

Left 3-Engel elements in groups of exponent 60

Gareth Tracey (G.M.Tracey@bath.ac.uk)
Gunnar Traustason (G.Traustason@bath.ac.uk) *

Department of Mathematical Sciences,
University of Bath,
Bath BA2 7AY, UK

Let G be a group and let $x \in G$ be a left 3-Engel element of order dividing 60. Suppose furthermore that $\langle x \rangle^G$ has no elements of order 8, 9 and 25. We show that x is then contained in the locally nilpotent radical of G . In particular all the left 3-Engel elements of a group of exponent 60 are contained in the locally nilpotent radical.

Keywords: Left Engel, Nilpotent, Presentation
Mathematics Subject Classification 2010: 20F45, 20F12

1 Introduction

Let G be a group. An element $a \in G$ is a left Engel element in G , if for each $x \in G$ there exists a non-negative integer $n(x)$ such that

$$[[[x, a], a], \dots, a] = 1.$$

$\underbrace{\hspace{10em}}_{n(x)}$

If $n(x)$ is bounded above by n then we say that a is a left n -Engel element in G . It is straightforward to see that any element of the Hirsch-Plotkin radical $HP(G)$ of G is a left Engel element and the converse is known to be true for some classes of groups, including solvable groups and finite groups (more generally groups satisfying the maximal condition on subgroups) [3,6]. The converse is however not true in general and this is the case even for bounded left Engel elements. In fact whereas one sees readily that a left 2-Engel element is always in the Hirsch-Plotkin radical this is still an open question for left 3-Engel elements. There is some substantial general progress by A. Abdollahi in [1] where he proves in particular that for any left 3-Engel p -element a in a group G one has that a^p is in $HP(G)$ (in fact he proves the stronger result that a^p is in the Baer radical), and that the subgroup generated by two left 3-Engel elements is nilpotent of class at most 4. Then in [11] it is shown that the left 3-Engel elements in groups of exponent 5 are in

*This research is supported through a standard grant from EPSRC

$HP(G)$. In this paper we will extend this result to groups of exponent 60. In fact we will prove something quite stronger. See also [2] for some results about left 4-Engel elements.

It was observed by William Burnside [4] that every element in a group of exponent 3 is a left 2-Engel element and so the fact that every left 2-Engel element lies in the Hirsch-Plotkin radical can be seen as the underlying reason why groups of exponent 3 are locally finite. For groups of 2-power exponent there is a close link with left Engel elements. If G is a group of exponent 2^n then it is not difficult to see that any element a in G of order 2 is a left $(n + 1)$ -Engel element of G (see the introduction of [11] for details). For sufficiently large n we know that the variety of groups of exponent 2^n is not locally finite [8,9]. As a result one can see [11] that it follows that for sufficiently large n we do not have in general that a left n -Engel element is contained in the Hirsch-Plotkin radical. Using the fact that groups of exponent 4 are locally finite [10], one can also see that if all left 4-Engel elements of a group G of exponent 8 are in $HP(G)$ then G is locally finite.

In this paper we continue our study of left 3-Engel elements started in [11]. We first make the observation that an element $a \in G$ is a left 3-Engel element if and only if $\langle a, a^x \rangle$ is nilpotent of class at most 2 for all $x \in G$ [1]. In [11] we introduced the following related class of groups.

Definition. A *sandwich* group is a group G generated by a set X of elements such that $\langle x, y^g \rangle$ is nilpotent of class at most 2 for all $x, y \in X$ and all $g \in G$.

Remark. In [11] it was shown that any sandwich group of rank 3 is nilpotent.

If $a \in G$ is a left 3-Engel element then $H = \langle a \rangle^G$ is a *sandwich* group and it is clear that the following statements are equivalent:

- (1) For every pair (G, a) where a is a left 3-Engel element in the group G we have that a is in the locally nilpotent radical of G .
- (2) Every sandwich group is locally nilpotent.

It is also clear that to prove (2), it suffices to show that every finitely generated sandwich group is nilpotent.

Left 3-Engel elements of finite order. For left 3-Engel elements of finite order some further reduction can be made. Suppose G is a group with a left 3-Engel element x of order $m = p_1^{n_1} \cdots p_r^{n_r}$ where p_1, \dots, p_r are distinct primes and n_1, \dots, n_r are positive integers. For $1 \leq j \leq r$, let $m_j = m/p_j^{n_j}$. Then m_1, \dots, m_r are coprime. Thus in order to show that $x \in HP(G)$, it suffices to show that $x^{m_1}, \dots, x^{m_r} \in HP(G)$. So the problem of showing that an element of finite order is in $HP(G)$ reduces to dealing with elements of prime power order. Further reductions can be made. First we recall a standard notion. Let G be a group. For any set π consisting of primes, we say that x is a π -element in G if the order of G only has numbers from π as prime factors.

Lemma 1.1 *Let π be a set of primes. Suppose that for all groups G and all primes $p \in \pi$ we have that all left 3-Engel elements of order p in G are contained in $HP(G)$. It then follows that for all groups G , all left 3-Engel elements in G that are π -elements are in*

$HP(G)$.

Proof Let G be any group. We have already seen that we only need to consider the case when x is a left 3-Engel element in G of some prime power exponent p^n where $p \in \pi$ and n is a positive integer. By the result of Alireza [1] mentioned above, we know that $x^p \in HP(G)$. As x^p is a p -element we then know that $N = \langle x^p \rangle^G$ is a locally finite p -group. By our assumption we know that $\langle x \rangle^G N/N$ is locally nilpotent and thus a locally finite p -group. Hence $\langle x \rangle^G$ is a locally finite p -group and thus locally nilpotent. We thus conclude that $x \in HP(G)$. \square

Remark. The problem of showing that all left 3-Engel elements of finite order are in the Hirsch-Plotkin radical thus reduces to only having to consider elements of prime order. Thus dealing with left 3-Engel elements of finite order reduces to working with sandwich groups generated by elements of prime order p . This is because the following are equivalent for any prime p :

(1 _{p}) For every pair (G, a) where a is a left 3-Engel element of order p in G we have that $a \in HP(G)$.

(2 _{p}) Every finitely generated sandwich group generated by elements of order p is nilpotent.

In sections 2 and 3. We will work with sandwich groups of rank 3 and 4 generated by elements of order 2.

Left 3-Engel elements in groups of finite exponent. Determining whether a left 3-Engel element of finite order is in the Hirsch-Plotkin radical seems a very difficult problem in general. One could thus consider adding further constraints on the group. For example one could require that for the given left 3-Engel element x in G we have that $\langle x \rangle^G$ is of finite exponent. In fact we will consider a weaker condition. Let $p_1^{n_1}, \dots, p_r^{n_r}$ be non-trivial powers where p_1, \dots, p_r are distinct primes. Consider the following statement.

$\mathcal{E}(p_1^{n_1}, \dots, p_r^{n_r})$: For all groups G and all left 3-Engel elements $x \in G$ of order dividing $p_1^{n_1} \cdots p_r^{n_r}$, where $\langle x \rangle^G$ has no elements of order $p_1^{n_1+1}, \dots, p_r^{n_r+1}$, we have that $x \in HP(G)$.

Remark. Notice that if $\mathcal{E}(p_1^{n_1}, \dots, p_r^{n_r})$ holds, then it would follow that a left 3-Engel element of G is in $HP(G)$ when $\langle x \rangle^G$ has exponent dividing $p_1^{n_1} \cdots p_r^{n_r}$. Thus in particular all left 3-Engel elements in a group G of exponent dividing $p_1^{n_1} \cdots p_r^{n_r}$ would be in the Hirsch-Plotkin radical.

We will next prove a reduction result that is similar in nature to Lemma 1.1. For a given prime p and positive integer n , consider the following statement

$\mathcal{Q}(p, n)$: For all groups G and all left 3-Engel elements $x \in G$, where x is of order p and $\langle x \rangle^G$ has no element of order p^{n+1} , we have that x is in $HP(G)$.

Proposition 1.2 Let $m = p_1^{n_1} \cdots p_r^{n_r}$ be an integer where p_1, \dots, p_r are distinct primes and n_1, \dots, n_r are positive integers. Then

$$\mathcal{Q}(p_1, n_1) \wedge \cdots \wedge \mathcal{Q}(p_r, n_r) \Rightarrow \mathcal{E}(p_1^{n_1}, \dots, p_r^{n_r}).$$

Proof Let G be any group and x a left 3-Engel element in G of order dividing m . Suppose that $\langle x \rangle^G$ has no elements of order $p_1^{n_1+1}, \dots, p_r^{n_r+1}$. For $1 \leq j \leq r$, let $m_j = m/p_j^{n_j}$. Then m_1, \dots, m_r are coprime. Thus in order to show that x is in $\text{HP}(G)$, it suffices to show that x^{m_1}, \dots, x^{m_r} are in $\text{HP}(G)$. It thus suffices to deal with the case when $m = p^n$ for a prime $p = p_j$ and the positive integer $n = n_j$. Let x_1, \dots, x_r be finitely many conjugates of x . We want to show that $H = \langle x_1, \dots, x_r \rangle$ is nilpotent. By the result of A. Abdollahi [1] mentioned above, we know that $N = \langle x_1^p, \dots, x_r^p \rangle^H$ is locally nilpotent. As any finitely generated subgroup of N is contained in a subgroup of N generated by finitely many conjugates of x^p and as x^p is of p -power order, it follows that N is a p -group. As N is a p -group and H contains no elements of order p^{n+1} , the same is true for H/N . As $\mathcal{Q}(p, n)$ holds by assumption, we thus have that $\langle x_i \rangle^H N/N$ is locally nilpotent for $1 \leq i \leq r$ and hence H/N is nilpotent. Thus H/N is a finite p -group that implies that N is finitely generated and thus also a finite p -group. We conclude that H is a finite p -group and thus nilpotent. We have thus shown that $\langle x \rangle^G$ is locally nilpotent and therefore that $x \in \text{HP}(G)$. \square

Remark. In particular, it follows from last proposition that in order to show that left 3-Engel elements in groups of finite exponent are in the Hirsch-Plotkin radical, it suffices to show that $\mathcal{Q}(p, n)$ holds for all primes and positive integers n .

