RIGHT ENGEL-TYPE SUBGROUPS
AND LENGTH PARAMETERS OF FINITE GROUPS

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ABSTRACT. Let g be an element of a finite group G and let $R_n(g)$ be the subgroup generated by all the right Engel values $[g, {}_n x]$ over $x \in G$. In the case when G is soluble we prove that if, for some n, the Fitting height of $R_n(g)$ is equal to k, then g belongs to the $(k+1)$th Fitting subgroup $F_{k+1}(G)$. For nonsoluble G, it is proved that if, for some n, the generalized Fitting height of $R_n(g)$ is equal to k, then g belongs to the generalized Fitting subgroup $F_{f(k,m)}^*(G)$ with $f(k,m)$ depending only on k and m, where $|g|$ is the product of m primes counting multiplicities. It is also proved that if, for some n, the nonsoluble length of $R_n(g)$ is equal to k, then g belongs to a normal subgroup whose nonsoluble length is bounded in terms of k and m. Earlier similar generalizations of Baer's theorem (which states that an Engel element of a finite group belongs to the Fitting subgroup) were obtained by the first two authors in terms of left Engel-type subgroups.

1. INTRODUCTION

Recall that the n-Engel word $[y, {}_n x]$ is defined recursively by $[y, {}_0 x] = y$ and $[y, {}_{i+1} x] = [[y, {}_i x], x]$. Let G be a finite group. In [4] the authors considered the subgroups

$$E_n(g) = \langle [x, {}_n g] : x \in G \rangle,$$

where $g \in G$ and n is a positive integer. By a well-known result of Baer [2] we know that if $E_n(g) = 1$, then g belongs to the Fitting subgroup of G. The authors of [4] generalized Baer's result in terms of various length parameters of $E_n(g)$. The aim of the present paper is to consider the right Engel analog of this setting and to prove the right Engel versions of the three main theorems in [4]. Instead of $E_n(g)$ we consider

$$R_n(g) = \langle [g, {}_n x] : x \in G \rangle.$$

Before stating our first main result, we recall some terminology. For a finite group G, the Fitting series is defined recursively by $F_0(G) = 1$ and $F_{i+1}(G)/F_i(G) = F(G/F_i(G))$ where $F(H)$ is the Fitting subgroup of H. If G is soluble, then the Fitting height $h(G)$ is the least integer h such that $F_h(G) = G$. We can now state our first result that is about finite soluble groups.

Theorem 1.1. *Let g be an element of a finite soluble group G and n a positive integer. If the Fitting height of $R_n(g)$ is equal to k, then g belongs to $F_{k+1}(G)$.*

For nonsoluble finite groups we prove similar results in terms of the nonsoluble length and the generalized Fitting height of $R_n(g)$. We recall the relevant definitions. The *generalized Fitting series* is defined recursively by $F_0^*(G) = 1$ and $F_{i+1}^*(G)/F_i^*(G) = F^*(G/F_i^*(G))$, where $F^*(H)$ is the generalized Fitting subgroup of H, which is the characteristic subgroup generated by the Fitting subgroup $F(H)$ and all the subnormal quasisimple subgroups (a group Q is quasisimple if $[Q, Q] = Q$ and $Q/Z(Q)$ is simple). The

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least number h such that $F_h^*(G) = G$ is called the *generalized Fitting height* $h^*(G)$ of G. Notice that if G is soluble, then $h^*(G) = h(G)$ is the ordinary Fitting height of G. We are now ready for the statement of the second main result.

Theorem 1.2. *Let m and n be positive integers, and let g be an element of a finite group G whose order $|g|$ is the product of m primes counting multiplicities. If the generalized Fitting height of $R_n(g)$ is equal to k, then g belongs to $F_{f(k,m)}^*(G)$ where*

$$f(k, m) = ((k + 1)m(m + 1) + 2)(k + 3)/2.$$

The proof of Theorem 1.2 follows from Theorem 1.1 and our last result which involves another length parameter, the nonsoluble length. We define the *upper nonsoluble series* of a finite group G recursively as follows. Let $B_0 = 1$ and let D_0 be the soluble radical of G. Then let B_{i+1}/D_i be the product of all the normal simple non-abelian subgroups of G/D_i, and D_{i+1}/B_{i+1} the soluble radical of G/B_{i+1}. The *nonsoluble length* $\lambda(G)$ is defined to be the first integer l such that $D_l = G$, so that we have a normal series

$$1 = B_0 \leqslant D_0 < B_1 \leqslant D_1 \cdots < B_l \leqslant D_l = G.$$

It is not difficult to see that $\lambda(G)$ is the minimum number of nonsoluble factors in a normal series where each factor is either soluble or a direct product of non-abelian simple groups. In particular, a group is soluble if and only if the nonsoluble length is 0. We can now state our final main result.

