Groups with boundedly many commutators of maximal order

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Abstract. By a result of Cocke and Venkataraman we know that if G is a group with at most m elements of maximal order, then $|G|$ is m-bounded. In this paper we consider the following related setting. Suppose G is a group with at most m commutators whose order is maximal among all commutators. What can we say about the structure of the group G?

1. Introduction

Our notation in this paper is fairly standard. In particular we will denote by $|G|$ the cardinality of the group G. Also if $n_1, \ldots, n_s$ are some non-negative integer parameters, then we say that a quantity is $(n_1, \ldots, n_s)$-bounded to abbreviate "is finite and bounded above in terms of $n_1, \ldots, n_s$ only".

This paper is motivated by the following result. We include a short proof for the convenience of the reader.

Theorem (Cocke and Venkataraman [**1**]). *Let G be a group with m elements of maximal order, where m is a positive integer. Then $|G|$ is m-bounded.*

Proof. Let a be an element of maximal order, say n. Obviously n is a positive integer. Let $H = \langle a \rangle$ and suppose $\{Hx_i : i \in I\}$ are the cosets of H in $C_G(a)$. Then

$$C_G(a) = \bigcup_{i \in I} Hx_i.$$

1991 *Mathematics Subject Classification.* 20F12, 20F24.
Key words and phrases. Commutators, FC-groups.

As a is of maximal order $\langle x_i, a\rangle = \langle x_i a^{n(i)}\rangle \times \langle a\rangle$ for some non-negative integer $n(i)$. Replacing x_i by $x_i a^{n(i)}$, we can assume that $\langle x_i, a\rangle = \langle x_i\rangle \times \langle a\rangle$ for all $i \in I$. Then each coset has at least $\phi(n)$ elements of order n, where ϕ is the Euler function. Namely the elements $a^k x_i,\ (k, n) = 1$. Thus $|I|\phi(n) \leq m$ and it follows that $|C_G(a)|$ is m-bounded. As $[G : C_G(a)] = |\{a^g : g \in G\}| \leq m$, it follows that $|G| = [G : C_G(a)] \cdot |C_G(a)|$ is m-bounded. □

Consider now the following setting. Suppose we have exactly m commutators whose order is maximal among all commutators, say n this order. What can we say about the structure of G?
Recall that an element $g \in G$ is a commutator if $g = [x, y] = x^{-1}y^{-1}xy$ for some $x, y \in G$. In Section 2, we will see that if a group contains commutators of infinite order, then there are infinitely many such commutators. We will also show that if G is residually finite with finitely many commutators of maximal order, then G is abelian-by-finite (if additionally G is finitely generated, then $[G, G]$ is finite). In the subsequent sections we focus on the following question.

Question. Suppose G is a group with only m commutators whose order, say n, is maximal among all commutators. Does it follow that G has a subgroup N of m-bounded index such that $[N, N]$ is of m-bounded order?

We will see that one can often show that this is the case when some further constraints are added. We will also see that if this is the case and G is finitely generated by at most r elements, then one furthermore has that $[G, G]$ is of (m, r)-bounded order. In some cases one can even prove the stronger property that $[G, G]$ is of m-bounded order.

The structure of the paper is otherwise as follows. In Section 3, we will show that the answer to the question is positive when G is metabelian. In Section 4, we will see that this is also the case in general when n is a prime power or when it is a product of two odd prime powers. From these results we are then able in Section 5 to show that the answer to the question is positive when the group is nilpotent. Finally in Section 6 we deal with the so-called A-groups, that is, finite groups all of whose Sylow subgroups are abelian.

2. Some preliminary results

In this section we state and prove some preliminary results. Many arguments rely on some standard properties and results on FC-groups and for these we refer the reader to [7].

Let G be a group with finitely many commutators of maximal order. Let $\mathcal{D}$ be the set of all the commutators of maximal order. Suppose $m = |\mathcal{D}|$ and that the maximal order of a commutator is n. We say that an element $a \in G$ is $\mathcal{D}$-related if there exists $b \in G$ such that $[a, b] \in \mathcal{D}$. We first show that n must finite.

Lemma 2.1. *If a group G contains commutators of infinite order, then there are infinitely many such commutators.*

Proof. We argue by contradiction and suppose that G has only m commutators of infinite order where m is a positive integer. Pick a and b in G such that $[a, b] \in \mathcal{D}$. We will get a contradiction by showing that $[a, b]$ must be of finite order. Notice that $D = \langle \mathcal{D} \rangle$ is contained in the FC-centre and is thus is an FC-group. As D is finitely generated $D/Z(D)$ is finite and thus $[D, D]$ is finite by Schur's Theorem [**8**, 10.1.4]. As $[a, b]$ is of finite order if and only if $[a, b][D, D]$ is of finite order in $D/[D, D]$, we can assume without loss of generality that $[D, D] = 1$. We can now suppose that D is a finitely generated abelian group. As the torsion part of D is finite, we can also assume that D is a torsion-free group. Let $g \in G$. Notice that $[D, g] \leq D$. We claim that $[D, g] = 1$. To see this let $d \in D$. Notice that, for every positive integer k, we have $[d^k, g] = [d, g]^{d^{k-1}}[d^{k-1}, g] = [d, g][d^{k-1}, g]$, thus, by induction on k it is easy to see that $[d^k, g] = [d, g]^k$. Consider the commutators

$$[d, g], [d^2, g] = [d, g]^2, \ldots, [d^k, g] = [d, g]^k, \ldots$$

If $[d, g] \neq 1$ then the set of these commutators would be infinite contradicting the assumption that are only finitely many. Thus $[d, g] = 1$. We have thus shown that $D \leq Z(G)$.

