Left 3-Engel elements of odd order in groups

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Let G be a group and let $x \in G$ be a left 3-Engel element of odd order. We show that x is in the locally nilpotent radical of G. We establish this by proving that any finitely generated sandwich group, generated by elements of odd orders, is nilpotent. This can be seen as a group theoretic analog of a well-known theorem on sandwich algebras by Kostrikin and Zel'manov.

We also give some applications of our main result. In particular, for any given word $w = w(x_1, \ldots, x_n)$ in n variables, we show that if the variety of groups satisfying the law $w^3 = 1$ is a locally finite variety of groups of exponent 9, then the same is true for the variety of groups satisfying the law $(x_{n+1}^3 w^3)^3 = 1$.

1 Introduction

Let G be a group. An element $a \in G$ is a left Engel element in G, if for each $x \in G$ there exists a non-negative integer $n(x)$ such that

$$[[[x, \underbrace{a], a], \ldots, a]}_{n(x)} = 1.$$

If $n(x)$ is bounded above by n then we say that a is a left n-Engel element in G. Recall that the *locally nilpotent radical* (also know as the Hirsch–Plotkin radical) of G is the product of all the normal locally nilpotent subgroups of G. It is straightforward to see that any element of the locally nilpotent radical $HP(G)$ of G is a left Engel element and the converse is known to be true for some classes of groups, including solvable groups and finite groups (more generally groups satisfying the maximal condition on subgroups) [3,5]. The converse, however, is not true in general and this is the case even for bounded left Engel elements. In fact while one sees readily that a left 2-Engel element is always in the locally nilpotent radical it is still an open question whether a left 3-Engel element of a group G is always contained in $HP(G)$. There is some substantial progress by A. Abdollahi in [1] where in particular, he proves that for any left 3-Engel p-element a in a group G, a^p is in $HP(G)$ (in fact he proves the stronger result that a^p is in the Baer radical), and the subgroup generated by two left 3-Engel elements is nilpotent of class at most 4. Then in [11] it is shown that the left 3-Engel elements in groups of exponent 5 are in $HP(G)$. In [10] this result is extended to groups of exponent 60. In fact something quite stronger is proved in [10]. Namely that if x is a left 3-Engel element of order dividing 60 and $\langle x\rangle^G$ has no elements of order 8, 9 or 25, then $x \in HP(G)$. See also [2] for some results about left 4-Engel elements.

It was observed by William Burnside [4] that every element in a group of exponent 3 is a left 2-Engel element and so the fact that every left 2-Engel element lies in the locally nilpotent radical can be seen as the underlying reason why groups of exponent 3 are locally finite. For groups of 2-power exponent there is a close link with left Engel elements. If G is a group of exponent 2^n then it is not difficult to see that any element a in G of order 2 is a left $(n+1)$-Engel element of G (see the introduction of [11] for details). For sufficiently large n we know that the variety of groups of exponent 2^n is not locally finite [6,8]. As a result one can see (for example in [11]) that for sufficiently large n we do not have in general that a left n-Engel element is contained in the locally radical. Using the fact that groups of exponent 4 are locally finite [9], one can also see that if all left 4-Engel elements of a group G of exponent 8 are in $HP(G)$ then G is locally finite.

In this paper we continue our study of left 3-Engel elements started in [11] and [10]. We first make the observation that an element $a \in G$ is a left 3-Engel element if and only if $\langle a, a^x\rangle$ is nilpotent of class at most 2 for all $x \in G$ [1]. In [11] we introduced the following related class of groups.

Definition. Let G be a group. We say that $X \subseteq G$ is a *sandwich set* in G if $\langle x, y^g\rangle$ is nilpotent of class at most 2 for all $x, y \in X$ and $g \in G$. A *sandwich group* is a group G that can be generated by a sandwich set X. The *rank* of a sandwich group G is the smallest possible cardinality of a sandwich set that generates G.

Remark. In [11] it was shown that any sandwich group of rank 3 is nilpotent.

If $a \in G$ is a left 3-Engel element then $H = \langle a\rangle^G$ is a *sandwich* group and it is clear that the following statements are equivalent:

(1) For every pair (G, a) where a is a left 3-Engel element in the group G, a is in the locally nilpotent radical of G.

