On $(n+\frac{1}{2})$-Engel groups

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Abstract

Let n be a positive integer. We say that a group G is an $(n+\frac{1}{2})$-Engel group if it satisfies the law $[x,_n y, x] = 1$. The variety of $(n+\frac{1}{2})$-Engel groups lies between the varieties of n-Engel groups and $(n+1)$-Engel groups. In this paper we study these groups and in particular we prove that all $(4+\frac{1}{2})$-Engel $\{2,3\}$-groups are locally nilpotent. We also show that if G is a $(4+\frac{1}{2})$-Engel p-group where $p \geq 5$ is a prime, then G^p is locally nilpotent.

1 Introduction

Let G be a group and $g, h \in G$. The commutator of g and h is the element $[g, h] = g^{-1}h^{-1}gh \in G$. We define recursively $[g,_n h]$ where n is a positive integer as follows: $[g,_1 h] = [g, h]$ and $[g,_{n+1} h] = [[g,_n h], h]$ for $n \geq 1$. A subset $S \subseteq G$ is an Engel set of G if for every $g, h \in S$ there is a positive integer $k = k(g, h)$ such that $[g,_k h] = 1$. If k is bounded above by some positive integer n we say that S is an n-Engel subset and if furthermore $G = S$, then G is an n-Engel group. Recall that every 2-Engel group is nilpotent of class at most 3. By a classic result of Heineken [4] every 3-Engel group is locally nilpotent and this result was later generalized to include all 4-Engel groups [3] (see also [10]).

Recall that an element $a \in G$ is said to be left n-Engel if $[x,_n a] = 1$ for all $x \in G$ ad right n-Engel if $[a,_n x] = 1$ for all $x \in G$. We denote the subset of left n-Engel elements by $L_n(G)$ and the right n-Engel elements by $R_n(G)$.

Definition. Let G be a group and n a positive integer.

(1) We say that $a \in G$ is a left $(n+\frac{1}{2})$-Engel element if $[x,_n a, x] = 1$ for all $x \in G$.

(2) We say that $a \in G$ is a right $(n+\frac{1}{2})$-Engel element if $[a,_n x, a] = 1$ for all $x \in G$.

(3) We say that G is an $(n+\frac{1}{2})$-Engel group if it satisfies the law $[x,_n y, x] = 1$.

We denote the subset of left $(n+\frac{1}{2})$-Engel elements by $L_{n+\frac{1}{2}}(G)$ and the right $(n+\frac{1}{2})$-Engel elements by $R_{n+\frac{1}{2}}(G)$. Thus G is an $(n+\frac{1}{2})$-Engel group if and only if $L_{n+\frac{1}{2}}(G) = G$ or equivalently $R_{n+\frac{1}{2}}(G) = G$. We denote the variety of m-Engel groups by $\mathcal{E}_m$.

Remark. It is not difficult to prove that $L_{1+\frac{1}{2}}(G) = R_2(G)$ and that $R_{1+\frac{1}{2}}(G) = L_2(G)$. Thus in particular $\mathcal{E}_{1+\frac{1}{2}} = \mathcal{E}_2$.

Lemma 1.1. *Let G be a group and n a positive integer. We have $L_n(G) \subseteq L_{n+\frac{1}{2}}(G) \subseteq L_{n+1}(G)$. In particular $\mathcal{E}_n \subseteq \mathcal{E}_{n+\frac{1}{2}} \subseteq \mathcal{E}_{n+1}$.*

Proof That $L_n(G) \subseteq L_{n+\frac{1}{2}}(G)$ is obvious. To see that $L_{n+\frac{1}{2}}(G) \subseteq L_{n+1}(G)$, let $a \in L_{n+\frac{1}{2}}(G)$. Then for any $x \in G$ we have

$$1 = [ax,_n a, ax] = [x,_n a, x][x,_{n+1} a]^x = [x,_{n+1} a]^x.$$

Thus $[x,_{n+1} a] = 1$ and $a \in L_{n+1}(G)$. $\Box$

Our main results on $(n+\frac{1}{2})$-Engel groups are the following.