Now notice that for each $x \in G$, we have $[a, bx] = [a, x][a, b]$. There are only finitely many possible values for $[a, x]$ such that $[a, x] \in \mathcal{D}$. If $[a, x] \notin \mathcal{D}$, i.e. of finite order, then $[a, bx] \in \mathcal{D}$ and again there are only finitely many possible values for $[a, x] = [a, bx][b, a]$. We have thus shown that $[G : C_G(a)]$ is finite and a is in the FC-centre. By symmetry the same is true for b. Thus both a, b are in the FC-centre Z of G and thus $[a, b] \in [Z, Z]$ is of finite order. This contradiction finishes the proof. □

We will now establish a quantitative version of the above lemma.

Proposition 2.2. *Let G be a group with m commutators of maximal order, say n, among all commutators. Then the number n is m-bounded.*

Proof. We know from Lemma 2.1 that n is finite. Choose $a, b \in G$ such that $[a, b] \in \mathcal{D}$. Without loss of generality we can assume that $G = \langle a, b\rangle$. In this case $[G, G] = \langle [a, b]^G\rangle$. Since the conjugacy class $[a, b]^G$ consists of at most m elements, it follows that the index $[G : C_G([G, G])]$ is m-bounded. Set $C = C_G([G, G])$ and observe that C is nilpotent of class at most 2. Thus, G is a finitely generated group having a nilpotent subgroup of finite index. We deduce that G is residually finite. The commutator subgroup $[G, G]$ is generated by finitely many FC-elements of finite order from the class $[a, b]^G$. It follows that $[G, G]$ is finite and therefore G has a finite-index normal subgroup N such that $N \cap [G, G] = 1$. Now we can pass to the quotient G/N and without loss of generality assume that G is finite.

Finite groups have a nice property that if x is a commutator, then each generator of the cyclic subgroup $\langle x\rangle$ is a commutator, too (see [**5**, page 45], or [**2**]). Therefore whenever r is coprime to n, the power $[a, b]^r$ is also a commutator and so $[a, b]^r \in \mathcal{D}$. It follows that $\phi(n) \leq m$ and thus n is m-bounded, as required. □

In view of Proposition 2.2 it will be assumed throughout the rest of the paper that all commutators in G have finite m-bounded order. Apart from the work of Cocke and Venkataraman the following observation is another motivation for our work.

Proposition 2.3. *A residually finite group G with only finitely many commutators of maximal order has an abelian subgroup of finite index. If G is finitely generated, then $[G, G]$ is finite (and hence G is central-by-finite).*

Proof. Notice first that by Dietzmann's Lemma (see for example [7, page 45]), we know that $D = \langle \mathcal{D}\rangle$ is finite. As G is residually finite there exists a normal subgroup N of finite index such that $N \cap D = 1$. Notice that $N \leq C_G(D)$. If $x \in N$ and $[a, b] \in \mathcal{D}$ then

$$[a, bx] = [a, x][a, b]$$

with $[a, x] \in N$. Thus the order of $[a, bx]N$ in G/N is the same as the order of $[a, b]N$ in G/N which we know is the same as the order of $[a, b]$ as $N \cap D = 1$. It follows that $[a, bx]$ is of maximal order. Hence $[a, x] = [a, bx][b, a] \in N \cap D = 1$. This shows that every $\mathcal{D}$-related element commutes with all the elements of N. It also shows that if b is $\mathcal{D}$-related then bx is $\mathcal{D}$-related for all $x \in N$. Now let $y, x \in N$. As y commutes with bx and b, it commutes with $x = b^{-1} \cdot bx$. Hence N is abelian.

Now assume that G is finitely generated. Being a subgroup of finite index, N is finitely generated, too. It follows that the torsion elements contained in N form a finite subgroup T. We pass to the quotient $\bar{G} = G/T$. Recall that all commutators in G have finite order. We conclude that $\bar{N} = N/T$ contains no nontrivial commutators and therefore $\bar{N} \leq Z(\bar{G})$. It follows from Schur's theorem that $[\bar{G}, \bar{G}]$ is finite and this of course implies finiteness of $[G, G]$. □

As we said in the introduction we will often be able to show that there exists a subgroup N of m-bounded index such that $[N, N]$ is of m-bounded order. If G is furthermore finitely generated, say by at most r elements, more can be said.