(2) Every sandwich group is locally nilpotent.

To prove (2), it suffices to show that every finitely generated sandwich group is nilpotent.

In this paper we prove the following.

Theorem 1.1 *Let G be any group and let $a \in G$ be a left 3-Engel element of odd order. Then a belongs to the locally nilpotent radical of G.*

From the discussion above we see that if we can show that any finitely generated sandwich group $\langle x_1, \ldots, x_r\rangle$, where x_i is of odd order for $i = 1, \ldots, r$, is nilpotent then the theorem follows.

In the last section we look at some applications. In particular, if for a given word $w = w(x_1, \ldots, x_n)$ in n variables the variety of groups satisfying the law $w^3 = 1$ is a locally finite variety of groups of exponent 9, then the same is true for the variety of groups satisfying the law $(x_{n+1}^3 w^3)^3 = 1$.

2 Proof of Theorem 1.1

In this section we prove Theorem 1.1 by showing that any sandwich group generated by a sandwich set, consisting of elements of odd order, is nilpotent.

Proposition 2.1 *Let $G = \langle x_1, \ldots, x_r \rangle$ be a sandwich group where $x_1, \ldots, x_r$ are of odd order. Then G is nilpotent.*

The proof of Proposition 2.1 relies on the following lemma. It is a critical for the rest of the paper.

Lemma 2.2 *Let G be a group and $X \subseteq G$ a sandwich set in G where $o(x)$ is an odd number for all $x \in X$. Let $x, y \in X$. Then $X \cup \{[x, y]\}$ is also a sandwich set in G and $o([x, y])$ is an odd number.*

Proof First notice that if the order of x is the odd number n, then $1 = [x^n, y] = [x, y]^n$ and thus $[x, y]$ is of odd order.

Let $z \in X$ and $g \in G$. We need to show that $\langle [x, y], z^g \rangle$ and $\langle [x, y], [x, y]^g \rangle$ are nilpotent of class at most 2. For the former subgroup, notice that $\langle x, y, z^g \rangle$ is a 3-generator sandwich group where the order of the generators is odd. From [11] we then know that $\langle x, y, z^g \rangle$ is nilpotent of class at most 3. Hence $\langle [x, y], z^g \rangle$ is nilpotent of class at most 2.

For the same reason $\langle [x, y], x^h \rangle$ and $\langle [x, y], y^h \rangle$ are nilpotent of class at most 2 for all $h \in G$. Hence $\langle [x, y], x^g, y^g \rangle$ is a 3-generator sandwich group and, as again all the generators are of odd order, nilpotent of class at most 3. Hence $\langle [x^g, y^g], [x, y] \rangle = \langle [x, y], [x, y]^g \rangle$ is nilpotent of class at most 2. $\Box$.

Suppose X is a sandwich set in G where $o(x)$ is odd for all $x \in X$. Let $\overline{X}$ be the closure of X with respect to the commutator operation. In other words $\overline{X}$ consists of all commutators in X (in any order and with any bracketing). It follows by iterated use of Lemma 2.2 that $\overline{X}$ is a sandwich set in G where $o(x)$ is odd for all $x \in \overline{X}$.

Lemma 2.3 *Let $u, v, w \in \overline{X}$. Then $[u, [v, w]] = [u, v, w] \cdot [u, w, v]^{-1}$.*

Proof. As $\langle u, v, w \rangle$ is a sandwich group and as $o(u), o(v), o(w)$ are odd we know that $\langle u, v, w \rangle$ is nilpotent of class at most 3. The result now follows from this and the Hall–Witt identity. $\Box$

Our proof makes use of the notion of standards words (see for example [12]) which played a crucial role in Chanyshev's proof of the theorem on sandwich algebras by Kostrikin and Zel'manov [7]. Our proof resembles the work of Chanyshev in outline although we work with group commutators instead of words in a Lie ring.