At this stage we don't even know if for a group G , satisfying the hypothesis in $\mathcal{Q}(p, n)$, we can conclude that $\langle x \rangle^G$ is a p -group. Let us thus consider the following weaker statement.

$\mathcal{R}_G(p, n)$: For all left 3-Engel elements $x \in G$, where x is of order p and $\langle x \rangle^G$ has no element of order p^{n+1} , we have that $\langle x \rangle^G$ is a p -group.

Remark Of course $\mathcal{R}_G(p, n)$ implies that $\langle x \rangle^G$ is of exponent dividing p^n . The next proposition gives us a sufficient condition for $\mathcal{R}_G(p, n)$.

Proposition 1.3 *Let p be a prime and n a positive integer. Let G be a group with the property that for any left 3-Engel element $x \in G$ of order p we have that $\langle x \rangle^G$ has no element of order p^{n+1} . Now suppose furthermore that for any left 3-Engel element x of order p and any element $g \in \langle x \rangle^G$ of order dividing p^n we have*

$$\langle x, x^g, \dots, x^{g^{p^n-1}} \rangle$$

is nilpotent. Then $\mathcal{R}_G(p, n)$ holds.

Proof Let $y \in \langle x \rangle^G$. Then $y = x_1 \cdots x_r$ for some r conjugates of x . We show by induction on r that $y^{p^n} = 1$. This is obvious when $r = 0$. Now let $r \geq 1$ and suppose that our claim holds for smaller values of r . By the induction hypothesis we know that $(x_1 \cdots x_{r-1})^{p^n} = 1$. Thus for $m = p^n$ and $g = x_1 \cdots x_{r-1}$ we have

$$\begin{aligned} y^m &= g^m x_r^{g^{m-1}} x_r^{g^{m-2}} \cdots x_r^g x_r \\ &= x_r^{g^{m-1}} \cdots x_r^g x_r. \end{aligned}$$

By our assumptions $\langle x_r, x_r^g, \dots, x_r^{g^{m-1}} \rangle$ is nilpotent and thus a finite p -group. By the assumptions $\langle x_r, x_r^g, \dots, x_r^{g^{m-1}} \rangle$ is then of exponent m . In particular it follows that

$$(x_r^{g^{m-1}} \cdots x_r^g x_r)^m = 1$$

and thus $y^{m^2} = 1$. As $\langle x \rangle^G$ has no element of order pm it follows that $y^m = 1$. \square

Remark. As any sandwich group of rank 3 is nilpotent [11], it follows from Proposition 1.3 that $\mathcal{R}_G(2, 1)$ and $\mathcal{R}_G(3, 1)$ hold in any group G . Groups of exponent 2 are abelian and from Burnside [4] we know that groups of exponent 3 are locally finite. It thus follows that $\mathcal{Q}(2, 1)$ and $\mathcal{Q}(3, 1)$ hold.

The main result of the paper is the following.

Theorem 1.4 *Let G be any group and let x be a left 3-Engel in G of order dividing 60. Suppose furthermore that $\langle x \rangle^G$ has no elements of order 8, 9 or 25. Then $x \in HP(G)$.*

Remark. From Proposition 1.2 it suffices to show that $\mathcal{Q}(2, 2)$, $\mathcal{Q}(3, 1)$ and $\mathcal{Q}(5, 1)$ hold. We have already seen from last remark that $\mathcal{Q}(3, 1)$ holds. It thus remains to see that $\mathcal{Q}(2, 2)$ and $\mathcal{Q}(5, 1)$ hold.

To prove $\mathcal{Q}(2, 2)$ we will need some preliminary work. This will be carried out in Sections 2 and 3. In [11] it was shown that all sandwich groups of rank 3 are nilpotent. The proof for the case when the group is generated by involutions is substantially simpler and thus we start by giving a short proof of this in Section 2. In Section 3 we will then deal with certain sandwich groups of rank 4 generated by involutions that are needed to prove $\mathcal{Q}(2, 2)$. In Sections 4 and 5 we will prove $\mathcal{Q}(2, 2)$ and $\mathcal{Q}(5, 1)$ respectively.

Remark. Our way of writing polycyclic presentations in this paper follows [7]. It reflects a polycyclic series

$$\langle x_1 \rangle \trianglelefteq \langle x_1, x_2 \rangle \trianglelefteq \cdots \trianglelefteq \langle x_1, \dots, x_m \rangle = G.$$

We also partition the set of generators into subsets X_1, \dots, X_r where $\langle X_1 \rangle \leq \langle X_1 \cup X_2 \rangle \leq \cdots \leq \langle X_1 \cup \cdots \cup X_r \rangle = G$ is a normal series with abelian factors.

2 Sandwich groups generated by 3 involutions

Let $F = \langle x, y, z \rangle$ be a 3-generator sandwich group generated by involutions x, y and z .

Theorem 2.1 *F is nilpotent of class at most 5.*

Proof We have that $a = x$ and $b = x^y$ commute. Thus a, b^z commute with a^z, b and

$$\langle a, b, z \rangle = (\langle a, b^z \rangle \cdot \langle a^z, b \rangle) \rtimes \langle z \rangle.$$

Then

$$\begin{aligned} [x, y, z, z] &= [ab, z, z] \\ &= [(ab^z)(ba^z), z] \\ &= (b^z a)^2 (a^z b)^2 \\ &= [b^z, a] \cdot [a^z, b]. \end{aligned}$$

This element clearly commutes with $a = x$ and z . As $\langle x, y \rangle$ is nilpotent of class at most 2, we have $1 = [x^2, y] = [x, y]^2$ and thus $[x, y, z, z] = [y, x, z, z]$. By symmetry we thus see that $[x, y, z, z] = [y, x, z, z]$ commutes with y and is thus in $Z(F)$. By symmetry it follows that

$$[x, y, z, z] = [y, x, z, z], [y, z, x, x] = [z, y, x, x], [z, x, y, y] = [x, z, y, y] \in Z(F). \quad (1)$$

Next notice that

$$[[x, y], [x, z]] = [ab, aa^z] = [ab, a^z] = [b, a^z]$$

commutes with $a = x$ and we have seen in (1) that modulo $Z(F)$ we have $[b, a^z] = [b^z, a]$. Thus, modulo $Z(F)$, we know that $[[x, y], [x, z]]$ commutes with x and z . By symmetry we then see that $[[x, y], [x, z]] = [[x, z], [x, y]]^{-1}$ also commutes with y modulo $Z(F)$. Hence $[[x, y], [x, z]] \in Z_2(F)$. By symmetry

$$[[x, y], [x, z]], [[y, z], [y, x]], [[z, x], [z, y]] \in Z_2(F). \quad (2)$$

Then, modulo $Z_2(F)$, we have

$$\begin{aligned} [x, y, z]^x &= [x, y, z[z, x]] \\ &= [x, y, [z, x]z] \\ &= [x, y, z] \cdot [x, y, [z, x]]^z \\ &= [x, y, z]. \end{aligned}$$

Thus from (1) and (2) we know that $[x, y, z] = [y, x, z]$ commutes with x, y, z modulo $Z_2(F)$. By symmetry we thus have

$$[x, y, z], [y, z, x], [z, x, y] \in Z_3(F). \quad (3)$$

As $\langle x, y \rangle, \langle y, z \rangle, \langle z, x \rangle$ are nilpotent of class at most 2, it follows from (3) that $[x, y], [y, z], [z, x] \in Z_4(F)$ from which it follows that $x, y, z \in Z_5(F)$. \square

Knowing that F is nilpotent, it is now easy to come up with a power-conjugation presentation for the largest such group. Notice first that

$$[z, x, y, [z, x]] = [z, x, y, z, x][z, x, y, x, z] = [z, x, [y, z], x][x, z, [y, x], z].$$

Calculating in $\langle a, b, x \rangle = \langle z, z^y, x \rangle$, we see that $[z, x, [y, z], x] = [a, x, ab, x] = [a^x, b, x] = [a^x, b][a, b^x] = [z, y, x, x]$. By symmetry we have $[x, z, [y, x], z] = [x, y, z, z]$. Then, calculating in $\langle a, b, y \rangle = \langle z, z^x, y \rangle$, we see that $[z, x, y, [z, x]] = [ab, y, ab] = [a^y b^y, ab] = [a^y, b][b^y, a] = [z, x, y, y]$. We thus have $[z, x, y, y] = [z, y, x, x][x, y, z, z]$ or

$$[z, x, y, y][x, y, z, z][y, z, x, x] = 1.$$

One can come up with a full presentation using for example the nilpotent quotient algorithm or by hand. It turns out that we get the group $F = \langle x, y, z \rangle$ of order 2^{13} where the generators and relations are as follows.