PROPOSITION 2.4. *Let G be an r-generator group with m commutators of maximal order. Assume that G contains a normal subgroup N of finite index such that $[N, N]$ is finite. Then $[G, G]$ has finite $(m, r, [G : N], |[N, N]|)$-bounded order.*

PROOF. Notice first that N is finitely generated and the minimal number of generators for N is bounded in terms of r and the index $[G : N]$. We can pass to the quotient $G/[N, N]$ and assume that N is abelian. Since all commutators in G have finite m-bounded order, we deduce that the subgroup T generated by all such commutators contained in N is finite with order bounded in terms of m, r, and $[G : N]$. Again we can pass to the quotient G/T and without loss of generality assume that $T = 1$. Then $N \leq Z(G)$. Hence $[G : Z(G)]$ is finite and thus in view of Schur's theorem we get that $[G, G]$ is finite. □

3. Metabelian groups

We will see that we are sometimes able to get the type of result we want if we either put some constraint on n or on the structure of G. In this section we will deal with the case when the group is metabelian. We start by proving a general lemma about abelian groups that will play a crucial role.

LEMMA 3.1. *Let G be a finite abelian group and let $x \in G$. There exist $y_1, \dots, y_t \in G$ of distinct prime power orders, where the primes divide the order of x, and $F \leq G$ such that $G = F \times \langle y_1 \rangle \times \cdots \times \langle y_t \rangle$ and $x \in \langle y_1 \rangle \times \cdots \times \langle y_t \rangle$.*

PROOF. Suppose $G = G_1 \times \cdots \times G_r$, where $G_1, \dots, G_r$ are the Sylow subgroups with respect to the primes $p_1, \dots, p_r$. Now suppose

$$x = w_1 \cdots w_r$$

with $w_s \in G_s$. Focusing on $w_1, \dots, w_r$, we can without loss of generality assume that G is a p-group. Suppose the exponent of G is p^l.

Now suppose

$$x = a_1 \cdots a_e b_1^{\beta_1} \cdots b_f^{\beta_f}$$

where $a_1, \ldots, b_f^{\beta_f}$ are non-trivial,

$$G = \langle a_1 \rangle \times \cdots \times \langle a_e \rangle \times \langle b_1 \rangle \times \cdots \times \langle b_f \rangle \times K$$

for some $K \leq G$ and where p divides $\beta_1, \ldots, \beta_f$. We can assume that $o(a_1) \leq \cdots \leq o(a_e)$ and $o(b_1^{\beta_1}) \leq \cdots \leq o(b_f^{\beta_f})$.

We prove by reverse induction on $1 \leq k \leq l$, where $p^k = min\{o(a_1), o(b_1^{\beta_1})\}$, that there exist $y_1, \ldots, y_t \in G$ of distinct prime power orders at least p^k and $F \leq G$ such that

$$G = F \times \langle y_1 \rangle \times \cdots \times \langle y_t \rangle$$

and $x \in \langle y_1 \rangle \times \cdots \times \langle y_t \rangle$.

For the induction basis suppose $k = l$. Then x is of maximal order and there exists $F \leq G$ such that $G = F \times \langle x \rangle$.

For the induction step suppose $1 \leq k \leq l - 1$ and that the result holds for larger values of k. If $e = 0$ and $\beta_i = p\gamma_i$ we have $x = z^p$ where

$$z = b_1^{\gamma_1} \cdots b_f^{\gamma_f}.$$

By the induction hypothesis there exist $y_1, \ldots, y_t \in G$ of distinct prime power orders at least p^{k+1} and $F \leq G$ such that

$$G = F \times \langle y_1 \rangle \times \cdots \times \langle y_t \rangle$$

and where z (and thus $x = z^p$) is in $\langle y_1 \rangle \times \cdots \times \langle y_t \rangle$.

We can thus assume that $e \geq 1$. Suppose that a_1 is of order p^j and that furthermore $a_2, \ldots, a_g$ are of order p^j and $b_1^{\beta_1}, \ldots, b_h^{\beta_h}$ are of order at most p^j. Let

$$\begin{aligned} x_1 &= a_1 \cdots a_g b_1^{\beta_1} \cdots b_h^{\beta_h} \\ x_2 &= a_{g+1} \cdots a_e b_{h+1}^{\beta_{h+1}} \cdots b_f^{\beta_f}. \end{aligned}$$

One sees that there exists $H \leq G$ such that $G = H \times \langle x_1 \rangle$ with $a_{g+1}, \ldots, a_e, b_{h+1}, \ldots, b_f \in H$ and among the generators for the decomposition of H into a direct product of cyclic groups (notice that the orders of these elements are greater than $o(x_1) = p^j \geq p^k$). By the induction hypothesis there exist $y_2, \dots, y_t \in H$ of distinct prime power orders greater than p^j such that

$$H = F \times \langle y_2 \rangle \times \cdots \times \langle y_t \rangle$$

and $x_2 \in \langle y_2 \rangle \cdots \langle y_t \rangle$. Now

$$G = F \times \langle x_1 \rangle \times \langle y_2 \rangle \times \cdots \times \langle y_t \rangle$$

and $x = x_1 x_2 \in \langle x_1 \rangle \times \langle y_2 \rangle \times \cdots \langle y_t \rangle$ where $o(x_1), o(y_2), \dots, o(y_t)$ are of distinct prime power orders at least p^k. This finishes the inductive proof. $\square$

We now proceed to examine metabelian groups containing boundedly many commutators of maximal order. We first handle the finite metabelian groups and then we will later lift our main result concerning these to any metabelian group.

