i, f_j) = (e_i f_j, v_m)$ and this is not zero if and only if $e_i f_j = w_m = e_4 f_n$. In particular $(v_m, e_4 f_n) = 1$ from which it follows that $v_m e_4 \in N$ and $v_m f_n \in M$ are both non-zero. The two vectors $v_m e_4, v_m f_n$ are then linearly independent and $v_m S$ has dimension 2 . A similar argument shows that the same holds for $v_{m-1} = e_1 f_2$, so that $Z(L) \leq \langle A \rangle$. To prove the reverse inclusion we choose $0 \neq g \in A$ and write $g = v + x + a$ with $v \in L, x \in M, a \in N$. If $x \neq 0$, then $\dim(gN) = \dim(xN) = 2n > 2$ so that x must be trivial. Similarly $\dim(gM) = \dim(aM) = 6$ if $a \neq 0$ and therefore $g \in L$. Since g is not 0 there must be a pair (i, j) for which $(g, e_i f_j) \neq 0$. This implies that $g e_i$ and $g f_j$ are not 0 and, since they belong respectively to M and N , they are linearly independent. If g does not belong to $Z(L)$, then gL has dimension at least 1 and $gS = gL + gM + gN$ turns out to have dimension at least 3 . This proves that g is in $Z(L)$ and the claim holds. □

Lemma 3.2 *Let* $B = \{a \mid \dim(aS + Z(L)) = 4\}$ *. Then* $Z(L) + \langle B \rangle = Z_2(L)$ *.*

Proof. We start by checking that $Z_2(L) \subseteq Z(L) + \langle B \rangle$. To this extent it is enough to see that $v_{m-2} = e_1 f_3$ belongs to B . By the structure properties of nilpotent SAA of maximal class it follows that $v_{m-2} L$ has dimension 2 (see [7]). The subspace $v_{m-2} M$ is contained in N so that, in order to understand its dimension, we calculate the values $(v_{m-2} e_i, f_j) = (e_i f_j, v_{m-2})$. This element is 0 unless $e_i f_j = w_{m-2}$. Since this happens for exactly one pair $(i, j) = (6, 2n)$ we have that $v_{m-2} M$ has dimension 1 and for the same reason $v_{m-2} N$ has dimension 1 . Thus $v_{m-2} S = v_{m-2} L + v_{m-2} M + v_{m-2} N$ has dimension 4 . Pick $g \in B$ and write $g = v + x + a$, for some $v \in L, x \in M$ and $a \in N$. Arguing like in the proof of Lemma 3.1 we see that $x = 0 = a$ whence we need only to show that $g \in Z_2(L)$. If g does not belong to $Z_2(L)$, there exists $u \in L$ such that $gu \notin Z(L)$. Hence $gL + Z(L) > Z(L)$ and we have $\dim(gS + Z(L)) = \dim(gL + Z(L)) + \dim(gN) + \dim(gM) \geq 3 + 2 = 5$. Therefore $g \in Z_2(L)$ and the claim is proved. □

The previous lemmas show that both $Z(L)$ and $Z_2(L)$ are characteristic subspaces of S. For this reason the subspace $T = \{g \mid gZ_2(L) \subseteq Z(L)\}$ is characteristic as well. If we write an element $g \in T$ as $g = v + x + a$, $v \in L, x \in M, a \in N$, and recall that $ML \leq N, LN \leq M$, it becomes clear that $T = L + U$, where $U = \{g \in M + N \mid gZ_2(L) = 0\}$. We also notice that $U = (U \cap M) + (U \cap N)$, so that we may describe T by identifying the two subspaces $U \cap M, U \cap N$.

