Unipotent Automorphisms of Solvable Groups

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Abstract

Let G be a solvable group and H a solable subgroup of $\mathrm{Aut}\,(G)$ whose elements are n-unipotent. When H is finitely generated, we show that it stabilizes a finite series in G and conclude that H is nilpotent. If G furthermore has a characteristic series with torion-free factors the conclusion as above holds without the extra assumption that H is fintiely generated.

1 Introduction

If G is a group and a is an automorphism of G, we say that a is a *nil-automorphism* if, for every $g \in G$ there exists $n = n(g)$ such that $[g,_n a] = 1$ (here commutators are taken in the holomorph $G\mathrm{Aut}(G)$ and, as usual, the n-fold commutator $[g,_n a]$ is defined recursively by $[g,_0 a] = g$ and, for $n > 0$, $[g,_n a] = [[g,_{n-1} a], a]$). When n can be chosen independently from g, we say that a is *n-unipotent* or simply *unipotent* if the integer n is clear. The notion of nil and unipotent automorphism was defined by Plotkin in [9] and, in recent years, some papers have dealt with this topic.

Nil and unipotent automorphisms can be regarded as a natural extension of the concept of Engel element, since a nil-automorphism a is just a left-Engel element in $G\langle a\rangle$. Another way to look at nil-automorphisms, is to consider them as a generalization of unipotent automorphisms of vector spaces. For these reasons there are several natural questions that can be asked about nil-automorphisms, which are suggested by known facts about Engel groups or unipotent linear groups. Maybe the first question we may ask is whether a group of automorphisms whose elements are n-unipotent, needs to be locally nilpotent. This question is just the *automorphism* version of the well known problem about the local nilpotency of n-Engel groups. This problem is still open, although some interesting partial results have been obtained either for small values of n or for groups in some particular classes. When n is smaller than 4, the problem has positive solution. The cases $n = 3$ was treated by Heineken in [6] . The case $n = 4$ is considerably more involved and was solved in [7].

Engel groups in some special classes have been considered by several authors. In [1] Baer studied the subset of left Engel elements in groups satysfying Max, proving that this subset is actually the Fitting subgroup. The set of Engel elements (right and left) in solvable groups was later investigated by Gruenberg in [5]. Another remarkable result was obtained in [10] where the authors show that profinite Engel groups are locally nilpotent. This was later extended to compact Hausdorff groups by Medvedev in [8].

Recent results on unipotent action are contained in [2] and[4]. The first paper is concerned with groups of unipotent automorphisms of finitely generated residually finite and profinite groups. In this situation the authors prove that the group of automorphisms is locally nilpotent. Frati looks at automorphisms of solvable groups. When the group acted upon is abelian-by-polycyclic or has finite Prüfer rank, then Frati shows that a group of n-unipotent automorphisms is locally nilpotent. Both papers [2] and [4] rely heavily on results by Crosby and Traustason on normal right-Engel subgroups (see [3]). In turn, in their work, Crosby and Traustason use some deep results on Lie algebras due to Zelmanov.

In this paper we continue the study of unipotent automorphisms of solvable groups proving the following theorem.

Theorem *Let G be a solvable group and H a solvable subgroup of* $\mathrm{Aut}(G)$ *whose elements are n-unipotent.*

1. *If H is finitely generated then it stabilizes a finite series in G and it is therefore nilpotent.*
2. *If G has a characteristic series with torsion-free factors, then H stabilizes a finite series in G and is therefore nilpotent.*

2 The results

We start with two definitions.

Definition 2.1 *Let G be a group and H a normal subgroup of G. We say that H is a* normal right-n-Engel *subgroup if, for every $h \in H$ and $g \in G$, we have $[h,_n g] = 1$.*

For a group G, subgroup H of G and a non-negative integer n, we define recursively $[H,_n G]$ by $[H,_0 G] = H$ and $[H,_n G] = [[H,_{n-1} G], G]$ for $n > 0$.