Let G be a finite metabelian group. Recall that $\mathcal{D}$ is the collection of all commutators of maximal order n and that there are m of these. Notice that the exponent of $[G, G]$ divides $n!$ and is therefore also m-bounded. Let $M = C_G(\mathcal{D})$. As $\mathcal{D}$ is invariant under conjugation we have $M \trianglelefteq G$. Any conjugation permutes the elements of $\mathcal{D}$ implying that $[G : M] \leq m!$.

Let $a, b \in G$ be such that $[a, b] \in \mathcal{D}$. Consider the subgroup

$$E = \langle [a, b], [a, x] : x \in M \rangle.$$

By Lemma 3.1 there exist $y_1, \dots, y_t \in E$ of distinct prime power orders, where the primes divide the order of $[a, b]$, and $F \leq E$ such that $E = F \times \langle y_1 \rangle \times \cdots \times \langle y_t \rangle$ and $[a, b] \in H = \langle y_1 \rangle \times \cdots \times \langle y_t \rangle$. For each $h \in H$ let

$$E_h = \{[a, x] : x \in M \text{ and } [a, x] = w_x h \text{ for some } w_x \in F\}.$$

Notice that if $[a, x], [a, y] \in E_h$ then

$$\begin{aligned}
[a, bxy^{-1}] &= [a, xy^{-1}][a, b] \\
&= [a, y^{-1}][a, x]^{y^{-1}}[a, b] \\
&= ([a, y]^{-1}[a, x][a, b])^{y^{-1}} \\
&= (w_y h)^{-1} w_x h [a, b])^{y^{-1}} \\
&= (w_y^{-1} w_x [a, b])^{y^{-1}}.
\end{aligned}$$

Now as $w_y^{-1} w_x \in F$ and $[a, b]$ has order n, we see therefore that $[a, bxy^{-1}]$ has order at least n. As this is a commutator, the order is then exactly n and thus this element is in $\mathcal{D}$. Notice that these calculations show that

$$[a, y]^{-1}[a, x] = [a, bxy^{-1}]^y [a, b]^{-1},$$

where the right hand side is a difference of two elements in $\mathcal{D}$. Let $[a, y]$ be some fixed element in E_h. We have seen that

$$E_h \subseteq [a, y]\{ef^{-1} : e, f \in \mathcal{D}\}$$

and has thus at most m^2 elements.

LEMMA 3.2. *If a is $\mathcal{D}$-related, then $[G : C_G(a)]$ is m-bounded.*

PROOF. We have

$$\{[a, x] : x \in M\} = \bigcup_{h \in H} E_h.$$

As there are m-boundedly many prime divisors of $o([a, b])$ and as the exponent of $[G, G]$ is m-bounded we see that $|H|$ is m-bounded. As $|E_h| \leq m^2$ we see thus that $|\{[a, x] : x \in M\}|$ is m-bounded. Therefore

$$|a^G| = |a^M| \cdot [G : M]$$

is m-bounded. (Notice that as we have seen before $[G : M] \leq m!$). □

PROPOSITION 3.3. *Let G be a finite metabelian group with at most m commutators of maximal order. Then there exists a normal subgroup N of m-bounded index where $[N, N]$ is of m-bounded order.*

Indeed there exist two functions f and g such that, if a is a $\mathcal{D}$-related element and $T = C_G(a)$, then $[G : T] \leq f(m)$ and $|[T, T]| \leq g(m)$.

PROOF. Let $[a, b] \in \mathcal{D}$. As a is $\mathcal{D}$-related we know from Lemma 3.2 that $[G : C_G(a)]$ is m-bounded, say at most $f(m)$. Let $t \in C_G(a)$. Then

$$[a, bt] = [a, b]^t \in \mathcal{D}$$

and thus both bt and b are $\mathcal{D}$-related. As $C_G(t) \supseteq C_G(bt) \cap C_G(b)$, we see that

$$[G : C_G(t)] \leq [G : C_G(bt)] \cdot [G : C_G(b)]$$

is m-bounded. It follows that $T = C_G(a)$ is a BFC-group where the conjugacy classes have m-bounded size. Hence by a well known result of B. H. Neumann [**6**] we see that $[T, T]$ is of m-bounded order, say at most $g(m)$. Finally replacing T by its core N in G we see that $[G : N] \leq f(m)!$ and thus N is of m-bounded index with $[N, N]$ of m-bounded order. □

We next extend this result to all metabelian groups that are finitely generated.

PROPOSITION 3.4. *Let G be a finitely generated metabelian group with at most m commutators of maximal order. Then there exists a normal subgroup N of m-bounded index such that $[N, N]$ is of m-bounded order.*

Indeed there exist two functions f and g such that, if a is a $\mathcal{D}$-related element and $T = C_G(a)$, then $[G : T] \le f(m)$ and $|[T, T]| \le g(m)$.