Generators

$$X_1: \quad x_1 = [z, x, y, y], \quad x_2 = [x, y, z, z], \quad x_3 = [y, z, x, x]$$

$$X_2: \quad x_4 = [z, x, [z, y]], \quad x_5 = [x, y, [x, z]], \quad x_6 = [y, z, [y, x]], \\ x_7 = [z, x, y], \quad x_8 = [z, y, x]$$

$$X_3: \quad x_9 = [z, x], \quad x_{10} = [z, y], \quad x_{11} = [x, y]$$

$$X_4: \quad x_{12} = x, \quad x_{13} = y, \quad x_{14} = z.$$

Relations

$$x_3 = x_2x_1, \\ x_1^2 = x_2^2 = x_4^2 = x_5^2 = x_6^2 = x_9^2 = x_{10}^2 = x_{11}^2 = x_{12}^2 = x_{13}^2 = x_{14}^2 = 1, \quad x_7^2 = x_1, \quad x_8^2 = x_2x_1,$$

$$x_4^{x_{12}} = x_4x_2x_1, \quad x_4^{x_{13}} = x_4x_1, \quad x_5^{x_{13}} = x_5x_1, \quad x_5^{x_{14}} = x_5x_2, \\ x_6^{x_{12}} = x_6x_2x_1, \quad x_6^{x_{14}} = x_6x_2, \\ x_7^{x_9} = x_7x_1, \quad x_7^{x_{10}} = x_7x_1, \quad x_7^{x_{11}} = x_7x_1, \quad x_7^{x_{12}} = x_7x_5x_1, \\ x_7^{x_{13}} = x_7x_1, \quad x_7^{x_{14}} = x_7x_4x_1, \\ x_8^{x_9} = x_8x_2x_1, \quad x_8^{x_{10}} = x_8x_2x_1, \quad x_8^{x_{11}} = x_8x_2x_1, \quad x_8^{x_{12}} = x_8x_2x_1, \\ x_8^{x_{13}} = x_8x_6x_2x_1, \quad x_8^{x_{14}} = x_8x_4x_2x_1, \\ x_9^{x_{10}} = x_9x_4, \quad x_9^{x_{11}} = x_9x_5, \quad x_9^{x_{13}} = x_9x_7, \quad x_{10}^{x_{11}} = x_{10}x_6, \quad x_{10}^{x_{12}} = x_{10}x_8, \\ x_{11}^{x_{14}} = x_{11}x_8x_7x_6x_5x_4x_2, \quad x_{12}^{x_{13}} = x_{12}x_{11}, \quad x_{12}^{x_{14}} = x_{12}x_9, \quad x_{13}^{x_{14}} = x_{13}x_{10}.$$

3 Sandwich groups generated by 4 involutions

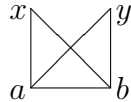
In this section we move on to 4-generator sandwiches. The ultimate aim is to show that these are nilpotent. We get here some partial results that will be sufficient to prove the main results of this paper. This is achieved by analysing various quotients of the largest sandwich group of rank 4 generated by involutions. The following definition will be useful.

Definition. Let G be a sandwich group generated by a finite set $X = \langle a_1, \dots, a_r \rangle$ of sandwich elements. The *commutativity graph* of G , $V(G)$, is an (undirected) graph whose set of vertices is the set of generators X and where a pair of distinct vertices a_i and a_j are joined by an edge if and only if a_i and a_j commute.

Remarks. (1) The commutativity graph of the free r -generator sandwich group has no edges and the largest r -generator sandwich group, whose commutativity graph is the complete graph, is the free abelian group of rank r .

(2) Let H and K be the largest r -generator sandwich groups with commutativity graphs $V(H)$ and $V(K)$ respectively. If $V(H) \subseteq V(K)$ then K is isomorphic to a quotient of H .

We now focus on sandwich groups generated by 4 involutions. It is clear that if we have a complete commutativity graph we get C_2^4 that is of order 16. There is only one type of a commutativity graph with 5 edges, namely



and the largest 4-generator sandwich group $\langle x, y, a, b \rangle$ with this commutativity graph, that is generated by involutions, is $\langle x, y \rangle \times \langle a, b \rangle = D_8 \times C_2^2$ that is of order 32. Moving next on to sandwich groups, whose commutativity graph has 4 edges, there are the following two types of graphs to consider (either the two removed edges are adjacent or not)



The largest sandwich group where x, y, a, b are involutions and with the former commutativity graph is $\langle x, y \rangle \times \langle a, b \rangle = D_8 \times D_8$ that has order 64. Moving to the latter group notice that a, b^c commute with b, a^c and we thus have

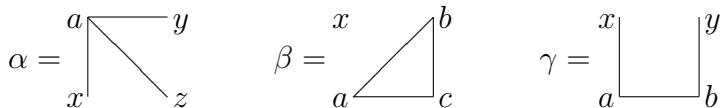
$$\langle a, b, c \rangle = \langle a, b^c \rangle \wr \langle c \rangle = D_8 \wr C_2$$

that is the standard wreath product of D_8 by C_2 . Thus the largest sandwich group generated by involutions a, b, c, x that have the latter graph as a commutativity graph is

$$\langle a, b, c \rangle \times \langle x \rangle = (D_8 \wr C_2) \times C_2$$

and is of order 256.

Next we consider the case when the commutativity graph has 3 edges. This is much more difficult and needs some care. Here there are three types of commutativity graphs. These are



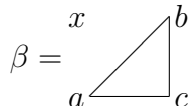
The largest sandwich group with the first commutativity graph is

$$G_\alpha = \langle x, y, z \rangle \times \langle a \rangle = R \times C$$

where R is the largest sandwich group generated by involutions x, y, z that we dealt with in last section. In the next two subsections we deal with the other two types.

3.1 Sandwich groups with commutativity graph β

Let $G_\beta = \langle x, a, b, c \rangle$ be a sandwich group where x, a, b, c are involutions and whose commutativity graph is



In this subsection we show that this group is nilpotent and obtain a consistent presentation for the group. We will use the fact that 3-generator sandwich groups are nilpotent. The following subgroups generated by 3 involutions will play a key role in the following: $H(c) = \langle x, x^{ab}, c \rangle$, $H(a) = \langle x, x^{bc}, a \rangle$, $H(b) = \langle x, x^{ca}, b \rangle$, $K = \langle x^a, x^b, x^c \rangle$.

Lemma 3.1 *We have that*

- (1) $[x, x^{ab}] = [x^a, x^b]$ commutes with x, a, b .
- (2) $[x, x^{bc}] = [x^b, x^c]$ commutes with x, b, c .
- (3) $[x, x^{ca}] = [x^c, x^a]$ commutes with x, c, a .

Proof We have that

$$\begin{aligned}
[x, x^{ab}] &= [x, [x, ab]] \\
&= [x, (a^x b)(b^x a)] \\
&= (b^x a)(a^x b)(a^x b)(b^x a) \\
&= [b^x, a][a^x, b]
\end{aligned}$$

and as $\langle b^x, a \rangle$, $\langle a^x, b \rangle$ are nilpotent of class at most 2, it follows that $[x, x^{ab}]$ commutes with a, b . As $[x, x^{ab}]$ clearly commutes with x we have that the first part follows and thus the others by symmetry. \square

Lemma 3.2 $\gamma_5(K) = \{1\}$.

Proof Using the fact that x commutes with x^a, x^b and x^c and that by Theorem 2.1 we know that $H(c)$ is nilpotent of class at most 5, we have

$$\begin{aligned}
[x^a, x^b, x^c, x^c] &= [x^a, x^b, [x, c], x^c] \\
&= [x^a, x^b, [x, c], [x, c]] \\
&\stackrel{L3.1}{=} [x, x^{ab}, [x, c], [x, c]] \\
&= 1.
\end{aligned}$$

By symmetry $[x^b, x^c, x^a, x^a] = [x^c, x^a, x^b, x^b] = 1$ and from the work on the 3-generator sandwich groups we know that implies that $\gamma_5(K) = \{1\}$. \square

Lemma 3.3 $\gamma_5(H(a)), \gamma_5(H(b)), \gamma_5(H(c)) \leq Z(\langle x \rangle^{G_\beta})$.

Proof From our analysis of the 3-generator groups we know that $\gamma_5(H(c))$ is generated by $[x, c, x^{ab}, x^{ab}]$ and $[x, x^{ab}, c, c]$ and that $[x^{ab}, c, x, x] = [x, c, x^{ab}, x^{ab}][x, x^{ab}, c, c]$. As $H(c)$ is nilpotent of class at most 5, we also know that these commutators all commute with x, x^{ab} and c . By Lemma 3.1 we also know that $[x, x^{ab}]$ commutes with a, b and thus $[x, x^{ab}, c, c]$ commutes with x, a, b, c and is thus in $Z(G_\beta)$. Hence

$$\gamma_5(H(c))Z(G_\beta) = \langle [x, c, x^{ab}, x^{ab}] \rangle Z(G_\beta).$$

In order to finish the proof of the lemma, it thus suffices to show that $[x, c, x^{ab}, x^{ab}]$ commutes with $x, x^a, x^b, x^c, x^{ab}, x^{ac}, x^{bc}, x^{abc}$. As we know already that it commutes with x, x^{ab}, c it only remains to show that it commutes with x^a, x^b . Now using the fact again that $H(c)$ is nilpotent of class at most 5 and that $[x, x^{ab}, x^{ab}] = [x^c, x^{ab}, x^{ab}] = 1$ we have

$$\begin{aligned}
[x, c, x^{ab}, x^{ab}] &= [xx^c, x^{ab}, x^{ab}] \\
&= [[x, x^{ab}][x, x^{ab}, x^c][x^c, x^{ab}], x^{ab}] \\
&= [x, x^{ab}, x^c, x^{ab}].
\end{aligned}$$

As $ab = ba$ we thus see from the symmetry that we only need to show that $[x, x^{ab}, x^c, x^{ab}]$ commutes with x^a . This will follow from the following calculations. We use there the fact from Lemma 3.1 that $[x, x^{ab}] = [x^a, x^b]$ and also that $K = \langle x^a, x^b, x^c \rangle$ is nilpotent of class at most 4. We have

$$\begin{aligned}
[x, x^{ab}, x^c, x^{ab}]^{x^a} &= [[x^a, x^b, x^c]^{x^a}, x^{ab}] \\
&= [x^a, x^b, x^c[x^c, x^a], x^{ab}] \\
&= [[x^a, x^b, [x^c, x^a]][x^a, x^b, x^c], x^{ab}].
\end{aligned}$$

Now we know by Lemma 3.1 that $[[x^a, x^b], [x^c, x^a]]$ commutes with a and as K is nilpotent of class at most 4, it follows that

$$1 = [x^a, x^b, [x^c, x^a], x^b]^a = [x^a, x^b, [x^c, x^a], x^{ab}].$$

From this and the calculations above it thus follows that

$$[x, x^{ab}, x^c, x^{ab}]^{x^a} = [[x^a, x^b, [x^c, x^a]][x^a, x^b, x^c], x^{ab}] = [x^a, x^b, x^c, x^{ab}] = [x, x^{ab}, x^c, x^{ab}],$$

Thus $[x, x^{ab}, x^c, x^{ab}]$ commutes with x^a and this finishes the proof. \square

Lemma 3.4 *We have that the following identities hold modulo $Z(\langle x \rangle^{G_\beta})$.*

- (1) $[x, x^{abc}, x^a] = [x^b, x^c, x^a]$.
- (2) $[x, x^{abc}, x^b] = [x^c, x^a, x^b]$.
- (3) $[x, x^{abc}, x^c] = [x^a, x^b, x^c]$.