In order to do this we first observe that $(e_1 f_1, e_4 f_n) = 1$ but $(e_1 f_1, e_r f_s) = 0$ for any other pair. Similary $(e_1 f_2, e_5 f_{2n}) = 1$ and $(e_1 f_3, e_6 f_{2n}) = 1$ but $(e_1 f_2, e_r f_s) = 0$ and $(e_1 f_3, e_r f_s) = 0$ for any other pair (r, s). We use this information to determine first $U \cap M$. Let $e = \sum_k \lambda_k e_k$. Then $e(e_1 f_1) = \lambda_4 e_4(e_1 f_1)$ and, as $e_4(e_1 f_1) \neq 0$, this implies that $\lambda_4 = 0$. Similarly, considering $e(e_1 f_2)$ and $e(e_1 f_3)$, we see that $\lambda_5 = \lambda_6 = 0$ and $e \in \mathbb{F}x_1 + \mathbb{F}x_2 + \mathbb{F}x_3$. Clearly $\mathbb{F}x_1 + \mathbb{F}x_2 + \mathbb{F}x_3 \leq U \cap M$. Hence $U \cap M = \mathbb{F}x_1 + \mathbb{F}x_2 + \mathbb{F}x_3$. Similarly we see that $V \cap M = \sum_{i \neq n, 2n} \mathbb{F}f_i$ and $T = L + \mathbb{F}x_1 + \mathbb{F}x_2 + \mathbb{F}x_3 + \sum_{i \neq n, 2n} \mathbb{F}f_i$. Then (note that $(f_n, f_{2n}) = 1$) the subspace

$$T \cap T^{\perp} = T \cap (\mathbb{F}x_1 + \mathbb{F}x_2 + \mathbb{F}x_3 + \mathbb{F}f_n + \mathbb{F}f_{2n}) = \mathbb{F}x_1 + \mathbb{F}x_2 + \mathbb{F}x_3$$

is characteristic.

Lemma 3.3 *The subalgebras L, M, N are characteristic in S.*

Proof. Since the subspace $\mathbb{F}x_1 + \mathbb{F}x_2 + \mathbb{F}x_3$ is characteristic, the same holds for the algebra it generates, which is M. This gives that $L + N = M^{\perp}$ is a characteristic subspace and therefore each subspace $(L + N)^i$ is characteristic as well. Using the fact that N is perfect, an easy inductive argument shows that $(L + N)^i \cap M^{\perp} = L^i + N$ and, if c is the nilpotency class of L, we get $(L + N)^{c+1} \cap M^{\perp} = N$. Hence N is characteristic and the same holds for L, since $L = (M + N)^{\perp}$. □

We are now in a position to describe the automorphism group of S.

Proposition 3.4 *The automorphism group of S is trivial.*

Proof. Let $\theta \in \mathrm{Aut}(S)$ be any automorphism. By Lemma 3.3 we have, for all $i = 1, \ldots, 6$ and $j = 1, \ldots, 2n$, expressions

$$e_i^{\theta} = \sum_{k=1}^{6} \lambda_{ik} e_k \text{ and } f_j^{\theta} = \sum_{l=1}^{2n} \mu_{jl} f_l.$$

Moreover the elements $e_1 f_1, e_1 f_2, \ldots, e_1 f_{2n}, e_2 f_{2n}, \ldots e_6 f_{2n}$ are left fixed by every automorphism of L hence they are centralized by the elements of $\mathrm{Aut}(S)$ because L is a characteristic subalgebra of S. Applying θ to e_if_j we have

$$(e_if_j)^\theta = e_i^\theta f_j^\theta = \big(\sum_{k=1}^{6} \lambda_{ik}e_k\big)\big(\sum_{l=1}^{2n} \mu_{jl}f_l\big) = \sum_{k,l} \lambda_{ik}\mu_{jl}e_kf_l.$$

When $i=1$ $(e_1f_j)^\theta = e_1f_j$ for all j and comparing coefficients we see that $\lambda_{11}\mu_{jj}=1$ while $\lambda_{1k}=0=\mu_{jl}$ whenever $(k,l)\neq(1,j)$. Setting $\lambda=\lambda_{11}$, the action of θ can be described as $e_1^\theta=\lambda e_1$ and $f_j^\theta=\lambda^{-1}f_j$ for all $j=1,\ldots,2n$. Similarly, from $(e_if_{2n})^\theta=e_if_{2n}$, it follows $e_i^\theta=\lambda e_i$ for all $i=1,\ldots,6$. Applying θ to the relation $x_1x_2=y_3$, we find

$$\lambda y_3 = y_3^\theta = (x_1x_2)^\theta = x_1^\theta x_2^\theta = \lambda^2 x_1x_2 = \lambda^2 y_3$$

whence $\lambda=1$ and θ fixes $M+N$ elementwise. Since S is generated, as an algebra, by $N+M$, θ must be the identity and the theorem is proved. □

Another interesting fact about S is the following.