Theorem 1.3. *Let m and n be positive integers, and let g be an element of a finite group G whose order $|g|$ is the product of m primes counting multiplicities. If the nonsoluble length of $R_n(g)$ is equal to k, then g belongs to $D_{f_1(k,m)}(G)$ where $f_1(k, m) = (k+1)m(m+1)/2$.*

Theorems 1.2 and 1.3 depend on the classification of finite simple groups in so far as they rely on the validity of the Schreier conjecture that the outer automorphism group of a finite simple group is soluble. Bounds for the nonsoluble length and generalized Fitting height of a finite group were implicitly used in the reduction of the Restricted Burnside Problem to soluble and nilpotent groups in the Hall–Higman paper [1]. Such bounds also find applications in the study of profinite groups; see for example [3, 5].

It is well-known that the inverse of a right Engel element is a left Engel element. But there is no straightforward connection between the right and left Engel-type subgroups $R_n(G)$ and $E_n(G)$. Therefore the results of the current paper do not follow directly from the results on the subgroups $E_n(G)$ in [4], although the proofs make use of several auxiliary propositions from that paper.

2. Preliminaries

First we mention some elementary radical properties of the subgroups $F_i(G)$, $F_i^*(G)$, and $D_i(G)$ that will be used throughout the paper. If N is a normal subgroup of G, then

$$F_i(N) = N \cap F_i(G), \quad F_i^*(N) = N \cap F_i^*(G), \quad D_i(N) = N \cap D_i(G).$$

It follows that the same equalities hold whenever N is a subnormal subgroup of G. Therefore the Fitting height (when G is soluble), the generalized Fitting height, and the nonsoluble length of N do not exceed the corresponding parameters of G when N is subnormal in G.

Let $S_i(G)$ be one of $F_i(G)$, $F_i^*(G)$, or $D_i(G)$. If N is a subnormal subgroup of G, then for the normal closures in G we have

$$\langle S_i(N)^G \rangle = \langle (N \cap S_i(G))^G \rangle \leqslant \langle N^G \rangle \cap S_i(G) = S_i(\langle N^G \rangle).$$

Since N is subnormal in $\langle N^G \rangle$, we also have $S_i(N) = N \cap S_i(\langle N^G \rangle)$. Hence the Fitting height (when G is soluble), the generalized Fitting height, and the nonsoluble length of N are the same as for $\langle N^G \rangle$.

We collect these properties in the following lemma for further references.

Lemma 2.1. *Let N be a subnormal subgroup of a finite group G.*

(a) The Fitting height (when G is soluble), the generalized Fitting height, and the non-soluble length of N do not exceed the corresponding parameters of G.

(b) The Fitting height (when N is soluble), the generalized Fitting height, and the nonsoluble length of the normal closure $\langle N^G \rangle$ are equal to the corresponding parameters of N.

Furthermore, $F_i(G) \cap H \leqslant F_i(H)$ whenever $H \leqslant G$, and for any normal subgroup N of G we have

$$F_i(G/N) \geqslant F_i(G)N/N, \quad F_i^*(G/N) \geqslant F_i^*(G)N/N, \quad D_i(G/N) \geqslant D_i(G)N/N.$$

When a group A acts by automorphisms on a group G, we regard A as a subgroup of the semidirect product GA and use the commutator and centralizer notation like $[G, A]$ and $C_G(A)$. We use without special references the well-known properties of coprime actions: if α is an automorphism of a finite group G of coprime order, $(|\alpha|, |G|) = 1$, then $C_{G/N}(\alpha) = C_G(\alpha)N/N$ for any α-invariant normal subgroup N, the equality $[G, \alpha] = [[G, \alpha,], \alpha]$ holds, and if G is in addition abelian, then $G = [G, \alpha] \times C_G(\alpha)$.