Theorem B. Let G be a $(4+\frac{1}{2})$-Engel $\{2,3\}$-group. Then G is locally nilpotent.

Theorem C. Let G be a $(4+\frac{1}{2})$-Engel p-group where p is a prime and $p \geq 5$. Then G^p is locally nilpotent.

A major ingredient to the proofs is a result on Engel sets that is also of independent interest. Let $e_n = [x,_n y]$ be the n-Engel word.

Theorem A. Let $R = \langle a, b \rangle$ be the largest 2-generator group satisfying the relations $e_3(a,b) = e_3(b,a) = e_3(a^{-1}, b^{-1}) = e_3(b^{-1}, a^{-1}) = 1$. Then R is nilpotent of class 4.

We will see later that these relations imply that $S = \{a, b, a^{-1}, b^{-1}\}$ is a 3-Engel subset of R.

2 Proof of Theorem A

Consider the n-Engel word $e_n(x,y) = [x,_n y] = 1$. As we will focus in particular on the 3-Engel word we will often use $e(x,y)$ instead of $e_3(x,y)$.

Lemma 2.1. *Suppose G is a group with elements a, b where $e(a,b) = e(a^{-1}, b^{-1}) = 1$. Then $\langle b, b^a \rangle = \langle b, [a,b] \rangle$ is nilpotent of class at most 2.*

Proof From the equations

$$\begin{aligned} 1 &= [a,b,b,b] = [b^{-a}b, b, b] = [b^{-a}, b, b]^b \\ 1 &= [a^{-1}, b^{-1}, b^{-1}, b^{-1}] = [b^{a^{-1}}, b^{-1}, b^{-1}]^{b^{-1}}, \end{aligned}$$

we see that $1 = [b^{-a}, b, b]$ and $1 = [b^{a^{-1}}, b^{-1}, b^{-1}]^a = [b, b^{-a}, b^{-a}]$. Thus $\langle b, b^a \rangle$ is nilpotent of class at most 2. $\Box$

Lemma 2.2. *Let G be a group with elements a, b where $e(a,b) = e(a^{-1}, b^{-1}) = 1$. Then $[a^{\epsilon}, b^{\epsilon_1}, b^{\epsilon_2}, b^{\epsilon_3}] = 1$ for all $\epsilon, \epsilon_1, \epsilon_2, \epsilon_3 \in \{1, -1\}$.*

Proof By symmetry it suffices to deal with the case when $\epsilon = 1$. As $[a, b^{\epsilon_1}, b^{\epsilon_2}, b^{-1}] = [a, b^{\epsilon_1}, b^{\epsilon_2}, b]^{-b^{-1}}$ we can also assume that $\epsilon_3 = 1$. Then from

$$[a, b^{\epsilon_1}, b^{-1}, b] = [[a, b^{\epsilon_1}, b]^{-1}, b]^{b^{-1}} = [a, b^{\epsilon_1}, b, b]^{-[a,b^{\epsilon_1},b]^{-1}b^{-1}},$$

we can also without loss of generality assume that $\epsilon_2 = 1$. We are thus only left with showing that $[a, b^{-1}, b, b] = 1$ but this follows from Lemma 2.1 and the fact that $[a, b^{-1}, b, b] = [b^a b^{-1}, b, b] \in \gamma_3(\langle b, b^a\rangle)$. □

Remark. It follows from Lemma 2.2 that if $e(a, b) = e(a^{-1}, b^{-1}) = e(b, a) = e(b^{-1}, a^{-1}) = 1$, then $S = \{a, b, a^{-1}, b^{-1}\}$ is a 3-Engel subset.