Standard words. Let $x_1, \ldots, x_r$ be free variables. Consider the set A of all words

$$x_{i(1)} \cdots x_{i(n)},\ 1 \leq i(1), \ldots, i(n) \leq r,\ n \geq 0.$$

We order these words as follows: $x_{i(1)} \cdots x_{i(n)} < x_{j(1)} \cdots x_{j(m)}$ if either for some $t < \min\{m, n\}$ we have $x_{i(1)} = x_{j(1)}, \ldots, x_{i(t)} = x_{j(t)}$ and $x_{i(t+1)} < x_{j(t+1)}$, or $m < n$ and $x_{j(1)} = x_{i(1)}, \ldots, x_{j(m)} = x_{i(m)}$. This gives us a total order on A.

Definition. We say that a word $x_{i(1)} \cdots x_{i(n)}$ is standard if for all $2 \leq t \leq n$ we have $x_{i(t)} \cdots x_{i(n)} x_{i(1)} \cdots x_{i(t-1)} < x_{i(1)} \cdots x_{i(n)}$.

We make the use of the following property for standard words. If c is a standard word of length at least 2, then $c = ab$ for some standard words a, b where $a > b$. Among such decompositions we pick the one where a is largest. Although the choice of a is irrelevant in what follows.

Definition. To each standard word c we associate a group commutator, denoted $[c]$, recursively as follows. Firstly $[x_i] = x_i$ for $i = 1, \ldots, r$. Then if the length $l(c)$ of c is at least 2 and $c = ab$ for standard words a, b, then $[c] = [[a], [b]]$.

Definition. To each word $c = x_{i(1)} \cdots x_{i(n)}$ in A we associate the left normed commutator $\mathrm{com}(c) = [x_{i(1)}, \ldots, x_{i(n)}]$.

We now turn to the proof of Proposition 2.1. Let $G = \langle x_1, \ldots, x_r \rangle$ be a finitely generated sandwich group where $o(x_i)$ is an odd number for $i = 1, \ldots, r$. Let

$$H = Z_\infty(G) = \bigcup_{i=0}^{\infty} Z_i(G)$$

be the hyper centre of G. To show that G is nilpotent it suffices to show that $\gamma_m(G) \leq H$ for some positive integer m. We argue by contradiction and suppose this not the be the case. We then get an infinite sequence $(x_{\alpha(i)})_{i=1}^{\infty}$ where $u_n = x_{\alpha(1)} \cdots x_{\alpha(n)}$ is the smallest word in A of length n such that $\mathrm{com}(u_n) \notin H$.

Lemma 2.4 *Let $n \geq 1$ and c be a standard word of length m.*

(1) If $u_n c < u_{n+m}$ then $[com(u_n), [c]] \in H$.
(2) If $u_n c = u_{n+m}$ then $[com(u_n), [c]]H = com(u_{n+m})H$.

Proof We prove this by induction on m. For $m = 1$, the statement (2) is obvious while (1) follows directly from our choice of the sequence $(x_{\alpha(i)})_{i=0}^{\infty}$. Let $m \geq 2$ and suppose (1) and (2) hold for smaller values of m. Consider first (1). Let $c = ab$ where a, b are standard words of lengths s, t. Then

$$\begin{aligned} [\mathrm{com}(u_n), [c]] &= [\mathrm{com}(u_n), [[a], [b]]] \\ &\overset{L2.3}{=} [\mathrm{com}(u_n), [a], [b]] \cdot [\mathrm{com}(u_n), [b], [a]]^{-1}. \end{aligned}$$

As $u_n ab = u_n c < u_{n+m}$ we must have $u_n a \leq u_{n+s}$. If $u_n a < u_{n+s}$ then we have $[\mathrm{com}(u_n), [a]]H = H$ by the induction hypothesis. If on the other hand $u_n a = u_{n+s}$ then $u_{n+s} b < u_{n+m}$ and thus by the induction hypothesis

$$[\mathrm{com}(u_n), [a], [b]]H = [\mathrm{com}(u_{n+s}), [b]]H = H.$$

As $c = ab$ is standard, we have $ba < c \leq u_{n+m}$. The same argument as before gives $[\mathrm{com}(u_n), [b], [a]]H = H$. Hence it follows that $[\mathrm{com}(u_n), [c]]H = H$ and we have proved (1) for m.