Proof By symmetry, we only need to deal with the last identity. Calculating modulo $Z(\langle x \rangle^{G_\beta})$ we see that

$$\begin{aligned} [x, x^{abc}, x^c] &= [x, x^{abc}, [x, c]] \\ &= [x, x^{ab}[x^{ab}, c], [x, c]] \\ &\stackrel{\text{L3.3}}{=} [x, x^{ab}, [x, c]] \\ &= [x, x^{ab}, x^c] \\ &\stackrel{\text{L3.1}}{=} [x^a, x^b, x^c]. \end{aligned}$$

This finishes the proof. \square

Lemma 3.5 $\gamma_4(K) \leq Z(\langle x \rangle^{G_\beta})$.

Proof Calculations modulo $Z(\langle x \rangle^{G_\beta})$ show that

$$\begin{aligned} 1 &\stackrel{\text{L3.3}}{=} [x, [x^{ab}, c], [x, x^{ab}]] \\ &= [x, x^{ab}x^{abc}, [x, x^{ab}]] \\ &\stackrel{\text{L3.3}}{=} [x, x^{abc}, [x, x^{ab}]] \\ &\stackrel{\text{L3.1}}{=} [x, x^{abc}, [x^a, x^b]] \\ &= [x, x^{abc}, (x^a x^b)^2] \\ &= [x, x^{abc}, x^a x^b]^2 [x, x^{abc}, x^a x^b, x^a x^b]. \end{aligned}$$

By Lemma 3.2 and the presentation for the largest sandwich group generated by 3 involutions, we know that K is nilpotent of class at most 4 and that $\gamma_3(K)^2 = \{1\}$. We also know that $[x^c, x^a, x^b, x^b] = [x^b, x^c, x^a, x^a] = 1$. By Lemma 3.4 we have that $[x, x^{abc}, x^a x^b]^2 \in \gamma_3(K)^3 = 1$ and thus

$$\begin{aligned} 1 &= [x, x^{abc}, x^a x^b, x^a x^b] \\ 1 &\stackrel{\text{L3.4, L3.2}}{=} [x, x^{abc}, x^a, x^b][x, x^{abc}, x^b, x^a][x, x^{abc}, x^a, x^a][x, x^{abc}, x^b, x^b] \\ &\stackrel{\text{L3.4}}{=} [x^b, x^c, x^a, x^b][x^c, x^a, x^b, x^a][x^b, x^c, x^a, x^a][x^c, x^a, x^b, x^b] \\ &= [[x^b, x^c], [x^a, x^b]][[x^c, x^a], [x^b, x^a]]. \end{aligned}$$

By symmetry we thus have that modulo $Z(\langle x \rangle^{G_\beta})$ we have

$$[[x^a, x^b], [x^b, x^c]] = [[x^b, x^c], [x^c, x^a]] = [[x^c, x^a], [x^a, x^b]].$$

By Lemma 3.2 we know that K is nilpotent of class at most 4 and thus these three elements all commute with x^a, x^b, x^c . They of course all commute with x as well. By Lemma 3.1 the first element $[[x^a, x^b], [x^b, x^c]]$ commutes with b . It follows that it commutes with x^{ab}, x^{bc} . As $[[x^b, x^c], [x^c, x^a]]$ and $[[x^c, x^a], [x^a, x^b]]$ are equal to $[[x^a, x^b], [x^b, x^c]]$ modulo $Z(\langle x \rangle^{G_\beta})$, it follows that they also commute with x^{ab}, x^{bc} . By symmetry the three elements all commute with x^{ca} as well. Then as $[[x^a, x^b], [x^b, x^c]]$ commutes with x^{ac} and b , it commutes with x^{abc} . The same is then true for $[[x^b, x^c], [x^c, x^a]]$ and $[[x^c, x^a], [x^a, x^b]]$. From the presentation for the 3 generators sandwich group and the fact that K is nilpotent of class at most 4, we know that $\gamma_4(K) = \langle [[x^a, x^b], [x^b, x^c]], [[x^b, x^c], [x^c, x^a]], [[x^c, x^a], [x^a, x^b]] \rangle$ and we have thus shown that $\gamma_4(K) \leq Z(\langle x \rangle^{G_\beta})$. \square

Proposition 3.6 $\langle x \rangle^{G_\beta}$ is nilpotent of class at most 4.

Proof It suffices to show that $x \in Z_4(\langle x \rangle^{G_\beta})$. As x commutes with x^a, x^b, x^c , it suffices to show that $[x, x^{abc}], [x, x^{ab}], [x, x^{bc}], [x, x^{ca}]$ are in $Z_3(\langle x \rangle^{G_\beta})$. By symmetry it suffices to show that $[x, x^{abc}], [x, x^{ab}]$ are in $Z_3(\langle x \rangle^{G_\beta})$. Again by symmetry, it then suffices to show that

$$[x, x^{abc}, x^c], [x, x^{abc}, x^{ab}], [x, x^{ab}, x^c], [x, x^{ab}, x^{abc}], [x, x^{ab}, x^{ac}]$$

are in $Z_2(\langle x \rangle^{G_\beta})$. By Lemma 3.4 and Lemma 3.1, we have modulo $Z(\langle x \rangle^{G_\beta})$ that

$$[x, x^{abc}, x^c] = [x^a, x^b, x^c] = [x, x^{ab}, x^c].$$

By Lemma 3.3 we know that $\gamma_5(H(c)) \leq Z(\langle x \rangle^{G_\beta})$ and thus module $Z(\langle x \rangle^{G_\beta})$, we have (using also the fact that $[x^{ab}, c]$ commutes with x^{ab})

$$\begin{aligned} [x, x^{abc}, x^{ab}] &= [x, [x^{ab}, c], x^{ab}] \\ &= [x, x^{ab}, [x^{ab}, c]] \\ &= [x, x^{ab}, x^{abc}]. \end{aligned}$$

By Lemma 3.1 we also have

$$\begin{aligned} [x, x^{ab}, x^{abc}] &= [x, x^{ab}, x^c]^{ab} \\ [x, x^{ab}, x^{ac}] &= [x, x^{ab}, x^c]^a. \end{aligned}$$

It thus only remains to show that $[x, x^{ab}, x^c] \in Z_2(\langle x \rangle^{G_\beta})$. Using the fact that $\gamma_5(H(c)) \leq Z(\langle x \rangle^{G_\beta})$ we see that the commutator of $[x, x^{ab}, x^c] = [x, x^{ab}, [x, c]]$ with x, x^{ab} and c is in $Z(\langle x \rangle^{G_\beta})$. Then using Lemma 3.5 we know that $\gamma_4(K) \leq Z(\langle x \rangle^{G_\beta})$ and thus (using Lemma 3.1) we see that the commutator of $[x, x^{ab}, x^c] = [x^a, x^b, x^c]$ with x^a, x^b, x^c is in $Z(\langle x \rangle^{G_\beta})$. Thus we have seen that the commutator of $[x, x^{ab}, x^c]$ with x, x^a, x^b, x^{ab} and $x^c, x^{ac}, x^{bc}, x^{abc}$ is in $Z(\langle x \rangle^{G_\beta})$. It follows that $[x, x^{ab}, x^c]$ is $Z_2(\langle x \rangle^{G_\beta})$ and this finishes the proof. \square

Theorem 3.7 G_β is finite.

Proof As $G_\beta / \langle x \rangle^{G_\beta}$ is abelian of order at most 8, we have that $\langle x \rangle^{G_\beta}$ is a finitely generated nilpotent torsion group and thus finite. From $G / \langle x \rangle^{G_\beta}$ and $\langle x \rangle^{G_\beta}$ being finite, it follows that G_β is finite. \square