Lemma 2.3. *Let G be a group with elements a, b satisfying $e(a, b) = e(a^{-1}, b^{-1}) = 1$. Then $[[a, b, b]^{-1}, a]$ commutes with b^a.*

Proof From the Hall-Witt identity and Lemma 2.1 we have

$$\begin{aligned} 1 &= [[a, b, b], a^{-1}, b]^a [a, b^{-1}, [a, b, b]]^b [b, [a, b, b]^{-1}, a]^{[a,b,b]} \\ &= [a, b, b, a^{-1}, b]^a = [[a, b, b, a]^{-1}, b^a]. \end{aligned}$$

It follows that $[a, b, b, a]$ commutes with b^a and thus, using Lemma 2.1 again, $[[a, b, b]^{-1}, a] = [a, b, b, a]^{-[a,b,b]^{-1}}$ commutes with b^a as well. □

Proof of Theorem A. Let $R = \langle a, b\rangle$ be the largest group satisfying the relations $e(a, b) = e(b, a) = e(a^{-1}, b^{-1}) = e(b^{-1}, a^{-1}) = 1$. By the remark after Lemma 2.2 we know that $S = \{a, b, a^{-1}, b^{-1}\}$ is a 3-Engel subset of R.

In order to show that R is nilpotent of class at most 4 we need to show that $a, b \in Z_4(R)$. This is equivalent to showing that $[b, a] \in Z_3(R)$. As $\langle a, [a, b]\rangle$ and $\langle b, [a, b]\rangle$ are nilpotent of class at most 2, we see that $[a, b, a] = [b, a, a]^{-1}$ and $[b, a, b] = [a, b, b]^{-1}$. In order to show that $[b, a] \in Z_3(R)$ we need to show that $[b, a, a]$ and $[b, a, b] = [a, b, b]^{-1}$ are in $Z_2(R)$. As $[b, a, a, a] = [a, b, b, b] = 1$ it suffices to show that $[[b, a, a]^{-1}, b], [[a, b, b]^{-1}, a] \in Z(R)$. In the following calculations we use again the fact that $\langle b, [a, b]\rangle$ and $\langle a, [a, b]\rangle$ are nilpotent of class at most 2. We have

$$\begin{aligned} [b, a, a][b, a, a, b] &= [b, a, a]^b \\ &= [[b, a]^b, a^b] \\ &= [[b, a, b][b, a], a[a, b]] \\ &= [[b, a, b][b, a], a]^{[a,b]} \\ &= [b, a, b, a]^{[b,a][a,b]}[b, a, a]^{[a,b]} \\ &= [[a, b, b]^{-1}, a][b, a, a]. \end{aligned}$$

Thus

$$[[a, b, b]^{-1}, a] = [b, a, a, b]^{[b,a,a]^{-1}} = [[b, a, a]^{-1}, b]^{-1}. \tag{1}$$

From (1) we thus see that it suffices to shows that $[[a, b, b]^{-1}, a] \in Z(R)$ and in fact it suffices to show that $[[a, b, b]^{-1}, a]$ commutes with b as then by symmetry, the RHS of (1) commutes with a and thus $[[a, b, b]^{-1}, a]$ commutes then with a as well.

From Lemma 2.2 we know that $e(a^\alpha, b^\beta) = e(b^\beta, a^\alpha) = 1$ for all $\alpha, \beta \in \{1, -1\}$. In particular the equation (1) holds if we replace a by a^{-1} or b by b^{-1}. Calculating in the group $\langle a, a^b\rangle$

that is nilpotent of class at most 2, we see that $[b, a^{-1}, a^{-1}] = [a^b, a^{-1}] = [a^{-b}, a] = [b, a, a]$. From this and (1) it follows that $[[a, b, b]^{-1}, a]$ is invariant under replacing a by a^{-1}. Notice also that $[a^{-1}, b, b] = [b^{-a^{-1}}, b] = [b^{-1}, b^a]^{a^{-1}} = [b^{-a}, b]^{-a^{-1}} = [a, b, b]^{-a^{-1}}$. Thus

$$[[a, b, b]^{-1}, a] = [[a^{-1}, b, b]^{-1}, a^{-1}] = [[a, b, b], a^{-1}]^{a^{-1}} = [a, b, b, a]^{-a^{-2}} = [[a, b, b]^{-1}, a]^{[a,b,b]a^{-2}}.$$