We next turn to (2). As $u_n ba < u_n ab = u_n c = u_{n+m}$ the same argument as above shows that $[\mathrm{com}(u_n), [b], [a]]H = H$. As $u_n a = u_{n+s}$ and $u_{n+s} b = u_{n+m}$ it thus follows from the induction hypothesis that

$$\begin{aligned} [\mathrm{com}(u_n), [c]]H &= [\mathrm{com}(u_n), [a], [b]]H \\ &= [\mathrm{com}(u_{n+s}), [b]]H \\ &= \mathrm{com}(u_{n+m})H. \end{aligned}$$

This finishes the proof of the lemma. □

We can now finish the proof of Proposition 2.1. By Theorem 3.1.10 in [12] we know that there exists an integer $N(r)$ such that any word in A of length greater than $N(r)$ must contain a subword cc, $cx_i c$ or $x_i c x_i$ where c is a standard word. In particular, the word $x_{\alpha(2)} \cdots x_{\alpha(2+N(r))}$ must contain one of these. If c has length m, one of the following must therefore hold

$$u_{n+2m} = u_n cc; \;\; u_{n+2m+1} = u_n c x_i c; \;\text{ or }\; u_{n+m+2} = u_n x_i c x_i$$

for some positive integers n, m. Now using the fact that $\langle \mathrm{com}(u_n), [c], x_i \rangle$ is a sandwich group, and therefore nilpotent of class at most 3, we see in the first case using Lemma 2.4 that

$$H = [\mathrm{com}(u_n), [c], [c]]H = [\mathrm{com}(u_{n+m}), [c]]H = \mathrm{com}(u_{n+2m})H.$$

This gives the contradiction that $\mathrm{com}(u_{n+2m}) \in H$. Similarly for the other cases we get contradictions from

$$H = [\mathrm{com}(u_n), [c], x_i, [c]]H = \mathrm{com}(u_{n+2m+1})H, \; H = [\mathrm{com}(u_n), x_i, [c], x_i]H = \mathrm{com}(u_{n+m+2})H.$$

From these contradictions we conclude that G must be nilpotent. This finishes the proof of Theorem 1.1.

3 Some applications

Let $F = \langle x, y \rangle$ be the free group of rank 2 with free generators x, y.

Theorem 3.1 *Let w be any word in $\langle x^9 y^9 \rangle^F$. Then any 3-group satisfying the law $(x^3y^3)^3 = w$ is locally finite. In particular, the variety of groups satisfying the law $(x^3y^3)^3 = 1$ is locally finite.*

Proof Let G be any 3-group satisfying the law $(x^3y^3)^3 = w$. Notice that G is locally finite if and only if it is locally nilpotent. We want to show that $G/HP(G)$ is trivial. As local finiteness is a property that is closed under taking extensions, we can replace G by $G/HP(G)$ and assume that $HP(G) = \{1\}$. We want to show that $G = \{1\}$. As a first step we show that $G^3 = \{1\}$. We argue by contradiction and suppose we have an element $g \in G$ of order 9. Let $h = g^3$. As G satisfies $(x^3y^3)^3 = w$ we then have for all $u \in G$ that

$$\begin{aligned} 1 &= (g^{3u}g^3)^3 = (h^u h)^3 = h^{h^{2u}} h^{h^u} h \\ 1 &= (g^{-3u}g^3)^3 = (h^{-u}h)^3 = h^{h^u} h^{h^{2u}} h. \end{aligned}$$

From this it follows that h commutes with h^{h^u} and therefore also $[h^u, h]$. The same argument using g^u and u^{-1} instead of g and u shows that h^u commutes with $[h, h^u]$. Thus $\langle h, h^u \rangle$ is nilpotent of class at most 2 and therefore $[u, h, h, h] = 1$. As u was arbitrary this shows that h is a left 3-Engel element and thus, by Theorem 1.1, $h \in HP(G) = \{1\}$. We therefore get the contradiction that $g^3 = h = 1$. Having shown that $G^3 = \{1\}$ it follows by Burnside [4] that G is locally finite. Hence $G = HP(G) = \{1\}$. □

The latter part Theorem 3.1 can be strengthened.