It is convenient to introduce a more specialized notation for right Engel-type subgroups. Let G be a group and $g \in G$. For a g-invariant subgroup H of G we set

$$R_{H,n}(g) = \langle [g, {}_n x] : x \in H \rangle.$$

Thus, for $H = G$ this is the subgroup $R_n(g) = \langle [g, {}_n x] : x \in G \rangle$ introduced above. It is clear from the definition that for g-invariant subgroups $K \leqslant H$ we have $R_{K,n}(g) \leqslant R_{H,n}(g)$, and if N is a g-invariant normal subgroup of H, then $R_{HN/N,n}(\bar{g})$ is the image of $R_{H,n}(g)$ in H/N. Another obvious property is that $C_H(g)$ normalizes $R_{H,n}(g)$. We shall use these properties of right Engel-type subgroups without special references.

We record here a simple property of abelian-by-cyclic subgroups.

Lemma 2.2. *If A is an abelian subgroup of a group G normalized by an element $g \in G$, then for any $a \in A$ we have $[g, {}_n ga] = [a^{-1}, {}_n g]$.*

Proof. We have $[g, ga] = [ga, g]^{-1} = [a, g]^{-1} = [a^{-1}, g]$ since A is abelian. For the same reason, the action of ga on A is the same as that of g, whence the result. $\square$

We shall also need a couple of lemmas on direct products of nonabelian finite simple groups; the first lemma was proved in [4].

Lemma 2.3 ([4, Lemma 3.8]). *Let $S = S_1 \times \cdots \times S_p$ be a direct product of p isomorphic non-abelian finite simple groups, where p is a prime, and let φ be the natural automorphism of S of order p that regularly permutes the S_i. Let n be a positive integer, and let*

$$F = \langle [x, {}_n \varphi] : x \in S_i, \ i = 1, 2, \ldots, r \rangle.$$

Then $F = S$.

As a corollary we get the following.

Lemma 2.4. *Let* $S = S_1 \times \cdots \times S_r$ *be a direct product of* r *isomorphic non-abelian finite simple groups, and let* φ *be the natural automorphism of* S *of order* r *that regularly permutes the* S_i. *Let* n *be a positive integer. Then* $S = R_{S\langle\varphi\rangle,n}(\varphi)$.

Proof. Let φ^k be of prime order (possibly, with $k = 1$). By Lemma 2.3 applied to each orbit of φ^k, we have

$$S = \langle [g, {}_n\varphi^k] : \, g \in S_i, \ i = 1, 2, \ldots, r \rangle. \tag{2.1}$$

For any $1 \leqslant i \leqslant r$ and any $g \in S_i$, the subgroup $A(g) = \langle g^{\langle\varphi\rangle} \rangle$ is abelian and φ-invariant. Using this fact and the commutator formulae $[b, \varphi^{i+1}] = [b, \varphi^i][b^{\varphi^i}, \varphi]$ and $[bc, \varphi^i] = [b, \varphi^i][c, \varphi^i]$ that hold for all $b, c \in A(g)$, we see that any commutator $[g, {}_n\varphi^k]$ belongs to the subgroup generated by commutators of the form $[x, {}_n\varphi]$ with $x \in A(g)$. By Lemma 2.2 we also have $[x, {}_n\varphi] = [\varphi, {}_n\varphi x^{-1}] \in R_{S\langle\varphi\rangle,n}(\varphi)$. Combined with (2.1), this proves that $S = R_{S\langle\varphi\rangle,n}(\varphi)$. $\qquad\square$

3. FITTING HEIGHT

In this section we prove Theorem 1.1 on finite soluble groups.

Proposition 3.1. *Let* α *be an automorphism of a finite soluble group* G *such that* $G = [G, \alpha]$. *Then* $G = R_{G\langle\alpha\rangle,n}(\alpha)$ *for all* $n \geqslant 1$.