Proof. By a well-known theorem of P. Hall [**3**] G is residually finite. For each commutator v of maximal order in G there exists a normal subgroup R_v of G that is of finite index and such that the intersection of R_v with $\langle v\rangle$ is trivial. As there are finitely many commutators of maximal order we can then find a normal subgroup R of G of finite index such that the intersection with any subgroup generated by a commutator of maximal order is trivial. It follows from this that if v is a commutator of maximal order in G then vS is of (same) maximal order in G/S for any normal subgroup of S of G of finite index contained in R. Thus if $\mathcal{D}$ is the set of all commutators in G of maximal order, then the set of commutators of maximal order in G/S is $\mathcal{D}_S = \{vS : v \in \mathcal{D}\}$. Let $a \in G$ be $\mathcal{D}$-related. We claim that $[G : C_G(a)] \le f(m)$ where $f(m)$ is as in Proposition 3.3. To see this, let S be a normal subgroup of finite index in G that is contained in R. Then aS is $\mathcal{D}_S$-related and by Proposition 3.3 we know that $[G : C_G(a)S] \le f(m)$. As this is true for all such S, it follows that $[G : C_G(a)] \le f(m)$.

Let $T = C_G(a)$. We next show that $[T, T]$ has order at most $g(m)$ where $g(m)$ is as in Proposition 3.3. By the proof of that proposition, we know that $[T, T]S/S$ is of order at most $g(m)$ for all S as above and hence it follows that $|[T, T]| \le g(m)$. Finally as in the proof of Proposition 3.3 we can replace T by its core N in G and observe that $[G : N] \le f(m)!$ and $|[N, N]| \le g(m)$. □

Theorem 3.5. *Let G be any metabelian group with at most m commutators of maximal order. Then there exists a normal subgroup N of m-bounded index such that $[N, N]$ is of m-bounded order.*

Proof. As before let $\mathcal{D}$ be the set of all commutators of maximal order. Let a be a $\mathcal{D}$-related element. We claim that $[G : C_G(a)] \le f(m)$ where $f(m)$ is as in Propositons 3.3.and 3.4. We argue by contradiction and suppose there are $f(m) + 1$ distinct conjugates $a^{g_1}, \ldots, a^{g_{f(m)+1}}$. Suppose the commutators of maximal order are $[a_1, b_1], \ldots, [a_m, b_m]$ (where $a_1 = a$). Consider the finitely generated subgroup

$$F = \langle a_1, b_1, \ldots, a_m, b_m, g_1, \ldots, g_{f(m)+1}\rangle.$$

But then we have $f(m)+1$ distinct conjugates of a in F that contradicts the fact that $f(m)$ should be an upperbound for the finitely generated case.

Let $T = C_G(a)$. We claim that the order of $[T, T]$ is at most $g(m)$ where $g(m)$ is as in the proof of Propositions 3.3 and 3.4. Let F be a finitely generated subgroup of T. Suppose that the commutators of maximal order are $[a_1, b_1], \ldots, [a_r, b_r]$ where $a_1 = a$. Consider the subgroup $H = \langle F, a_1, b_1, \ldots, a_r, b_r \rangle$. Then H is also finitely generated with at most m commutators of maximal order. By Propostion 3.4 we know that $[H \cap T, H \cap T]$ is of order at most $g(m)$ and thus the same is true $[F, F]$. As this is true for every finitely generated subgroup F of T it follows then that $[T, T]$ has order at most $g(m)$. Finally as in the proof of Proposition 3.3 we can replace T by its core N in G and get $[G : N] \leq f(m)!$ and $|[N, N]| \leq g(m)$. □

For the case when G is finite, n is the maximal order of a commutator and n is a prime power, one gets a stronger result.

THEOREM 3.6. *Let G be a finite metabelian group with at most m elements of maximal order n where $n = p^r$,paprime. Then $[G, G]$ is of m-bounded order.*

PROOF. As a first step we prove that all $\mathcal{D}$-related elements are of order at most $2m$. To see this let a be a $\mathcal{D}$-related element and $b \in G$ such that $[a, b] \in \mathcal{D}$. Now take any $g \in G$. Then

$$[a, bg]^{g^{-1}} = [a, g]^{g^{-1}}[a, b].$$

If $[a, g]^{g^{-1}} \notin \mathcal{D}$ then $[a, bg]^{g^{-1}} \in \mathcal{D}$. Thus

$$\{[a, g]^{g^{-1}} : g \in G\} = \{[a, g]^{g^{-1}} : [a, g] \in \mathcal{D}\} \cup \{[a, g]^{g^{-1}} : [a, bg]^{g^{-1}} \in \mathcal{D}\}.$$