Proposition 3.5 *The algebra S is simple.*

Proof. Assume, by contradiction, that I is a proper non-trivial ideal of S. Since $I^\perp$ is an ideal, we may assume, without loss of generality, that $\dim(I)\geq\dim(S)/2=7n+3$. The subalgebra L has dimension $12n$, so that $\dim(I\cap L)=\dim I+\dim L-\dim(I+L)\geq 5n-3>2$. Clearly $I\cap L$ is an ideal of L and, by Theorem 3.2 of [7], it contains $Z(L)$. Thus $Z(L)S\subseteq I$. In particular the vector v_mf_n is in I and, using the equations $(v_mf_n,e_i)=(v_m,e_if_n)$, we readily see that $v_mf_n=e_1$. Once we have got $e_1=x_1\in I$, it is clear that each e_i belongs to I. From $M\leq I$ we deduce $L=MN\leq I$ and finally $N=LM\leq I$. Thus $I=S$ contrary to the assumption that I was proper. This contradiction proves that S is simple, as claimed. □

As an immediate consequence of 3.5 we have:

Theorem 3.6 *Let N be any perfect SAA over a field $\mathbb{F}$. Then N can be embedded into a larger simple alternating algebra S of dimension $7\cdot(\mathit{dim}\, N)+6$ such that $Aut(S)$ is trivial.*

Remark. It follows in particular that there are infinitely many simple SAAs over $\mathbb{F}$.

4 Prescribing the automorphism group

In this section we will see how, using the algebra S described in Section 3, one can construct simple algebras whose automorphism group is any finite group G. We stick to the notation introduced in the previous section.

Let $\mathbb{F}$ be any field and $\mathbb{K}$ a finite dimensional Galois extention of $\mathbb{F}$. The trace map $\mathrm{tr}:\mathbb{K}\longrightarrow\mathbb{F}$, has image in $\mathbb{F}$ and the $\mathbb{F}$-bilinear form on $\mathbb{K}$ defined by $(a,b)=\mathrm{tr}(ab)$ is nondegenerate and symmetric. Set $\overline{L}=L\otimes\mathbb{K}$, where the tensor product is taken over $\mathbb{F}$. For every pair of elements $(s \otimes x), (t \otimes y)$ define $(s \otimes x)(t \otimes y) = (st) \otimes (xy)$. This can be extended to a product on $\overline{L}$ satisfying $uv = -uv$ for all $u, v \in \overline{L}$. The algebra $\overline{L}$ can be endowed with an alternating bilinear form defined by

$$(s \otimes x, t \otimes y) = (s, t)\mathrm{tr}(xy)$$

on elements $(s \otimes x), (t \otimes y)$, and then extended by bilinearity. This form is clearly nondegenerate. Choose $r \otimes u, s \otimes x, t \otimes y$. We have

$$((r \otimes u)(s \otimes x), t \otimes y) = ((rs) \otimes (ux), t \otimes y) = (rs, t)\mathrm{tr}(uxy) = (st, r)\mathrm{tr}(xyu) = ((s \otimes x)(t \otimes y), (r \otimes u))$$