Proof. Let $R = R_{G\langle\alpha\rangle,n}(\alpha)$. We argue by contradiction and let G be a counterexample of minimal order. Let M be a minimal normal α-invariant subgroup of G (and thus an elementary abelian p-group for some prime p). By the minimality of G we have $G/M = R_{G\langle\alpha\rangle/M,n}(\bar\alpha) = MR/M$ (where $\bar\alpha$ is the automorphism of G/M induced by α), and thus $G = MR$. Since $G \neq R$, we must have $M \not\leqslant R$ and therefore, $M \cap R \neq M$. Since M is abelian and normal, it follows that $M \cap R$ is normal in $MR = G$. Then by the minimality of M we must have $M \cap R = 1$. Thus,

$$G = MR \qquad \text{and} \qquad M \cap R = 1.$$

We claim that $C_M(\alpha) = 1$. If this is not the case, then, as $C_M(\alpha)$ normalizes R, we have $[C_M(\alpha), R] \leqslant M \cap R = 1$, so that $C_M(\alpha)$ centralizes R as well as M. Hence $C_M(\alpha)$ is an α-invariant subgroup contained in the centre of $G = MR$, and by the minimality of M it follows that $C_M(\alpha) = M$. But then $G = [G, \alpha] = [R, \alpha] \leqslant R$, a contradiction.

Thus, $C_M(\alpha) = 1$, which implies that the mapping $M \to M$ given by $m \mapsto [m, \alpha]$ is injective. Since M is finite, it follows that this mapping is surjective, and therefore every element in M is of the form $[m, {}_n\alpha]$ for some $m \in M$. Hence, using Lemma 2.2, we obtain

$$M = \langle [m, {}_n\alpha] : \, m \in M \rangle = \langle [\alpha, {}_n\alpha m] : \, m \in M \rangle \leqslant R.$$

Then $G = MR = R$, a contradiction. $\qquad\square$

Proof of Theorem 1.1. Let g be an element of a finite soluble group G, and n a positive integer. Suppose that the Fitting height of $R_n(g)$ is equal to k. We want to show that g then belongs to $F_{k+1}(G)$.

Consider the subnormal series

$$G \geqslant [G, g] \geqslant [[G, g], g] \geqslant \cdots \geqslant [\ldots[[G, \underbrace{g], g], \ldots, g}_{m}] = H,$$

where $[H, g] = H$ (can also be trivial). Let $N = \langle H^G \rangle$ be the normal closure of H. Since gN is a left Engel element in G/N, we know from Baer's theorem that $gN \in F(G/N)$. Hence it suffices to show that N is of Fitting height at most k. Since H is subnormal in G, by Lemma 2.1(b) it suffices to show that H has Fitting height at most k. Since $H = [H, g]$, we know from Proposition 3.1 that $H = R_{H\langle g\rangle,n}(g)$. We have $R_{H\langle g\rangle,n}(g) \leqslant R_n(g)$, and by hypothesis $R_n(g)$ has Fitting height k. Therefore H has Fitting height at most k, as required. $\qquad\square$

4. Nonsoluble length

In this section we prove Theorem 1.3 on nonsoluble length. First we introduce some notation. Suppose that the nonsoluble length of a finite group G is $\lambda(G) = \lambda$. This means that we have the upper nonsoluble series

$$1 = B_0 \leqslant D_0 < B_1 \leqslant D_1 \cdots < B_\lambda \leqslant D_\lambda = G. \tag{4.1}$$

Recall that each factor D_j/B_j is the soluble radical (possibly, trivial) of G/B_j. We fix the notation $U_j = B_j/D_{j-1}$ for the other factors, each of which is the (nontrivial) direct product of all subnormal nonabelian simple subgroups of G/D_{j-1}, say, $U_j = S_1 \times \cdots \times S_v$ (to lighten the notation, we do not use double indices for S_i here, but the groups and the number of factors may of course be different for different j). Acting by conjugation the group G permutes these subnormal factors S_i; for brevity we speak of *orbits of elements of G on U_j* meaning orbits in this permutational action. The stabilizer of a point S_i can also be denoted as the normalizer $N_G(S_i)$ of the section S_i. Let K_j be the kernel of the permutational action of G on $\{S_1, \ldots, S_v\}$. Clearly, $B_j \leqslant K_j$, and since K_j/B_j is soluble by the Schreier conjecture, we have $K_j \leqslant D_j$.

The following proposition is the main tool in the proof of Theorem 1.3; it is a right Engel analogue of Proposition 5.4 in [4].