But from Lemma 2.3 and (1) we know that $[[a, b, b]^{-1}, a] = [[b, a, a]^{-1}, b]^{-1}$ commutes with b^a and a^b. Replacing a by a^{-1} for the RHS we see that the common element also commutes with $b^{a^{-1}}$. Likewise it commutes with $a^{b^{-1}}$. As

$$[a, b, b]a^{-2} = [b, a]b^{-1}[a, b]ba^{-2} = a^{-b}ab^{-1}a^{-1}a^ba^{-2b^{-1}}b = a^{-b}b^{-a^{-1}}a^ba^{-2b^{-1}}b,$$

it follows that $[[a, b, b]^{-1}, a]$ commutes with b.

As R is nilpotent of class at most 4, it follows that R is metabelian and using the nilpotent quotient algorithm `nq` of Nickel [8] (which is implemented in `GAP`, [9]) one can see that the class is exactly 4. It turns out that R is torsion-free with $R' \cong \mathbb{Z}^4$. □

Remark. An interesting related result of [1] (Proposition 3.1) states that if $S = \{a, b\}$ with $[a, b, b] = [b, a, a, a] = 1$, then $\langle S\rangle$ is nilpotent of class at most 3.

The following examples show that the hypotheses of Theorem A cannot be weakened.

Example 1. (Example 4.2 of [1].) Let x and y be elements of S_{12} defined by

$$\begin{aligned} x =& (1, 2)(3, 4)(5, 6)(7, 9, 10, 8)(11, 12) \\ y =& (1, 3)(2, 4, 5, 7)(6, 8)(9, 11)(10, 12). \end{aligned}$$

Then $o(x) = 4 = o(y)$, $[x, y, y, y] = 1 = [y, x, x, x]$ and $G = \langle x, y\rangle$ has order $2^5 \cdot 3^4$, so, in particular G is not nilpotent.

Example 2. Let x and y be elements of S_{12} defined by

$$\begin{aligned} x =& (1, 2, 3, 4)(5, 6, 8, 10)(7, 9, 11, 12) \\ y =& (1, 3)(2, 4, 5, 7)(6, 9)(8, 11)(10, 12). \end{aligned}$$

Then $o(x) = 4 = o(y)$, $[x, y, y, y] = 1 = [x^{-1}, y, y, y]$, $[y, x, x, x, x] = 1 = [y^{-1}, x, x, x, x]$ and $G = \langle x, y\rangle$ has order $2^6 \cdot 3^4$, so, in particular G is not nilpotent.

3 Proofs of Theorem B and Theorem C

Lemma 3.1. *Let $G = \langle x, y\rangle$ be a group where $y \in L_{n+\frac{1}{2}}(G)$. Then $[x,_n y] \in Z(G)$.*

Proof That $[x,_n y]$ commutes with x is a direct consequence of $y \in L_{n+\frac{1}{2}}(G)$. Then $[yx,_n y, yx] = [x,_n y, yx] = [x,_n y, x][x_{n+1}y]^x = [x,_{n+1} y]^x$ shows that $[x,_n y]$ commutes also with y. □

Lemma 3.2. *Let G be a group and let $a, b \in G$. Suppose that for some $n \geq 2$ we have that $\{a, b, a^{-1}, b^{-1}\}$ is a n-Engel subset of G. Then $e_{n-1}(b^{-a}, b) = e_{n-1}(b, b^{-a}) = e_{n-1}(b^a, b^{-1}) = e_{n-1}(b^{-1}, b^a) = 1$.*