Having determined that the group G_β is finite, one can obtain the following power commutator presentation for it. In particular the group has order 2^{28} . Let

$$t(a) = [[x^c, x^a], [x^a, x^b]], t(b) = [[x^a, x^b], [x^b, x^c]], t(c) = [[x^b, x^c], [x^c, x^a]]$$

and

$$y(a) = [x, x^{bc}, x^a], y(b) = [x, x^{ca}, x^b], y(c) = [x, x^{ab}, x^c].$$

Generators

$$X_1 : b_1 = [t(a), b], b_2 = t(a), b_3 = t(b) \\ b_4 = y(b), b_5 = y(b)^a, b_6 = y(a), b_7 = y(a)^b, b_8 = y(a)^c,$$

$$X_2 : b_9 = [x, x^{ab}][x^c, x^{abc}], b_{10} = [x, x^{ab}], b_{11} = [x, x^{bc}][x^a, x^{abc}], \\ b_{12} = [x, x^{bc}], b_{13} = [x, x^{ac}][x^b, x^{abc}], b_{14} = [x, x^{ac}], b_{15} = [x, x^{abc}][x^c, x^{ab}], \\ b_{16} = [x, x^{abc}][x^a, x^{bc}], b_{17} = [x, x^{abc}],$$

$$X_3 : b_{18} = x, b_{19} = x^a, b_{20} = x^b, b_{21} = x^c, b_{22} = x^{ab}, b_{23} = x^{ca}, \\ b_{24} = x^{bc}, b_{25} = x^{abc},$$

$$X_4 : b_{26} = a, b_{27} = b, b_{28} = c.$$

Relations

$$b_1^2 = \dots = b_{28}^2 = 1.$$

$$b_2^{b_{27}} = b_2 b_1, b_2^{b_{28}} = b_2 b_1, b_3^{b_{26}} = b_3 b_1, b_3^{b_{28}} = b_3 b_1 \\ b_4^{b_{19}} = b_4 b_2, b_4^{b_{21}} = b_4 b_3 b_2, b_4^{b_{22}} = b_4 b_2 b_1, b_4^{b_{23}} = b_4 b_3, b_4^{b_{24}} = b_4 b_3 b_2 b_1, \\ b_4^{b_{25}} = b_4 b_3, b_4^{b_{26}} = b_5, b_4^{b_{28}} = b_8 b_6 b_4 \\ b_5^{b_{18}} = b_5 b_2, b_5^{b_{20}} = b_5 b_2 b_1, b_5^{b_{21}} = b_5 b_3 b_1, b_5^{b_{23}} = b_5 b_3 b_2 b_1, \\ b_5^{b_{24}} = b_5 b_3 b_1, b_5^{b_{25}} = b_5 b_3 b_2, b_5^{b_{26}} = b_4, b_5^{b_{28}} = b_8 b_6 b_5, \\ b_6^{b_{20}} = b_6 b_3, b_6^{b_{21}} = b_6 b_3 b_2, b_6^{b_{22}} = b_6 b_3 b_1, b_6^{b_{23}} = b_6 b_3 b_2 b_1, b_6^{b_{24}} = b_6 b_2, \\ b_6^{b_{25}} = b_6 b_2, b_6^{b_{27}} = b_7, b_6^{b_{28}} = b_8, \\ b_7^{b_{18}} = b_7 b_3, b_7^{b_{19}} = b_7 b_3 b_1, b_7^{b_{21}} = b_7 b_2 b_1, b_7^{b_{23}} = b_7 b_2 b_1 \\ b_7^{b_{24}} = b_7 b_3 b_2 b_1, b_7^{b_{25}} = b_7 b_3 b_2, b_7^{b_{27}} = b_6, b_7^{b_{28}} = b_8 b_7 b_6, \\ b_8^{b_{18}} = b_8 b_3 b_2, b_8^{b_{19}} = b_8 b_3 b_2 b_1, b_8^{b_{20}} = b_8 b_2 b_1, b_8^{b_{22}} = b_8 b_2 b_1, \\ b_8^{b_{24}} = b_8 b_3 b_1, b_8^{b_{25}} = b_8 b_3, b_8^{b_{27}} = b_8 b_7 b_6, b_8^{b_{28}} = b_6,$$

$$b_9^{b_{12}} = b_9 b_1, b_9^{b_{14}} = b_9 b_1, b_9^{b_{18}} = b_9 b_6 b_4, \\ b_9^{b_{19}} = b_9 b_6 b_5, b_9^{b_{20}} = b_9 b_7 b_4, b_9^{b_{21}} = b_9 b_6 b_4, \\ b_9^{b_{22}} = b_9 b_7 b_5, b_9^{b_{23}} = b_9 b_6 b_5, b_9^{b_{24}} = b_9 b_7 b_4, \\ b_9^{b_{25}} = b_9 b_7 b_5, \\ b_{10}^{b_{11}} = b_{10} b_1, b_{10}^{b_{12}} = b_{10} b_3, b_{10}^{b_{13}} = b_{10} b_1, b_{10}^{b_{14}} = b_{10} b_2, \\ b_{10}^{b_{16}} = b_{10} b_1, b_{10}^{b_{17}} = b_{10} b_3 b_2, b_{10}^{b_{21}} = b_{10} b_6 b_4, \\ b_{10}^{b_{23}} = b_{10} b_6 b_5, b_{10}^{b_{24}} = b_{10} b_7 b_4, b_{10}^{b_{25}} = b_{10} b_7 b_5, \\ b_{10}^{b_{28}} = b_{10} b_9, \\ b_{11}^{b_{14}} = b_{11} b_1, b_{11}^{b_{18}} = b_{11} b_6, b_{11}^{b_{19}} = b_{11} b_6, \\ b_{11}^{b_{20}} = b_{11} b_7, b_{11}^{b_{21}} = b_{11} b_8, b_{11}^{b_{22}} = b_{11} b_7, b_{11}^{b_{23}} = b_{11} b_8, b_{11}^{b_{24}} = b_{11} b_8 b_7 b_6,$$

$$\begin{aligned}
b_{11}^{b_{25}} &= b_{11}b_8b_7b_6, \\
b_{12}^{b_{13}} &= b_{12}b_1, \quad b_{12}^{b_{14}} = b_{12}b_3b_2, \quad b_{12}^{b_{15}} = b_{12}b_1, \\
b_{12}^{b_{17}} &= b_{12}b_2, \quad b_{12}^{b_{19}} = b_{12}b_6, \quad b_{12}^{b_{22}} = b_{12}b_7, \quad b_{12}^{b_{23}} = b_{12}b_8, \quad b_{12}^{b_{25}} = b_{12}b_8b_7b_6, \\
b_{12}^{b_{26}} &= b_{12}b_{11}, \\
b_{13}^{b_{18}} &= b_{13}b_4, \quad b_{13}^{b_{19}} = b_{13}b_5, \quad b_{13}^{b_{20}} = b_{13}b_4 \\
b_{13}^{b_{21}} &= b_{13}b_8b_6b_4, \quad b_{13}^{b_{22}} = b_{13}b_5, \quad b_{13}^{b_{23}} = b_{13}b_8b_6b_5, \quad b_{13}^{b_{24}} = b_{13}b_8b_6b_4 \\
b_{13}^{b_{25}} &= b_{13}b_8b_6b_5, \\
b_{14}^{b_{15}} &= b_{14}b_1, \quad b_{14}^{b_{16}} = b_{14}b_1, \quad b_{14}^{b_{17}} = b_{14}b_3, \quad b_{14}^{b_{20}} = b_{14}b_4, \\
b_{14}^{b_{22}} &= b_{14}b_5, \quad b_{14}^{b_{24}} = b_{14}b_8b_6b_4, \quad b_{14}^{b_{25}} = b_{14}b_8b_6b_5, \quad b_{14}^{b_{27}} = b_{14}b_{13}, \\
b_{15}^{b_{18}} &= b_{15}b_6b_4b_3b_2, \quad b_{15}^{b_{19}} = b_{15}b_6b_5b_3b_2b_1, \\
b_{15}^{b_{20}} &= b_{15}b_7b_4b_3b_2b_1, \quad b_{15}^{b_{21}} = b_{15}b_6b_4b_3b_2, \quad b_{15}^{b_{22}} = b_{15}b_7b_5b_3b_2, \quad b_{15}^{b_{23}} = b_{15}b_6b_5b_3b_2b_1, \\
b_{15}^{b_{24}} &= b_{15}b_7b_4b_3b_2b_1, \quad b_{15}^{b_{25}} = b_{15}b_7b_5b_3b_2, \\
b_{16}^{b_{18}} &= b_{16}b_6b_2, \quad b_{16}^{b_{19}} = b_{16}b_6b_2, \quad b_{16}^{b_{20}} = b_{16}b_7b_2b_1, \\
b_{16}^{b_{21}} &= b_{16}b_8b_2b_1, \quad b_{16}^{b_{22}} = b_{16}b_7b_2b_1, \quad b_{16}^{b_{23}} = b_{16}b_8b_2b_1, \quad b_{16}^{b_{24}} = b_{16}b_8b_7b_6b_2, \\
b_{16}^{b_{25}} &= b_{16}b_8b_7b_6b_2, \\
b_{17}^{b_{19}} &= b_{17}b_6b_2, \quad b_{17}^{b_{20}} = b_{17}b_4b_3, \quad b_{17}^{b_{21}} = b_{17}b_6b_4b_3b_2, \quad b_{17}^{b_{22}} = b_{17}b_7b_5b_3b_2, \\
b_{17}^{b_{23}} &= b_{17}b_8b_6b_5b_3, \quad b_{17}^{b_{24}} = b_{17}b_8b_7b_6b_2, \quad b_{17}^{b_{26}} = b_{17}b_{16}, \quad b_{17}^{b_{27}} = b_{17}b_{16}b_{15}, \\
b_{17}^{b_{28}} &= b_{17}b_{15},
\end{aligned}$$