Proposition 4.1. *Let g be an element of a finite group G whose order $|g|$ is a product of $m > 1$ primes. Suppose that $\langle g \rangle \cap K_{ms} = 1$ for some positive integer s. Then for any positive integer n the nonsoluble length of $R_n(g)$ is at least s.*

Before embarking on the proof of this proposition, we introduce some definitions. Let g be an element of G, and let $\{S_1, \ldots, S_r\}$ be a g-orbit on U_i, that is, on the set of simple factors the product of which is U_i. As in [4], we say that such an orbit is *pure* if the order of the automorphism of $S = S_1 \times \cdots \times S_r$ induced by g acting by conjugation is equal to r, which is the order of the permutation induced by g on this orbit. In other words, the orbit is pure if the stabilizer of a point in $\langle g \rangle$ acts trivially on S, that is, $N_{\langle g \rangle}(S_1) = C_{\langle g \rangle}(S)$. We introduce the following hypothesis, which is the same as Hypothesis 5.1 in [4].

Hypothesis 4.2. *Let g be an element of a finite group G, and let $\{S_1, \ldots, S_r\}$ be a g-orbit in a section $U_i = B_i/D_{i-1}$ of the series (4.1) with $i \geqslant 2$. Suppose that $\langle g \rangle \cap B_i = 1$, and that g acting by conjugation induces a nontrivial automorphism $\bar{g}$ of $S = S_1 \cdots S_r$. Let $t = |\bar{g}|$, so that $\langle g^t \rangle$ is the centralizer of S in $\langle g \rangle$ and $\langle \bar{g} \rangle = \langle g \rangle / \langle g^t \rangle$.*

Furthermore, let $\hat{S}$ be a minimal by inclusion g-invariant subgroup of G such that $\hat{S}D_{i-1}/D_{i-1} = S$. If in addition $\{S_1, \ldots, S_r\}$ is a pure g-orbit, then we choose $\hat{S}$ to be contained in $R_n(g)$, which is possible by Lemma 2.4. Let $\hat{S}_1$ be a minimal by inclusion subgroup of $\hat{S}$ such that $\hat{S}_1D_{i-1}/D_{i-1} = S_1$.

We use two technical propositions from [4]. The first one provides a passage from a pure orbit of g on U_i to an orbit of at least the same length on the preceding section U_{i-1}.

Proposition 4.3 ([4, Proposition 5.2]). *Assume Hypothesis* 4.2. *If* $\{S_1, \ldots, S_r\}$ *is a pure g-orbit, then* U_{i-1} *contains a g-orbit* $\{T_1, \ldots, T_l\}$ *of length divisible by* $r = |\bar{g}|$, *and for some choice of* $\hat{S}_1$ *there is an element* $x_0 \in \hat{S}_1 \setminus D_{i-1}$ *that does not belong to* $N_G(T_1)$.

The second technical proposition from [4] provides a passage from a non-pure orbit of g in U_i to an orbit in U_{i-1} that is strictly greater with respect to the following ordering. Namely, on the set of positive integers we introduce the lexicographical order with respect to the exponents of primes in the canonical prime-power decomposition: if $a = 2^{i_2} 3^{i_3} 5^{i_5} \cdots$ and $b = 2^{j_2} 3^{j_3} 5^{j_5} \cdots$, then by definition $a \prec b$ if, for some prime p, we have $i_q = j_q$ for all primes $q < p$ and $i_p < j_p$. In particular, if v is a proper divisor of u, then $v \prec u$.

Proposition 4.4 ([4, Proposition 5.3]). *Assume Hypothesis* 3.1. *If* $\{S_1, \ldots, S_r\}$ *is a non-pure g-orbit, then* U_{i-1} *contains a g-orbit* $\{T_1, \ldots, T_l\}$ *of length strictly greater than r with respect to the order* $\prec$.

Proof of Proposition 4.1. We have an element $g \in G$ of order equal to a product of m primes such that $\langle g \rangle \cap K_{ms} = 1$. (Recall that K_j is the kernel of the permutational action of G on the simple factors whose product is U_j.) We want to show that the nonsoluble length of $R_n(g)$ is at least s. With this in mind, we consider the upper nonsoluble series for $R_n(g)$ constructed in the same way as (4.1) was constructed for G, with its terms denoted by

$$1 = \beta_0 \leqslant \delta_0 < \beta_1 \leqslant \delta_1 \cdots < \beta_e \leqslant \delta_e = R_n(g). \tag{4.2}$$

Here each factor δ_j / β_j is the soluble radical (possibly, trivial) of G/β_j, and each factor β_j / δ_{j-1} is the (nontrivial) direct product of all subnormal nonabelian simple subgroups of G/δ_{j-1}. Our task is to show that $e \geqslant s$.