Definition 2.2 *Let G be a group and H a subgroup of G. We say that H is* residually hypercentral *in G, if $\bigcap_{k\in\mathbb{N}}[H,_k G] = 1$.*

Lemma 2.3 *Let A be an abelian group and H a polycyclic subgroup of* $\mathrm{Aut}(A)$*. If H acts n-unipotently on A then H stabilizes a finite series in A and is, therefore, nilpotent.*

Proof. The proof will be accomplished by induction on the derived length of H. When H is abelian the claim follows from a result from [4].

So assume H has derived length at least 2. By inductive hypothesis the subgroup H' is nilpotent and stabilizes a finite series

$$0 = A_k < A_{k-1} < \cdots < A_1 < A_0 = A.$$

Since $B_i = [A,_i H'] \leq A_i$ for all i, there is no loss of generality if we assume that each A_i is an H-module. Thus H acts on each factor A_i/A_{i+1}, $i = 0, 1, \ldots, k-1$ and every element induces an n-unipotent automorphism. On the other hand H' centralizes all these quotients so that the group of automorphisms induced by H on each such factor is abelian. But then, as pointed out before, in each factor A_i/A_{i+1} there is a finite series stabilized by H/H', hence by H, and the claim follows. □

Lemma 2.4 *Let A be an abelian group and H a finitely generated solvable subgroup of* $\mathrm{Aut}(A)$. *If H acts n-unipotently on A then H stabilizes a finite series in A and is, therefore, nilpotent.*

Proof. We argue by induction on the derived length of H.

When H is abelian then it is polycyclic and the result follows from Lemma 2.3

Assume that the claim holds for groups of derived length at most $d-1$ and let H have derived length d. The subgroup H' is locally nilpotent and, for every finitely generated subgroup $F \leq H'$, we have that F stabilizes a finite series in A. Of course the length of such series may depend on F. However the group AF is nilpotent and A is a normal *right n-Engel subgroup* of AF and the nilpotency of AF implies that A is residually hypercentral. By [3] there exist integers $e(n), c(n)$ such that $[A^{e(n)},_{c(n)} F] = 1$. Since this holds for every finitely generated subgroup of H', we get

$$[A^{e(n)},_{c(n)} H'] = 1.$$

Thus H' stabilizes a finite series in $A^{e(n)}$ and we may assume that the elements of such series are H-submodules of $A^{e(n)}$. The same argument used in the previous lemma shows that such series can be refined to a series stabilized by H. To finish the proof it is enough to show that H stabilizes a finite series in $A/A^{e(n)}$. To simplify notation we can therefore assume that A has finite exponent e and use additive notation for A. If e is a prime, then A is a vector space over the finite field with e elements. Define $m = e^k$ such that $n \leq m$ and notice that, for all $a \in A$ and $h \in H$, we have.

$$0 = [a,_m h] = a(h-1)^m = a(h^m - 1).$$

Hence $h^m = 1$ and H has finite exponent. On the other hand H is finitely generated and solvable, thus it is finite. In particular H is polycyclic and, by lemma 2.3, it stabilizes a finite series in A. The general case follows easily, because A has a finite characteristic series, whose factors have prime exponent. □

Lemma 2.5 *Let A be a torsion-free abelian group and H a solvable subgroup of* $\mathrm{Aut}(A)$*. If H acts n-unipotently on A then H stabilizes a finite series in A and is, therefore, nilpotent.*

Proof. By lemma 2.4 the group H is locally nilpotent and, if F is any finitely generated subgroup of H, the group AF is nilpotent. Again A is a residually hypercentral right n-Engel subgroup of AF and, by [3], there exists an integer $c(n)$ such that $[A,_{c(n)} F] = 1$. Since this holds for all finitely generated subgroups of H, it follows that $[A,_{c(n)} H] = 1$ and the claim is proved. □

We are now in a position to prove our main result.