so that, for all $\alpha, \beta, \gamma \in A$, the relation $(\alpha\beta, \gamma) = (\beta\gamma, \alpha)$ holds and $\overline{L}$ is a SAA.
We need to gain information about the structure of $\overline{L}$ when L is one of the algebras of maximal class considered in Section 2. So let L be one of these algebras and $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ be a standard basis as given in Section 2. It is not difficult to see that $I_2 \otimes \mathbb{K}, \ldots, I_{n-1} \otimes \mathbb{K}, I_{n-1}^{\perp} \otimes \mathbb{K}, \ldots, I_2^{\perp} \otimes \mathbb{K}$ are the terms of the upper central series for $\overline{L}$ and thus characteristic. Also $(I_4 \otimes \mathbb{K}) \cdot (I_2^{\perp} \otimes \mathbb{K}) = I_1 \otimes \mathbb{K}$ and $I_n \otimes \mathbb{K} = \{x \in I_{n-1}^{\perp} \otimes \mathbb{K} : \ x(I_{n-2}^{\perp} \otimes \mathbb{K}) \leq I_{n-3}^{\perp} \otimes \mathbb{K}$ and hence $I_1 \otimes \mathbb{K}, I_n \otimes \mathbb{K}$ and $I_1^{\perp} \otimes \mathbb{K} = (I_1 \otimes \mathbb{K})^{\perp}$ are also characteric. The same argument as in Section 2 shows that $\langle x_4 \rangle \otimes \mathbb{K}, \ldots, \langle x_n \rangle \otimes \mathbb{K}$ are characteristic suspaces and that $x_5 \otimes 1, \ldots, x_n \otimes 1$ are fixed by all $\theta \in \mathrm{Aut}\,(\overline{L})$. Using this we see that:

Lemma 4.1 *Let θ any automorphism of $\overline{L}$. There exist $\tau \in \mathrm{Gal}(\mathbb{K} \mid \mathbb{F})$ such that, for every $k = 5, \ldots, n$ and for all $a \in \mathbb{K}$, we have*

$$(x_k \otimes a)^{\theta} = x_k \otimes a^{\tau}$$

Proof. We know that $x_4 \otimes \mathbb{K}, \ldots, x_n \otimes \mathbb{K}$ as well as the subspaces $I_1^{\perp} \otimes \mathbb{K}, \ldots, I_n^{\perp} \otimes \mathbb{K}$ are characteristic. It follows that exist $\alpha_k, \beta_k : \mathbb{K} \longrightarrow \mathbb{K}$ such that, for every $a, b \in \mathbb{K}$ one has:

$$\begin{aligned} (x_{n-k+1} \otimes a)^{\theta} &= x_{n-k+1} \otimes a^{\alpha_{n-k+1}} \\ (y_{n-k+2} \otimes b)^{\theta} &= y_{n-k+2} \otimes b^{\beta_{n-k+2}} + s_k(b) \\ (x_{n-k+4} \otimes (ab))^{\theta} &= x_{n-k+4} \otimes (ab)^{\sigma_{n-k+4}} \end{aligned}$$

for $4 \leq k \leq n-1$, where $s_k(b) \in I_{k-1}^{\perp} \otimes \mathbb{K}$. Using the fact that θ preserves products and that $(I_k \otimes \mathbb{K})(I_{k-1}^{\perp} \otimes \mathbb{K}) = 0$, we see at once that $(ab)^{\sigma_{n-k+4}} = a^{\alpha_{n-k+1}} b^{\beta_{n-k+2}}$. Using the fact that θ fixes $x_5 \otimes 1, \ldots x_n \otimes 1$, we see that by choosing a or b equal 1 we get $\sigma_{n-k+4} = \alpha_{n-k+1} = \beta_{n-k+2}$ and $(ab)^{\alpha_{n-k+1}} = a^{\alpha_{n-k+1}} b^{\alpha_{n-k+1}}$ for all $a, b \in \mathbb{K}$. On the other hand θ preserves sums, so for $v_k = x_{n-k+1}$ and $\alpha = \alpha_{n-k+1}$ we have

$$v_k \otimes (a+b)^{\alpha} = (v_k \otimes (a+b))^{\theta} = (v_k \otimes a)^{\theta} + (v_k \otimes b)^{\theta} = v_k \otimes a^{\alpha} + v_k \otimes b^{\alpha} = v_k \otimes (a^{\alpha} + b^{\alpha}).$$

This shows that $(a+b)^{\alpha_k} = a^{\alpha_k} + b^{\sigma_k}$ and $(ab)^{\alpha_k} = a^{\alpha_k} b^{\alpha_k}$ for $k = 5, \ldots, n$ and thus α_k is a field automorphism for $k = 5, \ldots, n$.