$$\begin{aligned}
b_{18}^{b_{22}} &= b_{18}b_{10}, \quad b_{18}^{b_{23}} = b_{18}b_{14}, \quad b_{18}^{b_{24}} = b_{18}b_{12}, \quad b_{18}^{b_{25}} = b_{18}b_{17}, \quad b_{18}^{b_{26}} = b_{19}, \\
b_{18}^{b_{27}} &= b_{20}, \quad b_{18}^{b_{28}} = b_{21}, \\
b_{19}^{b_{20}} &= b_{19}b_{10}, \quad b_{19}^{b_{21}} = b_{19}b_{14}, \quad b_{19}^{b_{24}} = b_{19}b_{17}b_{16}, \quad b_{19}^{b_{25}} = b_{19}b_{12}b_{11}, \quad b_{19}^{b_{26}} = b_{18}, \\
b_{19}^{b_{27}} &= b_{22}, \quad b_{19}^{b_{28}} = b_{23}, \\
b_{20}^{b_{21}} &= b_{20}b_{12}, \quad b_{20}^{b_{23}} = b_{20}b_{17}b_{16}b_{15}, \quad b_{20}^{b_{25}} = b_{20}b_{14}b_{13}, \quad b_{20}^{b_{26}} = b_{22}, \quad b_{20}^{b_{27}} = b_{18}, \\
b_{20}^{b_{28}} &= b_{24}, \\
b_{21}^{b_{22}} &= b_{21}b_{17}b_{15}, \quad b_{21}^{b_{25}} = b_{21}b_{10}b_9, \quad b_{21}^{b_{26}} = b_{23}, \quad b_{21}^{b_{27}} = b_{24}, \quad b_{21}^{b_{28}} = b_{18}, \\
b_{22}^{b_{23}} &= b_{22}b_{12}b_{11}, \quad b_{22}^{b_{24}} = b_{22}b_{14}b_{13}, \quad b_{22}^{b_{26}} = b_{20}, \quad b_{22}^{b_{27}} = b_{19}, \quad b_{22}^{b_{28}} = b_{25}, \\
b_{23}^{b_{24}} &= b_{23}b_{10}b_9, \quad b_{23}^{b_{26}} = b_{21}, \quad b_{23}^{b_{27}} = b_{25}, \quad b_{23}^{b_{28}} = b_{19}, \\
b_{24}^{b_{26}} &= b_{25}, \quad b_{24}^{b_{27}} = b_{21}, \quad b_{24}^{b_{28}} = b_{20}, \\
b_{25}^{b_{26}} &= b_{24}, \quad b_{25}^{b_{27}} = b_{23}, \quad b_{25}^{b_{28}} = b_{22}.
\end{aligned}$$

3.2 Sandwich groups with commutativity graph γ

Let $G_\gamma = \langle a, b, x, y \rangle$ be a sandwich group generated by involutions whose commutativity graph is

$$\gamma = \begin{array}{ccc} & x & y \\ & | & | \\ a & \text{---} & b \\ & | & | \end{array}$$

Proposition 3.8 G_γ is finite.

Proof We have that $\langle y, b, y^a \rangle$ is abelian and thus $H = \langle y, b, y^a, x \rangle$ is a homomorphic image of G_β . As $H \trianglelefteq G_\gamma$ and $G/H = \langle aH \rangle$, it follows that G_γ is finite. \square

Again one can then obtain a power-conjugation presentation of G_γ . This turns out to be the following one. In particular G_γ has order 2^{20} .

Generators

$$\begin{aligned}
X_1 : \quad e_1 &= [x, b, [y, a], x, y, [x, b]], \quad e_2 = [x, b, [y, a], y, x, [y, a]], \\
e_3 &= [x, b, [y, a], x, y, x], \quad e_4 = [x, b, [y, a], y, x, y], \\
e_5 &= [x, b, [y, a], x, y], \quad e_6 = [x, b, [y, a], y, x], \\
e_7 &= [x, b, [y, a], x], \quad e_8 = [x, b, [y, a], y], \\
e_9 &= [x, b, y, x], \quad e_{10} = [x, [y, a], y]
\end{aligned}$$

$$\begin{aligned}
X_2 : \quad e_{11} &= [x, b, [y, a]], \quad e_{12} = [x, b, y], \quad e_{13} = [x, [y, a]], \\
e_{14} &= [x, y]
\end{aligned}$$

$$X_3 : \quad e_{15} = [x, b], \quad e_{16} = [y, a], \quad e_{17} = x, \quad e_{18} = y$$

$$X_4 : \quad e_{19} = a, \quad e_{20} = b.$$

Relations

$$e_1^2 = e_2^2 = \dots = e_{20}^2 = 1.$$

$$\begin{aligned}
e_3^{e_{20}} &= e_3 e_1, \quad e_4^{e_{19}} = e_4 e_2, \quad e_5^{e_{15}} = e_5 e_1, \quad e_5^{e_{17}} = e_5 e_3, \\
e_6^{e_{16}} &= e_6 e_2, \quad e_6^{e_{18}} = e_6 e_4, \quad e_7^{e_{12}} = e_7 e_1, \quad e_7^{e_{14}} = e_7 e_3, \quad e_7^{e_{18}} = e_7 e_5, \\
e_8^{e_{13}} &= e_8 e_2, \quad e_8^{e_{14}} = e_8 e_4, \quad e_8^{e_{17}} = e_8 e_6 \\
e_9^{e_{11}} &= e_9 e_1, \quad e_9^{e_{13}} = e_9 e_3, \quad e_9^{e_{16}} = e_9 e_5 e_4 e_2, \quad e_9^{e_{19}} = e_9 e_7 e_6 e_1, \\
e_{10}^{e_{11}} &= e_{10} e_2, \quad e_{10}^{e_{12}} = e_{10} e_4, \quad e_{10}^{e_{15}} = e_{10} e_6 e_3 e_1, \quad e_{10}^{e_{20}} = e_{10} e_8 e_5 e_2, \\
e_{11}^{e_{14}} &= e_{11} e_6 e_5 e_4 e_3, \quad e_{11}^{e_{17}} = e_{11} e_7, \quad e_{11}^{e_{18}} = e_{11} e_8, \\
e_{12}^{e_{13}} &= e_{12} e_6 e_5 e_4 e_3, \quad e_{12}^{e_{16}} = e_{12} e_8, \quad e_{12}^{e_{17}} = e_{12} e_9, \quad e_{12}^{e_{19}} = e_{12} e_{11} e_8, \\
e_{13}^{e_{15}} &= e_{13} e_7, \quad e_{13}^{e_{18}} = e_{13} e_{10}, \quad e_{13}^{e_{20}} = e_{13} e_{11} e_7, \\
e_{14}^{e_{15}} &= e_{14} e_9, \quad e_{14}^{e_{16}} = e_{14} e_{10}, \quad e_{14}^{e_{19}} = e_{14} e_{13} e_{10}, \quad e_{14}^{e_{20}} = e_{14} e_{12} e_9, \\
e_{15}^{e_{16}} &= e_{15} e_{11}, \quad e_{15}^{e_{18}} = e_{15} e_{12}, \quad e_{16}^{e_{17}} = e_{16} e_{13}, \\
e_{17}^{e_{18}} &= e_{17} e_{14}, \quad e_{17}^{e_{20}} = e_{17} e_{15}, \quad e_{18}^{e_{19}} = e_{18} e_{16}.
\end{aligned}$$

4 Proof of the main result

In this final section we give a proof of Theorem 1.4. As we saw in the introduction, this reduces to $\mathcal{Q}(2, 2)$ and $\mathcal{Q}(5, 1)$. For the proof of $\mathcal{Q}(2, 2)$ we will need the preliminary work from Section 2 and 3 whereas the proof of $\mathcal{Q}(5, 1)$ will be modeled on the approach in [11].

4.1 Proof of $\mathcal{Q}(2, 2)$

Let G be a group with a left 3-Engel element x of order 2. Suppose $\langle x \rangle^G$ has no elements of order 8. The aim is to show that $x \in HP(G)$. That is we want to show that $E = \langle x \rangle^G$ is locally nilpotent. For this we only need to show that $x \in HP(E)$. Without loss of generality we can thus replace G by E and assume that G has no elements of order 8. As

groups of exponent 4 are locally finite [10], it suffices to show that $\mathcal{R}_G(2, 2)$ holds.

Let $g \in G$ be an element of order 4. Let $H = \langle a, c, b, d \rangle = \langle x, x^g, x^{g^2}, x^{g^3} \rangle$. By Proposition 1.3 it suffices to show that H is nilpotent and this will be our aim.

Lemma 4.1 *If $h \in G$ is an involution and y is a conjugate of x , then $[y, y^h] = 1$.*

Proof We have $\langle y, h \rangle = \langle y, y^h \rangle \rtimes \langle h \rangle$ where $\langle y, y^h \rangle$ is nilpotent of class at most 2. Hence $\langle y, h \rangle$ is a 2-group. As G has no element of order 8, we have that

$$1 = (hy)^4 = h^4(y^{h^3}y^{h^2}y^hy) = (yy^h)^2 = [y^h, y].$$

This finishes the proof. \square

Lemma 4.2 *If y, z are two conjugates of x that commute and h is an involution in G , then*

$$[y^h, z] = [y, z^h].$$

Proof By Lemma 4.1 we have that y, z^h commute with y^h, z and

$$\langle h, y, z \rangle = \langle y, z, y^h, z^h \rangle \rtimes \langle h \rangle = (\langle y, z^h \rangle \cdot \langle y^h, z \rangle) \rtimes \langle h \rangle.$$

As $\langle y, z^h \rangle$ is nilpotent of class at most 2 it is clear that this group is finite of order at most $8^2 \cdot 2 = 128$. As there is no element of order 8, we then must have

$$1 = (h(yz))^4 = [(yz)^h(yz)]^2 = (y^hz)^2(z^hy)^2 = [y^h, z][z^h, y].$$

Hence the result. \square

Remark By Lemma 4.1 it follows in particular that $[x, x^{g^2}] = [x^g, x^{g^3}] = 1$. Thus if we pick any three of the generators x, x^g, x^{g^2}, x^{g^3} then we know that two of them must commute.

Lemma 4.3 *Let u, v, t be three of the generators of x, x^g, x^{g^2}, x^{g^3} where u and v commute. Then $\langle u, v, t \rangle$ is nilpotent of class at most 3.*

Proof We have $[u, t, v, v] = [v^{[u, t]}, v] = 1$ by Lemma 4.1 as $1 = [u, t^2] = [u, t]^2$. Similarly we have that $[v, t, u, u] = [u^{[t, v]}, u] = 1$. From the presentation of the largest 3-generator sandwich group, generated by involutions, we thus know that $\langle u, v, t \rangle$ is nilpotent of class at most 4. Finally $[[t, u], [t, v]] = [t^u, t^v] = [t^u, (t^u)^{uv}] = 1$ by Lemma 4.1 as $(uv)^2 = 1$. Again by the presentation for the largest 3-generator sandwich group we see that $\langle u, v, t \rangle$ is nilpotent of class at most 3. \square

Proposition 4.4 *Let $T = \langle a, b, x, y \rangle$ be a subgroup of G that is a sandwich group of type*

$$\gamma = \begin{array}{ccc} x & & y \\ | & & | \\ a & \text{---} & b \end{array}$$

where a, b, x, y are involutions. Then T is nilpotent of class at most 3 and $T^4 = \{1\}$.