Since $\langle g \rangle \cap K_{ms} = 1$, the element g has at least one nontrivial orbit on the set of simple factors of $U_{ms} = B_{ms}/D_{ms-1}$, say, $\{S_1, \ldots, S_r\}$. If the orbit is pure, then we apply Proposition 4.3 to this orbit. If the orbit is not pure, then we apply Proposition 4.4. Then we apply the same procedure to the orbit $\{T_1, \ldots, T_l\}$ in U_{ms-1} just obtained: this orbit takes the role of the orbit $\{S_1, \ldots, S_r\}$ in Proposition 4.3 or 4.4 depending on whether it is pure or not. We proceed with constructing this sequence of orbits, descending over the sections U_i making $ms - 1$ such steps. If we make such a step from a pure orbit by Proposition 4.3, then the length of the new orbit is divisible by the length of the old orbit and therefore does not decrease with respect to the order $\prec$. If we make such a step from a non-pure orbit by Proposition 4.4, then the length of the new orbit is strictly greater with respect to the order $\prec$ than the length of the old orbit. In the sequence of orbits thus constructed, some orbits may be pure, some not. Notice that there can be at most $m - 1$ passages from a non-pure orbit, since $|g|$ is a product of m primes. As a result, if the length of the sequence is at least $(s - 1)m + (m - 1) + 1 = ms$, then this sequence must necessarily contain s successive pure orbits, with $s - 1$ consecutive passages between them. Let $i = t, t - 1, \ldots, t - s + 1$ be the indices of the corresponding sections U_i.

Recall that if a g-orbit $\{S_1, \ldots, S_r\}$ in U_i is pure, then by Lemma 2.4 the subgroup $S = S_1 \cdots S_r$ is contained in the image of $R_n(g)$ in G/D_{i-1} (since this image is obviously equal to the analogous subgroup $R_n(g)$ constructed for G/D_{i-1}). The idea is to use each of these s consecutive pure orbits in $U_t, U_{t-1}, \ldots, U_{r-s+1}$ to 'mark' a nonsoluble factor of the series (4.2) and prove that the factor marked by the pure orbit in U_{i-1} is necessarily 'lower'

in (4.2) than the factor marked by the pure orbit in U_i, for every $i = t, t-1, \ldots, t-s+2$. Then the series (4.2) must realize nonsoluble length at least s.

Thus, let $\{S_1, \ldots, S_r\}$ be a pure g-orbit in U_i, and $\{T_1, \ldots, T_l\}$ the pure g-orbit in U_{i-1} obtained by Proposition 4.3. Recall that in accordance with Hypothesis 3.1, $\hat{S}$ is a minimal by inclusion g-invariant subgroup of $R_n(g)$ such that $S = \hat{S}D_{i-1}/D_{i-1}$, and $\hat{S}_1$ is a minimal by inclusion subgroup of $\hat{S}$ such that $\hat{S}_1 D_{i-1}/D_{i-1} = S_1$. Note that since S_1 is nonabelian simple, $\hat{S}_1$ has no nontrivial soluble homomorphic images:

$$\hat{S}_1 = [\hat{S}_1, \hat{S}_1]. \tag{4.3}$$

By Proposition 4.3 we have an element $x_0 \in \hat{S}_1 \setminus D_{i-1}$ such that

$$T_1^{x_0} \neq T_1. \tag{4.4}$$

Consider the image of the series (4.2) in $R_n(g)D_{i-1}/D_{i-1}$. Since $\hat{S}_1 D_{i-1}/D_{i-1} \cong S_1$ is a subnormal nonabelian simple subgroup of $R_n(g)D_{i-1}/D_{i-1}$, we obviously have a well-defined index j such that

$$\hat{S}_1 \leqslant \beta_j D_{i-1} \qquad \text{and} \qquad \hat{S}_1 \not\leqslant \delta_{j-1} D_{i-1}. \tag{4.5}$$