We now argue in a similar manner as we did in Section 2. As $x_1 \otimes \mathbb{K}, \ldots, x_n \otimes \mathbb{K}$ are characteristic, we have that $\sum_{i \in \Omega} x_i \otimes \mathbb{K}$ is charactistic for all $\Omega \subseteq \{4, \ldots, n\}$. It follows that the subspaces $U \otimes \mathbb{K}, W \otimes \mathbb{K}, T \otimes \mathbb{K}$ are characteristic where $U = \langle I_n, y_1, y_2, y_3, y_4, y_7 \rangle$, $W = \langle I_n, y_1, y_2, y_3, y_4, y_5, y_8 \rangle$ and $T = \langle I_n, y_1, y_2, y_3, y_4, y_5, y_6, y_9 \rangle$. This is the case since

$U \otimes \mathbb{K} = (\langle x_5, x_6, x_i : i \geq 8\rangle \otimes \mathbb{K})^{\perp}$, $W \otimes \mathbb{K} = (\langle x_6, x_7, x_i : i \geq 9\rangle \otimes \mathbb{K})^{\perp}$, $T \otimes \mathbb{K} = (\langle x_7, x_8, x_i : i \geq 10\rangle \otimes \mathbb{K})^{\perp}$. Thus $(x_4 \otimes \mathbb{K}) \cdot (U \otimes \mathbb{K}) = \langle x_4 y_7\rangle \otimes \mathbb{K}$ is characteristic as well. Similarly we see that $\langle x_5 y_8\rangle \otimes \mathbb{K}$ and $\langle x_6 y_9\rangle \otimes \mathbb{K}$ are characteristic because these can be expressed as the products $(x_5 \otimes \mathbb{K}) \cdot (W \otimes \mathbb{K})$ and $(x_6 \otimes \mathbb{K}) \cdot (T \otimes \mathbb{K})$.

Thus there exist functions $\mu, \eta, \tau : \mathbb{K} \to \mathbb{K}$ such that

$$\begin{aligned}
(x_4 y_7 \otimes a)^{\theta} &= x_4 y_7 \otimes a^{\mu} = -(x_5 + \textstyle\sum_{i=8}^{n} x_i) \otimes a^{\mu} \\
(x_5 y_8 \otimes a)^{\theta} &= x_5 y_8 \otimes a^{\eta} = -(x_6 + \textstyle\sum_{i=9}^{n} x_i) \otimes a^{\eta} \\
(x_6 y_9 \otimes a)^{\theta} &= x_6 y_9 \otimes a^{\tau} = -(x_7 + \textstyle\sum_{i=10}^{n} x_i) \otimes a^{\tau}.
\end{aligned}$$

But we also get

$$(x_4 y_7 \otimes a)^{\theta} = (-(x_5 + \sum_{i=8}^{n} x_i) \otimes a)^{\theta} = -(x_5 \otimes a^{\alpha_5} + \sum_{i=8}^{n} x_i \otimes a^{\alpha_i}).$$

Matching the two expressions of $(x_4 y_7 \otimes a)^{\theta}$, we see that $\mu = \alpha_5 = \alpha_i$ for $8 \leq i \leq n$. Similar arguement can be applied to the images of $x_5 y_8 \otimes a$ and $x_6 y_9 \otimes a$ to see that $\eta = \alpha_6 = \alpha_i$ for $9 \leq i \leq n$ and $\tau = \alpha_7 = \alpha_i$ for $10 \leq i \leq n$. It follows that $\mu = \eta = \tau = \alpha_i$ or $5 \leq i \leq n$. This finishes the proof. □