Proof We have $1 = [a,_n b] = [b^{-a}b,_{n-1} b] = [b^{-a},_{n-1} b]^b$ and therefore $[b^{-a},_{n-1} b] = 1$. Replacing b by b^{-1} we see that $[a,_n b^{-1}] = 1$ implies $[b^a,_{n-1} b^{-1}] = 1$. Next we use $[a^{-1},_n b] = 1$ that implies that $[b^{-a^{-1}},_{n-1} b] = 1$ and thus after conjugation by a that $[b^{-1},_{n-1} b^a] = 1$. Replacing b by b^{-1} we see that $[a^{-1},_n b^{-1}] = 1$ implies $[b,_{n-1} b^{-a}] = 1$. $\Box$

Proposition 3.3. *Let G be a $(4+\frac{1}{2})$-Engel 2-group. Then G is locally nilpotent.*

Proof Taking the quotient of G by the Hirsch-Plotkin radical, we can assume that $\mathrm{HP}(G) = 1$ and we want to show that $G = 1$. We argue by contradiction and suppose $G \neq 1$. As groups of exponent 4 are locally finite, there must be an element $g \in G$ of order 8. We get a contradiction by showing that $\langle g^4 \rangle^G$ is abelian and thus $g^4 \in \mathrm{HP}(G) = 1$.

Let $h \in G$ and consider the subgroup $H = \langle g, g_1 \rangle$ where $g_1 = g^{-h}$. Let $\overline{H} = H/Z(G) = \langle \bar{g}, \bar{g_1} \rangle$ where $\bar{g} = gZ(H)$ and $\bar{g_1} = g_1 Z(H)$. By Lemma 3.2 we know that $e(g, g_1) = e(g_1, g) = e(g^{-1}, g_1^{-1}) = e(g_1^{-1}, g^{-1}) = 1$. By Theorem A we then know that $\bar{H}$ and therefore H is finite. Using GAP or MAGMA one can then check that $[g^{-4h}, g^4] = [g_1^4, g^4] = 1$ and thus we have shown that $\langle g^4 \rangle^G$ is abelian. $\Box$

Proposition 3.4. *Let G be a $(4+\frac{1}{2})$-Engel 3-group. Then G is locally nilpotent.*

Proof As before we can assume that $\mathrm{HP}(G) = 1$ and the aim is then to show that $G = 1$. We argue by contradiction and suppose that $G \neq 1$. As groups of exponent 3 are locally finite, there must be an element $g \in G$ of order 9. Let $h \in G$ and $g_1 = g^{-h}$. As in the proof of Proposition 1 one sees that H is finite and then with the help of GAP or MAGMA that $[g_1^3, g^3, g^3] = 1$. Thus $[h, g^3, g^3, g^3] = 1$ for all $h \in G$ and thus g^3 is a left 3-Engel element of G. By the main result of [5] we then know that $g^3 \in \mathrm{HP}(G) = 1$ that contradicts the fact that $o(g) = 9$. $\Box$

Lemma 3.5. *Let G be a group and let $a, b \in G$ be two elements of finite order such that $S = \{a, b, a^{-1}, b^{-1}\}$ is a 4-Engel set. Then every prime divisor of $o([a,b])$ is a divisor of $o(a)$ and $o(b)$. In particular if a and b are of coprime order, then $[a,b] = 1$.*

Proof By Lemma 3.2 together with Lemma 2.2, we know that $S_1 = \{b^a, b, b^{-a}, b^{-1}\}$ and $S_2 = \{a^b, a, a^{-1}, a^{-b}\}$ are 3-Engel subsets of G. By Theorem A we know that $H_1 = \langle a, a^b \rangle$ and $H_2 = \langle b, b^a \rangle$ are nilpotent. As these groups are nilpotent we know that every prime divisor of $|H_1|$ divides $o(a)$ and every prime divisor of $|H_2|$ divides $o(b)$. Now $[a,b] \in H_1 \cap H_2$ and thus $o([a,b])$ divides $|H_1|$ and $|H_2|$ and thus $o(a)$ and $o(b)$ from the discussion above. $\Box$