As $[a, g]^{g^{-1}} = [b, a][a, bg]^{g^{-1}}$, we see that both sets have at most m elements. Thus

$$2m \geq |\{[a, g]^{g^{-1}} : g \in G\}| = |\{[a, g^{-1}]^{-1} : g \in G\}| =$$

$$|\{[a, g^{-1}] : g \in G\}| = [G : C_G(a)].$$

In order to show that $[G, G]$ is m-bounded it suffices (by B. H. Neumann [**6**]) to show that if $d \in G$ is not $\mathcal{D}$-related we still have that $[G : C_G(d)]$ is m-bounded. But in this case

$$[a, bd] = [a, d][a, b]^d.$$

As $[a, d] \notin \mathcal{D}$, we see that $[a, bd] \in \mathcal{D}$. Thus bd and b are $\mathcal{D}$-related. From $C_G(d) \geq C_G(bd) \cap C_G(b)$ we then get as before

$$[G : C_G(d)] \leq [G : C_G(b)][G : C_G(bd)] \leq (2m)^2.$$

This finishes the proof. □

4. Adding constraints on n

In this section we will see that we are able to get the result we want provided the maximal order n is a power of p without any other constraints on the structure of the group. We will show that the same result holds if n is a product of two distinct prime powers where both primes are odd.

4.1. The case when $n = p^{\alpha}$.

THEOREM 4.1. *Suppose G has at most m commutators of maximal order which is a prime power. Then G has a normal subgroup M of m-bounded index such that $[M, M]$ is of m-bounded order.*

PROOF. As before let $\mathcal{D}$ be the set of all the commutators of maximal order and let $M = C_G(\mathcal{D})$. We know that M is a normal subgroup of G and that $[G : M] \leq m!$. We finish the proof by showing that M is a BFC-group where the size of any conjugacy class is m-bounded. It then follows from the classic result of B. H. Neumann [**6**] that the order of $[M, M]$ is m-bounded.

We first prove the following claim.

Claim: If a is $\mathcal{D}$-related, then $[G : C_G(a)] \leq 2m \cdot m!$.

To prove the claim let $b \in G$ such that $[a, b] \in \mathcal{D}$. For all $x \in M$, we have

$$[a, bx] = [a, x][a, b]. \tag{1}$$

Notice that, as $M \trianglelefteq G$, $[a, x]$ commutes with $[a, b]$. As the order of commutators in $\mathcal{D}$ is a prime power, if $[a, x]$ is not in $\mathcal{D}$ it must follow that $[a, bx] \in \mathcal{D}$. Thus

$$\{[a, x] : x \in M\} =$$
$$\{[a, x] : x \in M \text{ and } [a, x] \in \mathcal{D}\} \cup \{[a, x] : x \in M \text{ and } [a, bx] \in \mathcal{D}\}.$$

As $[a, x] = [a, bx][b, a]$, both the subsets on the right hand side have order at most m and thus $|\{[a, x] : x \in M\}| \leq 2m$. Hence $|a^M| \leq 2m$ and thus $[G : C_G(a)] = |a^G| \leq |a^M| \cdot [G : M] \leq 2m \cdot m!$.

This finishes the proof of the claim. We finish the proof of the proposition by using this to show that $[G : C_G(x)] \leq (2m \cdot m!)^2$ for all $x \in M$. If x is $\mathcal{D}$-related this follows immediately from the claim. Now suppose x is not $\mathcal{D}$-related. By (1) and the fact that the maximal order of a commutator is a prime power, it follows that b, bx are $\mathcal{D}$-related and thus $[G : C_G(b)], [G : C_G(bx)] \leq 2m \cdot m!$. As $C_G(x) \geq C_G(b) \cap C_G(bx)$ it follows that $[G : C_G(x)] \leq [G : C_G(b)] \cdot [G : C_G(bx)] \leq (2m \cdot m!)^2$. $\square$

4.2. The case when $n = p^\alpha q^\beta$ and n odd. Suppose G has at most m commutators of maximal order $n = p^\alpha q^\beta$ where p and q are different odd primes. As before we let $\mathcal{D}$ be the collection of all the commutators of maximal order and $M = C_G(\mathcal{D})$.

LEMMA 4.2. *Let $a \in G$ be $\mathcal{D}$-related. Then $\{[a,x] : x \in M\} \subseteq \mathcal{D} \cup \mathcal{D}^2$. Also if $[a,b] \in \mathcal{D}$ and $x \in M$, then one of x, bx, bx^{-1} is $\mathcal{D}$-related.*

PROOF. As a is $\mathcal{D}$-related, there exists $b \in G$ such that $[a,b] \in \mathcal{D}$. If $[a,x] \in \mathcal{D}$ the claim is obvious. Thus suppose this is not the case. Then one of p^α, q^β does not divide $o([a,x])$. Without loss of generality we can assume that p^α does not divide $o([a,x])$. Then from

$$\begin{aligned} [a,bx] &= [a,x][a,b] \\ [a,bx^{-1}]^x &= [a,x]^{-1}[a,b], \end{aligned}$$