Choose a finite group G of order d and a field extension $\mathbb{K} \mid \mathbb{F}$ such that $G = \mathrm{Gal}(\mathbb{K} \mid \mathbb{F})$. This is indeed possible and a well known fact from classical Galois Theory. To see this consider any field $\mathbb{T}$. Let $t_1, \ldots, t_m$ be indeterminates and consider the field of fractions $\mathbb{K} = \mathbb{T}(t_1, \ldots, t_m)$. Let $e_1, \ldots, e_m$ be the elementary symmetric polynomials in $t_1, \ldots, t_m$ and consider the subfield $\mathbb{E} = \mathbb{T}(e_1, \ldots, e_m)$. Then the field extension $\mathbb{T}(t_1, \ldots, t_n) \mid \mathbb{E}$ is a Galois extension with Galois group S_m. Then using the Galois correspondence, we can for any $G \leq S_m$ find a field $\mathbb{F}$ such that $\mathbb{E} \subseteq \mathbb{F} \subseteq \mathbb{T}(t_1, \ldots, t_m)$ and where the extension $\mathbb{K} \mid \mathbb{F}$ has Galois group G. Notice that we can choose $\mathbb{K}$ here to be of any characteristic.

Let $n \geq 7$ be any integer and let S be the simple SAA algebra over $\mathbb{F}$ of dimension $12n + 6$ constructed in Section 3. Define $Q = S \otimes \mathbb{K}$. We will prove that $\mathrm{Aut}(Q) \simeq G$.

Our first aim is to see that the subalgebras $L \otimes \mathbb{K}$, $M \otimes \mathbb{K}$ and $N \otimes \mathbb{K}$ are characteristic in Q. To this end we follow the outline of Section 3.

Lemma 4.2 *Let $A = \{a \in Q \mid \dim(aQ) = 2d\}$. Then $\langle A\rangle = Z(L) \otimes \mathbb{K}$.*

Proof. The inclusion $\langle A\rangle \geq Z(L) \otimes \mathbb{K}$ is clear. To prove the reverse inclusion we choose $g \in A$ and write $g = v + x + a$ with $v \in L \otimes \mathbb{K}, x \in M \otimes \mathbb{K}, a \in N \otimes \mathbb{K}$. Assume $x \neq 0$. If this is the case we can write $x = \sum_{i=1}^{3} (x_i \otimes \lambda_i + y_i \otimes \mu_i)$ where, for at least one index i, either λ_i or μ_i is not 0. It is then easy to see that $\dim(xQ) \geq 2nd > 2d$ so that $\dim(gQ) > 2d$. A similar argument gives, when $a \neq 0$, that $\dim(xQ) \geq 6d > 2d$ proving that g belongs to $L \otimes \mathbb{K}$. If g is not in $Z(L) \otimes \mathbb{K}$, then $g(L \otimes \mathbb{K})$ has dimension at least d and the same is true for the subspaces $g(M \otimes \mathbb{K})$ and $g(N \otimes \mathbb{K})$. A look at the proof of Lemma 3.1 readily shows that the subspaces $g(L \otimes \mathbb{K})$, $g(M \otimes \mathbb{K})$, $g(N \otimes \mathbb{K})$ generates their direct sum, so that $\dim(gQ) \geq 3d$. Thus $g \in Z(L) \otimes K$ as claimed. □

Lemma 4.3 *Let* $B = \{a \mid \dim(aQ + Z(L) \otimes \mathbb{K}) = 4d\}$ *. Then* $Z(L) \otimes \mathbb{K} + \langle B \rangle = Z_2(L) \otimes \mathbb{K}$ *.*