Notice that then also

$$\hat{S}_1 \leqslant \beta_j(D_{i-1} \cap R_n(g)). \tag{4.6}$$

It is clear that the index j depends only on S_1 (that is, it is independent of the choice of $\hat{S}_1$ such that $\hat{S}_1 D_{i-1}/D_{i-1} = S_1$). Since $\{T_1, \ldots, T_l\}$ is a pure g-orbit in U_{i-1}, the product $T = T_1 \cdots T_l$ is also covered by $R_n(g)$ by Lemma 2.4. If $\hat{T}_1$ is any subgroup of $R_n(g)$ such that $\hat{T}_1 D_{i-2}/D_{i-2} = T_1$, then again, there is a well-defined index u depending only on T_1 such that

$$\hat{T}_1 \leqslant \beta_u(D_{i-2} \cap R_n(g)) \qquad \text{and} \qquad \hat{T}_1 \not\leqslant \delta_{u-1} D_{i-2}. \tag{4.7}$$

We will need the following lemma to finish the proof of Proposition 4.1.

Lemma 4.5. *Under the above hypotheses, $j > u$.*

Proof. We argue by contradiction and suppose that $j \leqslant u$. Then (4.6) implies that $\hat{S}_1 \leqslant \beta_u(D_{i-1} \cap R_n(g))$. The image of β_u/δ_{u-1} in $R_n(g)/(D_{i-2} \cap R_n(g))$ is a direct product of nonabelian simple groups, one of which is $\bar{T}_1 = \hat{T}_1/(\hat{T}_1 \cap D_{i-2}) \cong T_1$ by (4.7). Acting by conjugation the group $R_n(g)$ permutes these factors. Consider the permutational action of $R_n(g)$ on the orbit containing $\bar{T}_1$. Clearly, β_u is contained in the kernel of this action. The subgroup $B_{i-1} \cap R_n(g)$ normalizes $\hat{T}_1$ modulo $D_{i-2} \cap R_n(g)$ (since B_{i-1} normalizes $\hat{T}_1$ modulo D_{i-2}). As a normal subgroup of $R_n(g)$, then $B_{i-1} \cap R_n(g)$ is contained in the kernel of that action on the orbit containing $\hat{T}_1$. Since D_{i-1}/B_{i-1} is soluble, we obtain that the image of $\hat{S}_1 \leqslant \beta_u(D_{i-1} \cap R_n(g))$ in this action is soluble. But $\hat{S}_1 = [\hat{S}_1, \hat{S}_1]$ by minimality of $\hat{S}_1$, as noted in (4.3). Therefore this image must actually be trivial; in particular,

$$\hat{T}_1^{x_0} \equiv \hat{T}_1 \pmod{\delta_{u-1}(D_{i-2} \cap R_n(g))}. \tag{4.8}$$

On the other hand, since $T_1^{x_0} \neq T_1$ by (4.4), it follows that $[\hat{T}_1, \hat{T}_1^{x_0}] \leqslant D_{i-2} \cap R_n(g)$. Together with (4.8) this implies

$$\bar{T}_1 = [\bar{T}_1, \bar{T}_1] \equiv 1 \pmod{\delta_{u-1}(D_{i-2} \cap R_n(g))},$$

contrary to (4.7). Lemma 4.5 is proved. □

We now finish the proof of Proposition 4.1. In our sequence of orbits constructed above, each pure orbit $\{S_1, \ldots, S_r\}$ in U_i for $i = t, \ldots, t - s + 1$ marks a nonsoluble quotient β_j / δ_{j-1} of the series (4.2) in the sense of (4.5). By Lemma 4.5 the next pure orbit in U_{i-1} marks a strictly lower section. Therefore there must be at least s different nonsoluble sections in (4.2), since their indices must be strictly descending as we go over the s consecutive pure orbits. As a result, the nonsoluble length of $R_n(g)$ is at least s. $\square$

We are now ready to finish the proof of Theorem 1.3.

Proof of Theorem 1.3. Recall that we have $\lambda(R_n(g)) = k$ and $|g|$ is a product of m primes counting multiplicities. We want to show that $g \in D_{(k+1)m(m+1)/2}(G)$. We proceed by induction on m. As a base of this induction we can take the case of $m = 0$, when $g = 1$ and certainly $1 \in D_{(k+1)\cdot 0 \cdot (0+1)/2}(G) = D_0(G)$.