Proof of Theorem B. Let G be a $\{2,3\}$-group that is $(4+\frac{1}{2})$-Engel. Let H_2 be the set consisting of all elements in G whose order is a power of 2 and H_3 of those elements whose order is a power of 3. In view of Propositions 3.3 and 3.4, it suffices to show that H_2 and H_3 are subgroups and that G is a direct product of H_2 and H_3. Now take any two elements $a, b \in G$ of coprime orders and let $T = \langle a, b \rangle$. By Lemma 3.1 we know that $e(a^\alpha, b^\beta) = e(b^\beta, a^\alpha) \in Z(T)$ for all $\alpha, \beta \in \{1, -1\}$. By Lemma 3.5 it follows that $[a,b] \in Z(T)$. Thus T is nilpotent and as a and b are of coprime order, it follows that $[a,b] = 1$. Now let $a \in H_2$ and $b \in \langle H_2 \rangle$ that has odd order. By the argument above we know that $[a,b] = 1$ and as $a \in H_2$ was arbitrary we see that $b \in Z(\langle H_2 \rangle)$. Thus $\langle H_2 \rangle / Z(\langle H_2 \rangle)$ is a 2-group and by Proposition 3.3 it is locally nilpotent and thus also $\langle H_2 \rangle$. As $\langle H_2 \rangle$ is generated by 2-elements, it is then a 2-group and thus $\langle H_2 \rangle = H_2$ and thus H_2 is a subgroup. The proof that H_3 is a subgroup is similar using Proposition 3.4. Now let $a \in H_2$ and $b \in H_3$ then $[a,b] \in H_2 \cap H_3$ and thus trivial. Hence G is a direct product of H_2 and H_3 and thus locally nilpotent. $\Box$

Lemma 3.6. *Let $p \geq 5$ be a prime and consider the group $G = \langle x, y \rangle$ where $x^{p^2} = y^{p^2} = 1$ and that $\{x, y, x^{-1}, y^{-1}\}$ is a 3-Engel set. Then G has exponent p^2 and G^p is abelian.*

Proof By Theorem A we know that G is nilpotent of class at most 4. Then G is regular and it follows easily that $G^{p^2} = 1$ and then that $[G^p, G^p] = 1$ (for definition and properties of regular p-groups see §12.4 of [2], in particular Theorem 12.4.3). □

Proof of Theorem C. Let $p \geq 5$ be a prime and let G be a $(4 + \frac{1}{2})$-Engel p-group. Consider $H = G/\mathrm{HP}(G)$ where $\mathrm{HP}(G)$ is the Hirsch-Plotkin radical of G. The aim is to show that H is of exponent p. Passing from G to H we can thus without loss of generality assume that the Hirsch-Plotkin radical of G is trivial and the aim is to show that G is then of exponent p. We argue by contradiction and suppose that G has an element g of order p^2. Let $h \in G$ and consider the subgroup $H = \langle g, g_1 \rangle$ where $g_1 = g^{-h}$. Let $\overline{H} = H/Z(H) = \langle \bar{g}, \bar{g_1} \rangle$ where $\bar{g} = gZ(H)$ and $\bar{g_1} = g_1 Z(H)$. By Lemma 3.2 we know that $e(g, g_1) = e(g_1, g) = e(g^{-1}, g_1^{-1}) = e(g_1^{-1}, g^{-1}) = 1$. By Theorem A we then know that $\overline{H}$ is finite and thus also H is finite. By Lemma 3.6 we know that $[\bar{g_1}^p, \bar{g}^p] = 1$, that is $[g_1^p, g^p] \in Z(H)$ and thus in particular $[h, g^p, g^p, g^p] = [g_1^p, g^p, g^p] = 1$. Thus g^p is a left 3-Engel element of odd order in G. By the main result of [5] it follows that $g^p \in \mathrm{HP}(G) = 1$ that contradicts our assumption that $o(g) = p^2$. □

Remark. The variety $\mathcal{E}_{n+\frac{1}{2}}$ seems to be the "engelization" of the variety of groups satisfying the law $[y, x_1, x_2, \ldots, x_n, y] = 1$ studied by Macdonald in [6] and [7].

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