Proof. If $B_0 = \{b \in S \mid \dim(bS + Z(L)) = 4\}$ we know, by Lemma 3.2, that

$$Z(L) \otimes \mathbb{K} + \langle B_0 \rangle \otimes \mathbb{K} = (Z(L) + \langle B_0 \rangle) \otimes \mathbb{K} = Z_2(L) \otimes \mathbb{K}$$

and, since $\dim((s \otimes \lambda)Q) = \dim(sQ)d$ for all $s \in S$, we see that the set $B_0 \otimes \mathbb{K}$ is contained in B. Conversely, if we choose $g \in B$, the same argument used in Lemma 4.2 shows that $g \in L \otimes \mathbb{K}$. At this stage the last part of the proof of Lemma 3.2 can be applied in order to see that, if $g \notin Z_2(S) \otimes \mathbb{K}$, then $\dim(gQ + Z(L) \otimes \mathbb{K}) \geq 5d$. □

The previous lemmas show that $Z(L) \otimes \mathbb{K}$ and $Z_2(L) \otimes \mathbb{K}$ are characteristic subspaces and, as we did in the previous section, we use this information to single out other relevant characteristic subspaces of Q.

As a first step we shall describe the subspace $\overline{T} = \{g \in Q \mid g(Z_2(L) \otimes \mathbb{K} \leq Z(L) \otimes \mathbb{K}\}$ which is clearly characteristic in Q. Of course $T \otimes \mathbb{K} \leq \overline{T}$ and the reverse inclusion is also easy to check.

At this stage we can mimic the proof in Section 3 and show that the subalgebras $\overline{L} = L \otimes \mathbb{K}$, $\overline{M} = M \otimes \mathbb{K}$ and $\overline{N} = N \otimes \mathbb{K}$ are characteristic in Q.

We are now in a position to prove the main result of this section.

Theorem 4.4 *The automorphism groups of* Q *is isomorphic to* G *.*

Proof. Once again we follow the same ideas of previous section and try to modify the proof of Theorem 3.4. The notation is the one we have set up in the previous section. Let θ be any automorphism of Q and choose $1 \leq i \leq 6$, $1 \leq j \leq n$ and $a, b \in \mathbb{K}$. Thus

$$\begin{array}{l} (e_i \otimes a)^\theta = \sum_{l=1}^{6} e_l \otimes \alpha_l(a, i) \\ (f_j \otimes b)^\theta = \sum_{k=1}^{n} f_k \otimes \beta_k(b, j) \end{array}$$

because $\overline{M} = M \otimes \mathbb{K}$ and $\overline{N} = N \otimes \mathbb{K}$ are characteristic subalgebras of Q. By Lemma 4.1 there exists $\sigma \in \mathrm{Gal}(\mathbb{K} \mid \mathbb{F})$ such that

$$((e_i f_j) \otimes c)^\theta = e_i f_j \otimes c^\sigma$$

when $(i, j) \in \{(1,1), (1,2), \ldots (1,2n), (2,2n), \ldots (6,2n)\}$. For such a pair (i, j) we have

$$e_i f_j \otimes (ab)^\sigma = (e_i f_j \otimes ab)^\theta = (e_i \otimes a)^\theta (f_j \otimes b)^\theta = \sum_{l,k} e_l f_k \otimes (\alpha_l(a, i)\beta_k(b, j))$$

Choose $i = 1$. It follows that $\alpha_1(a, 1)\beta_j(b, j) = (ab)^\sigma$ while $\alpha_l(a, i) = 0 = \beta_k(b, j)$ whenever $(l, k) \neq (1, j)$. In particular, choosing $a = b = 1$ we find $\beta_j(1, j) = \alpha_1(1, 1)^{-1} = c$ for all j. From this we see that $\alpha_1(a, 1) = ca^\sigma$ for all $a \in \mathbb{K}$. Choosing $j = 2n$ one obtains $\alpha_i(a, i)\beta_{2n}(b, 2n) = (ab)^\sigma$ and $\alpha_i(a, l) = 0$ if $l \neq i$. As before it follows that $\alpha_i(a, i) = ca^\sigma$. It is now possible to describe completely the action of θ on $\overline{M}$. For every e_i and all $a \in \mathbb{K}$ one has $(e_i \otimes a)^\theta = e_i \otimes ca^\sigma$. Thus, for any pair $a, b \in \mathbb{K}$