Proof We already know from Section 3 that the group T is a finite 2-group and as there are no elements of order 8 we must have $T^4 = \{1\}$. In the following calculations we use the presentation for the largest group of type γ generated by involutions. With a slight abuse of notation we will use e_1, e_2, \dots, e_{20} for the values of this free group in T under the natural homomorphism. As $(xa)^2 = (yb)^2 = 1$ we can deduce from Lemma 4.1 that

$$1 = [y^x, y^a] = [y, x, [y, a]] = [e_{14}, e_{16}] = e_{10}$$

and

$$1 = [x^y, x^b] = [x, y, [x, b]] = [e_{14}, e_{15}] = e_9.$$

It is easy to see that from this it follows that $e_1 = \dots = e_8 = 1$. Also

$$1 = (byxa)^4 = [(xa)^{by}xa]^2 = (x^{by}a^yxa)^2 = (x^{by}aa^yx)^2.$$

Notice that $1 = [x^y, x^b]^y = [x, x^{by}]$. Now we thus get

$$1 = [(xx^{by})(aa^y)]^2 = [(aa^y)^{xx^{by}}(aa^y)] = [aa^y, xx^{by}] = [a, xx^{by}]^{a^y} [a^y, xx^{by}] = [a, x^{by}][a^y, x].$$

Hence

$$1 = [a^y, x^b][a, x^y] = [a^y, x]^b [a, [x, y]] = [e_{16}, e_{17}]^{e_{20}} [e_{19}, e_{14}] = e_{11}.$$

From the presentation one now observes that $e_1 = \dots = e_{11} = 1$ implies that T is nilpotent of class at most 3. \square

We now return to the main task of this section. Let $H = \langle a, b, c, d \rangle = \langle x, x^{g^2}, x^g, x^{g^3} \rangle$. We know that $[a, b] = [c, d] = 1$. We want to show this group is a finite 2-group.

Lemma 4.5 *We have that $[[a, c], [b, d]], [[a, d], [b, c]] \leq Z_2(H)$.*

Proof We first show that $\langle [a, c], [b, d] \rangle$ and $\langle [a, d], [b, c] \rangle$ are nilpotent of class at most 2. We first turn to the first one. Calculating in $\langle a, x, b, y \rangle = \langle a, a^c, b, b^d \rangle$ we get a group of type γ . Hence using the presentation for the largest such group and the fact from the last proposition that this group has class at most 3, we see that

$$[a, c, [b, d], [a, c]] = [ax, by, ax] = [a, y, x][x, y, a] = [e_{16}, e_{17}][e_{14}, e_{19}] = 1.$$

By symmetry we also have that $[[a, c], [b, d]]$ commutes with $[b, d]$. Similarly one sees that $\langle [a, d], [b, c] \rangle$ is nilpotent of class at most 2. Notice that $1 = [a, c]^2 = [b, d]^2 = [a, d]^2 = [b, c]^2$ and thus $[a, c] = [c, a], [b, d] = [d, b], [a, d] = [d, a], [b, c] = [c, b]$. From what we have seen above we also know that

$$[[a, c], [b, d]] = [[b, d], [a, c]], \quad [[a, d], [b, c]] = [[b, c], [a, d]].$$

This symmetry implies that in order to show that $[[a, c], [b, d]], [[a, d], [b, c]]$ are in $Z_2(H)$, it suffices that their commutator with a is in $Z(H)$. Calculating again in $\langle a, x, b, y \rangle = \langle a, a^c, b, b^d \rangle$, we see that

$$[a, c, [b, d], a] = [ax, by, a] = [x, y, a] = [a^c, b^d, a].$$

As $[c, d] = [a, b] = 1$, we have by Lemma 4.2 that $[a^c, b^d] = [a^d, b^c]$. Thus

$$[a, c, [b, d], a] = [a^c, b^d, a] = [a^d, b^c, a] = [a, d, [b, c], a]. \quad (4)$$

Notice the joint element commutes with a and b as $\langle a, a^c, b, b^d \rangle$ is nilpotent of class at most 3. Next notice that in $\langle a, x, b, y \rangle = \langle a, a^c, b, b^d \rangle$ we have

$$[a, c, [b, d], a] = [x, y, a] = [e_{14}, e_{19}] = e_{13} = [e_{17}, e_{16}] = [x, [y, a]] = [ax, [by, a]] = [a, c, [b, d, a]].$$

Notice also that $[b, d, a, a] = [by, a, a] = [y, a, a] = 1$ and thus $1 = [b, d, a^2] = [b, d, a]^2 [b, d, a, a] = [b, d, a]^2$. By Lemma 4.1 we thus have that

$$\langle c, c^a, c^{[b,d,a]}, c^{a[b,d,a]} \rangle$$

is abelian. Hence

$$\begin{aligned} [a, c, [b, d, a]]^c &= [a, c, [b, d, a][b, d, a, c]] \\ &= [a, c, [b, d, a, c]] \cdot [a, c, [b, d, a]]^{[b,d,a,c]} \\ &= [a, c, [b, d, a]]. \end{aligned}$$

Thus $[a, c, [b, d], a] = [a, c, [b, d, a]]$ commutes with c and by symmetry $[a, d, [b, c], a]$ commutes with d . From (4) we thus now know that $[a, c, [b, d], a] = [a, d, [b, c], a]$ commutes with a, b, c, d and is thus in $Z(G)$. \square

Proposition 4.6 *We have that $\langle a, b, c, d \rangle$ is finite.*

Proof By Lemma 4.1 we know that $\langle c, c^a, c^b, c^{ab} \rangle$ is abelian and thus $[c, a, b]$ commutes with c . Clearly $[c, a, b]$ commutes with a, b . By symmetry $[c, b, a]$ commutes with a, b, c and thus $\langle a, b, c \rangle$ is nilpotent of class at most 3. From this and the fact that $[a, b] = 1$, it follows that $[c, a, b] = [c, b, a]$. In order to show that $[c, a, b] \in Z_3(\langle a, b, c, d \rangle)$ it then only remains to see that $[c, a, b, d] = [c, b, a, d] \in Z_2(\langle a, b, c, d \rangle)$. It clearly commutes with c as $[c, d] = 1$. As $[c, a, b]^2 = 1$ it follows from Lemma 4.1 that it commutes with d . Then

$$[c, a, b, d, b] = [c, a, b, d]^{-1} [c, a, b, d [d, b]] = [c, a, b, d]^{-1} [c, a, b, [d, b]] [c, a, b, d]^{[d,b]}.$$

By the last lemma we know that $[c, a, b]^{[d,b]} = [[c, a][[c, a], [d, b]], b] = [c, a, b]$ modulo $Z(\langle a, b, c, d \rangle)$. Thus $[c, a, b, d, b] = [c, a, b, d]^{-1} [c, a, b, d]^{[d,b]} = [c, a, b, d]^{-1} [c, a, b, d] = 1$ modulo $Z(\langle a, b, c, d \rangle)$. By symmetry $[c, a, b, d, a] = [c, b, a, d, a] \in Z(\langle a, b, c, d \rangle)$. We have thus shown that $[c, a, b] \in Z_3(\langle a, b, c, d \rangle)$. By symmetry this is true for any commutator of weight 3 in a, b, c, d . Hence $\langle a, b, c, d \rangle$ is nilpotent of class at most 6. \square

4.2 Proof of $\mathcal{Q}(5, 1)$

Let G be a group with a left 3-Engel element x of order 5 and suppose furthermore that $H = \langle x \rangle^G$ has no element of order 25. The aim is to show that x is in the locally nilpotent radical of G . The proof of this will be modeled on [11], where this is proved under the stronger hypothesis that G is of exponent 5. A key ingredient is a certain variant of a similar result from [12]. Before stating it we recall some terminology from [5]. We say that a group $\langle a, b, c \rangle$ is of type (r, s, t) if $\langle a, b \rangle$, $\langle a, c \rangle$ and $\langle b, c \rangle$ are of class at most r, s and t respectively.

Proposition 4.7 *Let $K = \langle a, b, c \rangle$ be a subgroup of H that is of type $(1, 2, 3)$ and where a, b, c are of order 5. Suppose furthermore that c is a left 3-Engel element of H and that $[b, c, c] = 1$. The $\langle a, b, c \rangle$ is nilpotent of class at most 4 and of exponent 5.*

Proof From the proof of Proposition 2.2 in [11] (see Step 2), we know that $\langle c \rangle^K$ is nilpotent of class at most 3. As c is of order 5 it follows in particular that $\langle c \rangle^K$ is a 5-group. Now $K/\langle c \rangle^K$ is abelian of order dividing 25. Thus $\langle c \rangle^K$ is a finitely generated nilpotent 5-group and thus a finite 5-group. Hence K is a finite 5-group. As $K \leq H$, it has no elements of order 25 and thus must be of exponent 5. The fact that K is nilpotent of class at most 4 now follows from Proposition 2.3 in [11]. \square

In order to show that x is in the locally nilpotent radical of G it suffices to show that $\langle x \rangle^G$ is locally nilpotent. It thus suffices to prove the following result that is again a variant of the corresponding result in [11] for groups of exponent 5. That proof was also modelled on a similar result in [12].