Theorem 2.6 *Let G be a solvable group and H a solvable subgroup of* $\mathrm{Aut}(G)$ *whose elements are n-unipotent.*

1. *If H is finitely generated then it stabilizes a finite series in G and it is therefore nilpotent.*
2. *If G has a characteristic series with torsion-free factors, then H stabilizes a finite series in G and is therefore nilpotent.*

Proof. Both statements are true when G is abelian, so we argue by induction on the derived length of G. Let $1 = G_0 < G_1 < \cdots < G_k = G$ be a characteristic series with abelian factors. The group H acts as a group of n-unipotent automorphisms on each factor G_{i+1}/G_i for all $i = 0, \ldots, k-1$. If H is finitely generated then, by lemma 2.4, it stabilizes a finite series in each G_{i+1}/G_i and this implies that G has a finite series stabilized by H. If all the factors G_{i+1}/G_i are torsion-free, then we can invoke lemma 2.5 to get the same conclusion without assuming that H is finitely generated. □

It is worth pointing out that, in general, the fact that a group of automorphisms acts n-unipotently, is not enough to ensure that it stablizes a finite series.

Example 2.7 *There exists a solvable group G and an abelian group H acting 3-unipotently on G, such that G has no finite series stabilized by H.*

Clearly in the example we are going to discuss, the group H will not be finitely generated and G will not have any series with torsion-free factors (indeed G is going to be a torsion group). Let H be a countable elementary abelian 2-group and A a cyclic group of order 2. Let $G = A\mathrm{wr}H$ the standard wreath product of A and H. The group H acts on G as a group of 3-unipotent automorphisms but it does not stabilize any finite series.

Example 2.8 *There exist a group G with a cyclic automorphism acting n-unipotently for some n, which does not stabilize any finite series of G.*

Let m be the smallest integer such that the Burnside variety $\mathcal{B}(2^m)$ of groups of exponent dividing 2^m is not locally finite. Choose any finitely generated infinite group G in $\mathcal{B}(2^m)$ and pick an involution $a \in G$. For every $g \in G$ the subgroup $D = \langle a, a^g \rangle$ is a dihedral group and $x = a^g a = [g, a]$ generates a normal subgroup of D. Moreover $x^a = x^{-1}$ and a straightforward calculation shows that, for every k, one has

$$[g,_k a] = [x,_{k-1} a] = x^{(-2)^{k-1}} = (a^g a)^{(-2)^{k-1}}.$$

For $k = m + 1$ we obtain

$$[g,_k a] = (a^g a)^{(-2)^{k-1}} = (a^g a)^{(-2)^m} = 1.$$

This means that a is an n-unipotent automorphism of G for $n = m + 1$. We claim that there exists at least one involution a that does not stabilize any finite series in G. By way of contradiction assume that this is not true. If u is any element of G, then $a = u^{2^{n-1}}$ is either 1 or an involution. If a is not trivial then it stabilizes a finite series

$$1 = G_r < G_{r-1} < \cdots < G_0 = G.$$

Therefore we have a decreasing chain

$$G \geq [G, \langle a \rangle] \geq [G,_2 \langle a \rangle] \geq \cdots \geq [G,_{r-1} \langle a \rangle] \geq [G,_r \langle a \rangle] = 1$$

from which we get the subnormal series

$$\langle a \rangle \trianglelefteq \langle a \rangle^{[G,_{r-1} \langle a \rangle]} \trianglelefteq \langle a \rangle^{[G,_{r-2} \langle a \rangle]} \trianglelefteq \cdots \trianglelefteq \langle a \rangle^{[G, \langle a \rangle]} \trianglelefteq \langle a \rangle^G.$$

Since every subnormal cyclic group is contained in $H(G)$, the Hirsch-Plotkin radical of G, we have shown that

$$G^{2^{n-1}} \leq H(G).$$

Therefore $G/H(G)$ has exponent dividing 2^{n-1} and, by our choice of n, it is finite. Hence $H(G)$ is finitely generated whence nilpotent and finite, being a torsion group. This, in turn, implies that G is finite, a contradiction.

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