For $m > 0$, by Proposition 4.1 we have

$$\langle g \rangle \cap D_{(k+1)m} \geqslant \langle g \rangle \cap K_{(k+1)m} \neq 1.$$

The corresponding subgroup $R_n(\bar{g})$ constructed for the image $\bar{g}$ of g in $\bar{G} = G/D_{(k+1)m}$ is clearly a quotient of $R_n(g)$ and therefore has nonsoluble length at most k. As $|\bar{g}|$ is a product of at most $m - 1$ primes, we have $\bar{g} \in D_{(k+1)(m-1)m/2}(\bar{G})$ by the induction hypothesis. The result now follows, since $D_{(k+1)(m-1)m/2}(\bar{G})$ is the image of $D_{(k+1)m(m+1)/2}$ in $G/D_{(k+1)m}$. $\square$

5. Generalized Fitting height

In this section we prove Theorem 1.2 on generalized Fitting height $h^*(G)$. This is achieved by combination of Theorem 1.3 on nonsoluble length and Theorem 1.1 on Fitting height.

Proof of Theorem 1.2. Recall that g is an element of a finite group G whose order is a product of m primes such that the generalized Fitting height of the right Engel sink $R_n(g)$ is equal to k. We need to show that g belongs to $F^*_{((k+1)m(m+1)+2)(k+3)/2}(G)$. As in the proof of Theorem 1.1, we consider the subnormal series

$$G \geqslant [G, g] \geqslant [[G, g], g] \geqslant \cdots \geqslant [\ldots[[G, \underbrace{g], g], \ldots, g}_{m}] = H,$$

where $[H, g] = H$ (can also be trivial). Let $N = \langle H^G \rangle$ be the normal closure of H. Since gN is a left Engel element in G/N, we know from Baer's theorem that $gN \in F(G/N)$. Hence it suffices to show that the generalized Fitting height of N is at most $((k+1)m(m+1)+2)(k+3)/2 - 1$. Since H is subnormal in G, by Lemma 2.1(b) it suffices to obtain the given bound for the generalized Fitting height of H.

Since the nonsoluble length of $R_n(g)$ does not exceed its generalized Fitting height, we know from Theorem 1.3 that g belongs to $D_{(k+1)m(m+1)/2}(G)$. Since $H = [H, g]$, it follows that $H \leqslant D_{(k+1)m(m+1)/2}(G)$. Since H is a subnormal subgroup, it follows that $\lambda(H) \leqslant (k+1)m(m+1)/2$ by Lemma 2.1(a). It remains to obtain the necessary bounds for the Fitting height of the $(k+1)m(m+1)/2+1$ soluble factors of the upper nonsoluble series of H, which are g-invariant (because $H = [H, g]$ is g-invariant).

Let $X = Y/Z$ be one of those soluble factors, where Y and Z are g-invariant and normal in H and therefore subnormal in G. Since Y is subnormal, we have $h^*(R_n(g) \cap Y) \leqslant h^*(R_n(g)) = k$ by Lemma 2.1(a). Then the image of $R_n(g) \cap Y$ in Y/Z, which is soluble,

has Fitting height at most k. Since, clearly, $R_{Y\langle g\rangle,n}(g) \leqslant R_n(g) \cap Y$, we obtain that $R_{X\langle g\rangle,n}(g)$, which is the image of $R_{Y\langle g\rangle,n}(g)$, also has Fitting height at most k.

Consider the semidirect product $X \rtimes \langle g\rangle$. By applying Theorem 1.1 we obtain that $g \in F_{k+1}(X \rtimes \langle g\rangle)$. Therefore, $[X,g] \leqslant F_{k+1}(X \rtimes \langle g\rangle) \cap X \leqslant F_{k+1}(X)$. In other words, g acts trivially on $X/F_{k+1}(X)$. Since $H = [H,g]$, it follows that H also acts trivially on $X/F_{k+1}(X)$, that is, $X/F_{k+1}(X)$ is a central section of H. In particular, $X = F_{k+2}(X)$ and thus the Fitting height of X is at most $k + 2$. Hence the generalized Fitting height of H is at most

$$((k+1)m(m+1)/2+1)(k+2)+(k+1)m(m+1)/2 = ((k+1)m(m+1)+2)(k+3)/2-1,$$

as required. $\square$

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