Proposition 4.8 *Let k be a positive integer and let a_1, \dots, a_k be conjugates of x . Then $A = \langle a_1, \dots, a_k \rangle$ is nilpotent of class at most k and of exponent 5. Furthermore $\langle a_i \rangle^A$ is abelian for $i = 1, \dots, k$.*

Proof From the proof of Theorem 3.1 in [11], we know that this result holds when A is of exponent 5. Thus it suffices to show that A is nilpotent as then A is a 5-group and the assumption that H has no elements of order 25 implies then that A is of exponent 5.

We have that the case $k = 2$ holds by the assumption that x is a left 3-Engel element and the case $k = 3$ follows from the fact that 3-generator sandwich groups are nilpotent. Now suppose that $k \geq 3$. Let $u = [a_1, a_2, \dots, a_{k-2}]$. Then the subgroup $\langle a_{k-1}, a_{k-1}^u, a_k \rangle$ is generated by 3-conjugates of x and is thus nilpotent of class at most 3. By this and the fact that any two conjugates generate a subgroup of class at most 2, it follows that

$$[a_1, a_2, \dots, a_{k-1}, a_k, a_k] = [a_{k-1}^{-u} a_{k-1}, a_k, a_k] = 1$$

and

$$[a_{k,3} [a_1, a_2, \dots, a_{k-1}]] = [a_{k,3} a_{k-1}^{-u} a_{k-1}] = 1.$$

We thus have the following identities which hold for any conjugates a_1, a_2, \dots, a_k of x and for any $k \geq 3$.

$$\begin{aligned} [a_1, a_2, \dots, a_{k-1}, a_k, a_k] &= 1, & (5) \\ [a_k, [a_1, a_2, \dots, a_{k-1}], [a_1, a_2, \dots, a_{k-1}], [a_1, a_2, \dots, a_{k-1}]] &= 1. \end{aligned}$$

We now proceed with the induction step. Let $k \geq 4$ and suppose that the result is true for all smaller values of k . We first show that if $1 \leq r \leq k$, then

$$[[a_1, a_2, \dots, a_r], [a_1, a_k, a_{k-1}, \dots, a_{r+1}]] = [a_1, a_2, \dots, a_k, a_1]^{(-1)^{k-r}}. \quad (6)$$

This is obvious when $r = k$. Now consider the case $r = k - 1$. Let $u = [a_1, \dots, a_{k-1}]$. By the induction hypothesis and (4) we have that $\langle a_1, u, a_k \rangle$ is of type (1, 2, 3) and satisfies the condition for Proposition 4.7. Hence $\langle a_1, u, a_k \rangle$ is nilpotent of class at most 4. Using the fact that u commutes with a_1 and the first identity in (4) one sees easily that all commutators of weight (2, 1, 1) and (1, 1, 2) in a_1, u, a_k are trivial. The only commutators that one needs to consider are $[u, a_k, a_1, a_1]$ and $[u, a_k, a_1, a_k]$ but as $[u, [a_k, a_1, a_1]] = [u, [a_1, a_k, a_k]] = 1$ we get by expanding these that

$$\begin{aligned} 1 &= [u, a_k, a_1, a_1], \\ 1 &= [u, a_k, a_1, a_k]. \end{aligned}$$

From this one sees that $[[u, [a_1, a_k]] = [u, a_k, a_1]^{-1}$ that gives us identity (5) when $r = k - 1$. This argument also tells us that

$$[[a_1, a_k, \dots, a_3], [a_1, a_2]] = [a_1, a_k, \dots, a_2, a_1]^{-1}$$

and thus

$$[a_1, [a_1, a_k, \dots, a_2]] = [[a_1, a_2], [a_1, a_k, \dots, a_3]]^{-1}$$

that shows that the case $r = 1$ follows if it holds for $r = 2$. To establish (5) it is thus sufficient to show that for $2 \leq r \leq k - 2$ we have

$$[[a_1, a_2, \dots, a_r], [a_1, a_k, \dots, a_{r+1}]] = [[a_1, a_2, \dots, a_{r+1}], [a_1, a_k, \dots, a_{r+2}]]^{-1}.$$

Let $u = [a_1, a_2, \dots, a_r]$ and $v = [a_1, a_k, \dots, a_{r+2}]$. By the induction hypothesis we have that u and v commute and that $\langle u, a_{r+1} \rangle, \langle v, a_{r+1} \rangle$ are nilpotent of class at most 2. Thus $\langle u, v, a_{r+1} \rangle$ is of type (1, 2, 2). From induction hypothesis we also know that $u^5 = v^5 = 1$. By Section 2.1.2. in [11] we know that the group $\langle u, v, a_{r+1} \rangle$ is nilpotent and thus of exponent 5. The presentation given for this group in Section 2.1.2 in [11] shows that the group is then nilpotent of class at most 3. Thus $[u, [v, a_{r+1}]] = [u, a_{r+1}, v]^{-1}$ as required. This establishes (5).

We want to show that A is nilpotent. We will show that A is nilpotent of class at most $k + 1$. The rest will then follow from the fact that A is of exponent 5 and [11].

Consider a commutator $c = [b_1, b_2, \dots, b_{k+1}]$ where $b_1, \dots, b_{k+1} \in \{a_1, \dots, a_k\}$. We want to show that $c \in Z(A)$. By induction $c = 1$ unless $\{b_1, \dots, b_k\} = \{a_1, \dots, a_k\}$. Also by (4) we have that $c = 1$ if $b_k = b_{k+1}$. So there is no loss of generality in assuming that $b_{k+1} = a_1, b_k = a_k$ and that $\{b_1, \dots, b_{k-1}\} = \{a_1, \dots, a_{k-1}\}$. Then, using the inductive hypothesis, we see that $[b_1, b_2, \dots, b_{k-1}]$ can be expressed as a product $u_1 u_2 \cdots u_r$ where each u_i is a commutator of the form $[a_1, a_{\sigma(2)}, a_{\sigma(3)}, \dots, a_{\sigma(k-1)}]^{\epsilon_i}$ for some permutation σ of $\{2, 3, \dots, k-1\}$ and where $\epsilon_i = \pm 1$. So

$$c = [b_1, \dots, b_{k+1}] = [u_1 \cdots u_r, a_k, a_1] = \left[\prod_{i=1}^r [u_i, a_k]^{u_{i+1} u_{i+2} \cdots u_r}, a_1 \right].$$

Now the inductive hypothesis implies that u_1, u_2, \dots, u_r commute with a_1 . So c is the product of conjugates of the commutators $[u_1, a_k, a_1], \dots, [u_r, a_k, a_1]$. Notice also that $[u_i, a_k]$ commutes with a_1 if and only if $[u_i^{-1}, a_k]$ commutes with a_1 . To show that $c \in Z(A)$ it thus clearly suffices to show that $[a_1, a_2, \dots, a_k, a_1] \in Z(A)$.

So consider $d = [a_1, a_2, \dots, a_k, a_1, a_i]$, where $1 \leq i \leq k$. If $i = 1$ then $d = 1$ by (4). If $i = k$, let $u = [a_1, a_2, \dots, a_{k-1}]$. Then, using the induction hypothesis, u is of order 5, $\langle a_1, u, a_k \rangle$ is of type (1, 2, 3) and satisfies the conditions given in Proposition 4.7. It is thus nilpotent of class at most 4 (and then exponent 5). Thus

$$1 = [u, [a_1, a_k, a_k]] = [u, a_k, a_1, a_k]^{-2}.$$

This implies that $[u, a_k, a_1, a_k] = 1$ and thus $d = 1$ when $i = k$. Now let $1 < i < k$. To show that $d = 1$, it suffices by (5) to show that $[u, a_i, v, a_i] = 1$ when $u = [a_1, a_2, \dots, a_{i-1}]$ and $v = [a_1, a_k, a_{k-1}, \dots, a_{i+1}]$. Now by the induction hypothesis $\langle u, v, a_i \rangle$ is of type

$(1, 2, 3)$ with $u^5 = v^5 = 1$ satisfies the criteria from Proposition 4.7. Thus it is nilpotent of class at most 4. Hence again

$$1 = [u, [v, a_i, a_i]] = [u, a_i, v, a_i]^{-2}$$

that implies that $[a_1, a_2, \dots, a_k, a_1]$ commutes with a_i . This finishes the proof that A is nilpotent and thus the inductive step. \square

References

- [1] A. Abdollahi, Left 3-Engel elements in groups, *J. Pure Appl. Algebra* **188** (2004), 1-6.
- [2] A. Abdollahi and H. Khosravi, On the right and left 4-Engel elements, *Comm. Algebra* **38** (2010), 993-943.
- [3] R. Baer, Engelsche Elemente Noetherscher Gruppen, *Math. Ann.* **133** (1957), 256-270.
- [4] W. Burnside, On an unsettled question in the theory of discontinuous groups, *Quart. J. Pure Appl. Math.* **37** (1901), 230-238.
- [5] G. Endimioni and G. Traustason, Groups that are pairwise nilpotent, *Comm. Alg.* **36** (2008), 4413-4435.
- [6] K. W. Gruenberg, The Engel elements of a soluble group, *Illinois J. Math.* **3** (1959), 151-169.
- [7] R. Hartung and G. Traustason, Refined solvable presentations for polycyclic groups, *Int. J. Group Theory*, **1** no. 2 (2012), 1-17.
- [8] S. V. Ivanov, The free Burnside groups of sufficiently large exponents, *Int. J. Algebra and Comp.* **4** (1994), 1-308.
- [9] I. G. Lysenok, Infinite Burnside groups of even period, *Izv. Math.* **60** (1996), 453-654.
- [10] I. N. Sanov, Solution of Burnside's Problem for Exponent Four, Leningrad State Univ., *Ann. Maths. Ser.* **10** (1940), 166-170.
- [11] G. Traustason, Left 3-Engel elements in groups of exponent 5, *J. Algebra*, **414** (2014), 41-71.
- [12] M. Vaughan-Lee, Engel-4 groups of exponent 5, *Proc. London Math. Soc.* **74** (1997), 306-334.