we see that p^α divides $o([a,bx])$ and $o([a,bx^{-1}])$. If q^β also divides $o([a,bx])$ or $o([a,bx^{-1}])$, then in the former case $[a,bx] \in \mathcal{D}$ and $[a,x] = [a,bx][a,b]^{-1} \in \mathcal{D}^2$, whereas in the latter case $[a,bx^{-1}] \in \mathcal{D}$ and $[a,x] = [a,b][a,bx^{-1}]^{-x} \in \mathcal{D}^2$. We are thus left with the case where q^β divides neither $o([a,bx])$ nor $o([a,bx^{-1}])$. We will see that this cannot happen. If this was the case, then there would exist integers h, k coprime to q where $[a,bx]^{q^{\beta-1}h} = [a,bx^{-1}]^{q^{\beta-1}k} = 1$. But then we would get

$$[a,b]^{2q^{\beta-1}kh} = [a,bx]^{q^{\beta-1}hk}([a,bx^{-1}]^x)^{q^{\beta-1}hk} = 1.$$

This would then imply that q^β must divide $q^{\beta-1}hk$ that would give the contradiction that q divides hk. As we have seen that one of $[a,x], [a,bx]$ and $[a,bx^{-1}]$ is in $\mathcal{D}$, the second part of the lemma follows. $\square$

THEOREM 4.3. *Let G be a group that has at most m commutators of maximal order $n = p^\alpha q^\beta$ where p and q are different odd primes. Then G has a normal subgroup M of m-bounded index such that $[M,M]$ is of m-bounded order.*

PROOF. Notice first that by Lemma 4.2 we have that if a is $\mathcal{D}$-related, then $|a^M| \leq m^2 + m$. By the second part of the Lemma we also know that if $x \in M$ then x is a product of a most two $\mathcal{D}$-related elements. It follows that $|x^M| \leq (m^2+m)^2$. Hence M is a BFC-group where every element has m-boundedly many conjugates and it follows as before that $[M,M]$ is of m-bounded order. As we have seen previously, we furthermore have $[G : M] \leq m!$. $\square$

5. Nilpotent groups

In this section we deal with nilpotent groups. We start with finite groups.

PROPOSITION 5.1. *Let G be a finite nilpotent group with at most m commutators of maximal order. Then there exists a normal subgroup M of m-bounded index such that the order of $[M,M]$ is m-bounded.*
Indeed $|[M,M]| \leq h(m)$, for some function h, where $M = C_G(\mathcal{D})$ and $\mathcal{D}$ is the set of all commutators of maximal order.

PROOF. Let $P_1, \ldots, P_r$ be the non-abelian Sylow subgroups with respect to the distinct primes $p_1, \ldots, p_r$. Suppose $[a,b]$ is a commutator of maximal order $n = p_1^{n_1} \cdots p_r^{n_r}$. Notice that as before n is m-bounded and then also r is m-bounded. Suppose $a = a_1 a_2 \cdots a_r$ and $b = b_1 b_2 \cdots b_r$ where $a_i, b_i \in P_i$, then

$$[a,b] = [a_1,b_1] \cdots [a_r,b_r].$$

where $[a_i,b_i]$ has order $p_i^{n_i}$. Conversely suppose $[a_i,b_i]$ is a commutator in P_i of maximal order (and thus $p_i^{k_i}$ where $k_i \geq n_i$), then for $a = a_1 a_2 \cdots a_r$ and $b = b_1 b_2 \cdots b_r$ we have that

$$[a,b] = [a_1,b_1] \cdots [a_r,b_r]$$

has order $p_1^{k_1} \cdots p_r^{k_r} \geq n$ and thus $[a,b]$ is of maximal order that implies that the order of $[a_i,b_i]$ is $p_i^{n_i}$. Notice also that $[a_i,b_i]$ is a power of $[a,b]$. Let $\mathcal{D}_i$ be the set of the commutators of maximal order in P_i. We have shown above that $\mathcal{D} = \mathcal{D}_1 \cdots \mathcal{D}_r$. By Theorem 4.1 we know that for $M_i = C_{P_i}(\mathcal{D}_i)$ we have that M_i is of m-bounded index and $[M_i,M_i]$ of m-bounded order. Now recall that r is m-bounded (notice that the argument above shows that $n_i \geq 1$ for all $1 \leq i \leq r$) and if we take $M = C_G(\mathcal{D}) = M_1 \cdots M_r$ then $[G:M] \leq m!$ and $[M,M]$ of m-bounded order, say $h(m)$. □

We next extend this result to all nilpotent groups that are finitely generated.

PROPOSITION 5.2. *Let G be a finitely generated nilpotent group with at most m commutators of maximal order n. Then there exists a normal subgroup M of m-bounded index such that $[M,M]$ is of m-bounded order.*
Indeed $|[M,M]| \leq h(m)$, for some function h, where $M = C_G(\mathcal{D})$ and $\mathcal{D}$ is the set of all commutators of maximal order.