$$\begin{aligned} y_1 \otimes c(ab)^\sigma = (y_1 \otimes (ab))^\theta = ((x_2 x_3) \otimes (ab))^\theta = (x_2 \otimes a)^\theta (x_3 \otimes b)^\theta = (x_2 \otimes ca^\sigma)(x_3 \otimes cb^\sigma) = \\ = (x_2 x_3) \otimes c^2 (ab)^\sigma = y_1 \otimes c^2 (ab)^\sigma \end{aligned}$$

showing that $c = 1$. It readily follows that $\beta_j(b, j) = b^\sigma$ for all $b \in \mathbb{K}$ and all $j = 1, \ldots, 2n$ and we have

$$\begin{aligned}(e_i \otimes a)^\theta &= e_i \otimes a^\sigma \qquad &&\forall\, i = 1, \ldots, 6 \quad \forall a \in \mathbb{K}\\ (f_j \otimes b)^\theta &= f_j \otimes b^\sigma \qquad &&\forall j = 1, \ldots, 2n \quad \forall b \in \mathbb{K}.\end{aligned}$$

If v is any member of the basis $\mathcal{B} = \{e_1, \ldots, e_6, f_1, \ldots f_{2,n}, e_i f_j \mid 1 \le i \le 6\, 1 \le j \le 2n\}$ and $a \in \mathbb{K}$, then $(v \otimes a)^\theta = v \otimes a^\sigma$. On the other hand, for any given $\sigma \in \mathrm{Gal}(\mathbb{K} \mid \mathbb{F})$, let $\theta = \Theta(\sigma)$ be the map defined by setting $(v \otimes a)^\theta = v \otimes a^\sigma$ for $v \in \mathcal{B}$ and $a \in \mathbb{K}$ and extending by linearity. This map is easily seen to be an automorphism of Q and, since $\mathrm{tr}((ab)^\sigma) = \mathrm{tr}(ab)$, θ is also a symplectic map. The function $\Theta : \mathrm{Gal}(\mathbb{K} \mid \mathbb{F}) \longrightarrow \mathrm{Aut}(Q)$ sending σ to $\Theta(\sigma)$, is then a surjective homomorphism, and its kernel is clearly trivial. Therefore $\mathrm{Aut}(Q) \simeq \mathrm{Gal}(\mathbb{K} \mid \mathbb{F}) \simeq G$, proving the claim. □

References

[1] A. Abdollahi, A. Faghihi, S. A. Linton, E. A. O'Brien, Finite 3-groups of class 3 whose elements commute with their automorphic images, *Arch. Math. (Basel)*, **95** (2010), no. 1, 1-7.

[2] R. Y. Donaghi, On the geometry of Grassmanians, *Duke Math. J.*, **44** no.4 (1977), 795–837.

[3] The Kourovka Notebook, Unsolved problems in group theory, 18th edition, Novosibirsk, 2014.

[4] P. Moravec and G. Traustason, Powerful 2-Engel groups, *Comm. Algebra* **36** (2008), 3301–3323.

[5] J. A. Schouten, Klassifizierung der alternierenden Grössen dritten Grades in 7 Dimensionen, *Red. Circ. Math. Palermo*, **55** (1931), 137–156.

[6] C. Segre, Sui complessi lineari di piani nello spazio a ciqnue dimensioni, *Annali di Mat. pura ed applicata*, **27** (1917), 75–123.

[7] L. Sorkatti and G. Traustason, Nilpotent symplectic alternating algebras I, *J. Algebra* **423** (2015), 615–635.

[8] G. Traustason, Powerful 2-Engel groups II, *J. Algebra* **319** (2008), 3301–3323.

[9] G. Traustason, Symplectic alternating algebras, *Internat. J. Algebra Comput.* **18** (2009) no. 4, 719–757.

[10] A. Tortora, M. Tota and G. Traustason, Symplectic alternating nil-algebras, *J. Algebra* **357** (2012), 193–202.