Theorem 3.2 *Let w be a law in n variables $x_1, \ldots, x_n$ where the variety of groups satisfying the law $w^3 = 1$ is a locally finite variety of groups of exponent 9. Then the same is true for the variety of groups satisfying the law $(x_{n+1}^3 w^3)^3 = 1$.*

Proof Let G be any group satisfying the law $(x_{n+1}^3 w^3)^3 = 1$. We argue as in the proof of Theorem 3.1. We assume that $HP(G) = \{1\}$ and the aim is to show that it follows that $G = \{1\}$. Let $u, g_1, \ldots, g_n$ be arbitrary $n+1$ elements of G. Then

$$\begin{aligned} &(w(g_1, \ldots, g_n)^{3u} w(g_1, \ldots, g_n)^3)^3 = 1, \\ &(w(g_1, \ldots, g_n)^{-3u} w(g_1, \ldots, g_n)^3)^3 = 1, \end{aligned}$$

imply as in the proof of Theorem 3.1 that $[u, w(g_1, \ldots, g_n)^3, w(g_1, \ldots, g_n)^3, w(g_1, \ldots, g_n)^3] = 1$. As this is true for all $u, g_1, \ldots, g_n \in G$, it follows that $w(g_1, \ldots, g_n)^3$ is a left 3-Engel element and thus, by Theorem 1.1, in $HP(G) = \{1\}$ for all $g_1, \ldots, g_n \in G$. Hence G satisfies the law $w^3 = 1$ and is locally finite by the hypothesis of the theorem. Hence $G = \{1\}$. □

Remark. We can use Theorem 3.2 to come up with an explicit sequence of words. Define the word $w_n = w_n(x_1, \ldots, x_n)$ in n variables recursively by $w_1 = x_1$ and $w_{n+1} = x_{n+1}^3 w_n^3$. The variety of groups satisfying the law $x_1^3 = 1$ is locally finite by Burnside and by repeated application of Theorem 3.2 we see that, for each $n \geq 1$, the variety of groups satisfying the law w_n^3 is a locally finite variety of groups of exponent 9.

Corollary 3.3 *A 6-Engel 3-group is locally finite if and only if every 2-generator subgroup if finite.*

Proof. Let $E = \langle x, y \rangle$ be the largest 6-Engel group of rank 2 and let E/N be the nilpotent residual. Using GAP or MAGMA one can show that modulo $(x^9y^9)^E N$ we have $(x^3y^3)^3 = 1$. Thus we have

$$(x^3y^3)^3 = yw$$

for some $y \in N$ and $w \in (x^9y^9)^E$. Now let G be a 6-Engel 3-group where all 2-generator subgroups are finite or equivalently nilpotent. Then G satisfies the law $(x^3y^3)^3 = w$. By Theorem 3.1 it is then locally finite. □

Corollary 3.4 *Let G be a 3-group. Suppose that every 2-generator subgroup is nilpotent of class at most 9. Then G is locally finite.*

Proof We use a similar approach to the proof of Theorem 3.1. Let G be any 3-group with all 2-generator subgroups nilpotent of class at most 9. Suppose that $HP(G) = \{1\}$. We want to show that $G = \{1\}$. As a first step we show that $G^9 = \{1\}$. We argue by contradiction and suppose that there is an element g of order 27. Using GAP or MAGMA one can show that $(g^{9y}g^9)^3 = (g^{-9y}g^9)^3 = 1$ and the same argument as in the proof of Theorem 3.1 shows that g^9 is a left 3-Engel element and thus, by Theorem 1.1, in $HP(G) = \{1\}$. By this contradiction $G^9 = \{1\}$. Again GAP or MAGMA calculations show that $[x, g^3, g^3, g^3] = 1$ for all $x, g \in G$. Hence g^3 is a left 3-Engel element, and thus in $HP(G) = \{1\}$, for all $g \in G$. This shows that G is of exponent 3 and thus locally finite. We conclude that $G = \{1\}$. $\square$

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