PROOF. Let $\mathcal{D}$ be the set of all commutators of maximal order and let $M = C_G(\mathcal{D})$. As before we know that M is of m-bounded index. We show that $[M, M]$ is of order at most $h(m)$ where h is as in Proposition 5.1. It is well known that G is residually finite. For each commutator of maximal order $[a, b]$ there exists a normal subgroup $N_{[a,b]}$ of G, contained in M, that is of finite index and such that the intersection of $N_{[a,b]}$ with $\langle [a, b] \rangle$ is trivial. As there are finitely many commutators of maximal order we can then find a normal subgroup N of G of finite index, contained in M, such that the intersection with any subgroup generated by a commutator of maximal order is trivial. It follows from this that if $[a, b]$ is a commutator of maximal order in G then $[a, b]N$ is of (same) maximal order in G/N. By Proposition 5.1 we then know that $[M/N, M/N]$ is of order at most $h(m)$. This holds for any normal subgroup S in G that is of finite index and contained in N. Hence the order of $[M, M]$ is at most $h(m)$. $\square$

THEOREM 5.3. *Let G be any nilpotent group with at most m commutators of maximal order n. Then there exists a normal subgroup M of m-bounded index such that $[M, M]$ is of m-bounded order.*

PROOF. As before let $\mathcal{D}$ be the set of all commutators of maximal order and let $M = C_G(\mathcal{D})$. We know that M is a normal subgroup of G of m-bounded index. We show that every finitely generated subgroup of M has commutator subgroup of order at most $h(m)$, where h is the function from Propositions 5.1 and 5.2. Let F be a finitely generated subgroup of M. Suppose the commutators of maximal order are $[a_1, b_1], \ldots, [a_r, b_r]$. Consider the subgroup $H = \langle F, a_1, b_1, \ldots, a_r, b_r \rangle$. Then H is also finitely generated with at most m commutators of maximal order. Also every element in $M \cap H$ centralizes every commutator of maximal order. By Proposition 5.2 we then know that $[H \cap M, H \cap M]$ is of order at most $h(m)$ and then of course the same is true for $[F, F]$. As this is true for every finitely generated subgroup F of M it follows then that $[M, M]$ has order at most $h(m)$. $\square$

6. A-groups

Recall that finite groups all of whose Sylow subgroups are abelian are called A-groups. Notice that if G is an A-group, then $Z(G) \cap [G, G] = \{1\}$, by an application of transfer theory (see for example [**4**, Chapter VI]). We will also use the following remark.

PROPOSITION 6.1. *Let $a, b, x \in G$, with $[a, b]$ of maximal order n and $x, x^a \in C_G([a, b])$. If $\langle [a, x] \rangle \cap \langle [a, b] \rangle = \{1\}$, then $[a, bx]$ has order n and $[a, x]$ has order that divides n.*

PROOF. We have $[a, bx] = [a, x][a, b]$. Let t be the order of $[a, bx]$, then $1 = [a, x]^t [a, b]^t$, hence $[a, x]^t = 1 = [a, b]^t$ and the orders of $[a, x]$

and $[a, b]$ divide t. Therefore the maximality of n implies $n = t$, and we have the result. □

THEOREM 6.2. *Let G be an A-group with at most m commutators of maximal order. Then G has a normal subgroup N of m-bounded index such that $[N, N]$ is of m-bounded order.*

PROOF. As before let $\mathcal{D}$ be the set all commutators of maximal order and let $M = C_G(\mathcal{D})$. We know that M is a normal subgroup of G of m-bounded index. Let $x \in [M, M]$, $[a, b] \in \mathcal{D}$. Then $[x, a] \in [M, M]$, thus $\langle [x, a] \rangle \cap \langle [a, b] \rangle \subseteq [M, M] \cap Z(M) = \{1\}$. Then $[a, bx] = [a, x][a, b]$, and by Proposition 6.1 $[a, bx]$ has maximal order. Hence $[a, x] \in Z(M) \cap [M, M] = \{1\}$. Therefore $[M, M] \subseteq C_G(a)$, for each $\mathcal{D}$-related element a. Now write D the subgroup generated by all $\mathcal{D}$-related elements. Obviously D is a normal subgroup of G. Hence $M \cap D$ is a normal subgroup of G. Moreover we have $[M \cap D, M \cap D] \subseteq [M, M] \subseteq C_G(D)$, hence $[M \cap D, M \cap D] \subseteq Z(M \cap D)$. Thus $M \cap D$ is nilpotent and then abelian since all Sylow subgroups of G are abelian. Therefore the set $\{[a, x] \mid a \in D, x \in M\} \subseteq D \cap M$ is abelian. Arguing as in Section 3, we can now prove that $[G : C_G(a)]$ is m-bounded for every $\mathcal{D}$-related element a of G. It follows that G has a normal subgroup N of G of m-bounded index such that the order of $[N, N]$ is m-bounded, as required. □

7. Acknowledgment

This work was supported by the National Group for Algebraic and Geometric Structures, and their Applications (GNSAGA - INDAM), Italy.

The work of the third author was supported by FAPDF and CNPq-Brazil.

The third author would likewise like to thank the Department of Mathematical Sciences of the University of Bath, where some initial work was done during his visit to Bath, that was supported by an EPSRC grant held by the fourth author.

The fourth author is grateful to the Department of Mathematics of the University of Salerno for its hospitality and support, while this investigation was carried out.

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