On triple homomorphisms of Lie algebras

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Abstract

Let L and K be two Lie algebras over a commutative ring with identity. In this paper, under some conditions on L and K, it is proved that every triple homomorphism from L onto K is the sum of a homomorphism and an antihomomorphism from L into K. We also show that a finite-dimensional Lie algebra L over an algebraically closed field of characteristic zero is nilpotent of class at most 2 iff the sum of every homomorphism and every antihomomorphism on L is a triple homomorphism.

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1. Introduction

Throughout this paper, assume that R is a commutative ring with identity and L, K are two Lie algebras over R. An R-linear map $f : L \to K$ is a homomorphism if

$$f([x, y]) = [f(x), f(y)],$$

for any $x, y \in L$, is called an antihomomorphism if

$$f([x, y]) = [f(y), f(x)],$$

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for any $x, y \in L$, and is called a triple homomorphism if

$$f([x,[y,z]]) = [f(x),[f(y),f(z)]],$$

for any $x, y, z \in L$.

The set of all homomorphisms, all antihomomorphisms, and all triple homomorphisms from L into K is denoted by $\mathrm{Hom}(L,K)$, $\mathrm{AHom}(L,K)$, and $\mathrm{THom}(L,K)$, respectively. It can be easily seen that

$$\mathrm{AHom}(L,K) = -\mathrm{Hom}(L,K),$$

$$\mathrm{THom}(L,K) = -\mathrm{THom}(L,K),$$

and

$$\mathrm{Hom}(L,K) \cup \mathrm{AHom}(L,K) \subseteq \mathrm{THom}(L,K).$$

For a subset X of L, the centralizer of X in L is denoted by $C_L(X)$ and is defined as follows:

$$C_L(X) = \{y \in L |\ [x,y] = 0, \forall x \in X\}.$$

In particular, $Z(L) = C_L(L)$ is called the center of L. Now let $\{L^{(n)}\}_{n\geq 0}$, $\{L^n\}_{n\geq 1}$, and $\{Z_n(L)\}_{n\geq 0}$ denote, respectively, the derived series of L, the lower central series of L, and the upper central series of L, that is,

$$L^{(0)} = L, \quad L^{(n+1)} = [L^{(n)}, L^{(n)}],$$

$$L^1 = L, \quad L^{n+1} = [L, L^n],$$

$$Z_0(L) = 0, \quad Z_{n+1}(L)/Z_n(L) = Z(L/Z_n(L)).$$

A Lie algebra L is said to be simple if it is nonabelian and has only two ideals. Also a Lie algebra L is said to be perfect if $L = L^2$. Thus any simple Lie algebra is perfect. Finally a Lie algebra L is called solvable if $L^{(n)} = 0$ for some $n \geq 0$, and is called nilpotent if $L^n = 0$ for some $n \geq 1$ or equivalently $Z_n(L) = L$ for some $n \geq 0$.

Algebraic systems with derivations and their generalizations are a popular object of study nowadays. In particular, the algebras of derivations and generalized derivations are important in the study of algebraic systems of Lie type. Triple derivations, which are sometimes called prederivations, and their generalizations, Leibniz-derivations of order n, were used to study nilpotent Lie algebras, see [5] and the references therein for more information.

Some work has been done on triple homomorphisms of associative rings and algebras, see [3] and the references therein. Triple homomorphisms, which are sometimes called prehomomorphisms, and their generalizations, Leibniz-derivations of order n, were used to study nilpotent Lie algebras. In recent years, triple homomorphisms of Lie algebras were also studied in [7] and [8]. In [7], the authors proved that every triple automorphism of a parabolic subalgebra of a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic zero is either an automorphism or an antiautomorphism. Also, Zhou in [8] proved that every triple homomorphism from a perfect Lie algebra L onto a centerless Lie algebra K is the sum of a homomorphism and an antihomomorphism from L into K.

The paper is organized as follows. First some basic properties of triple homomorphisms are given in section 2, and then, under some conditions on L and K, it is shown that $\mathrm{THom}(L, K) = \{0\}$. In section 3, among other things, we generalize the main result of [8], by weakening the assumptions on L and K. Finally in section 4, some converse results related to section 3 are studied. Specially, we prove that if L is a finite-dimensional Lie algebra over an algebraically closed field of characteristic zero, then L is nilpotent of class at most 2 iff

$$\mathrm{Hom}(L, L) + \mathrm{AHom}(L, L) \subseteq \mathrm{THom}(L, L).$$

2. General properties of triple homomorphisms

Suppose that $f : L \to K$ is an R-linear map and $M = M_f$ is the subalgebra of K generated by the image of f. It is obvious that f maps $Z(L)$ into $Z(M)$ and L^2 into M^2, if f is either a homomorphism or an antihomomorphism.

We begin this section with a lemma which shows some properties of triple homomorphisms. We omit the easy proof because it relies only on the definitions.

Lemma 2.1. *Let* $f \in \mathrm{THom}(L, K)$ *and* $M = M_f$. *Then*
(i) $M = f(L) + [f(L), f(L)]$ *containing* $[f(L), f(L)]$ *as a subalgebra.*
(ii) $f(L^3) \subseteq M^3$ *and* $f(Z_2(L)) \subseteq Z_2(M)$.

(iii) *if* L *is perfect, then* $\ker f$ *is an ideal of* L *and* M *is a perfect subalgebra of* K*.*

It should be mentioned that in general, it is not true that $Z(L)$ and L^2 are mapped by f into $Z(M)$ and M^2, respectively, if f is a triple homomorphism. Also, the image and the kernel of a triple homomorphism need not always be a subalgebra, as the following example shows.

Example 2.2. *Let* L *be the Heisenberg Lie algebra, i.e. the Lie algebra of all* 3×3 *strictly upper triangular matrices over* R *with the standard basis* $\{e_{12}, e_{13}, e_{23}\}$*. Then one can easily see that* $Z(L) = L^2 = Re_{13}$*,* $L^3 = 0$*,* $Z_2(L) = L$*, and every* R*-linear map on* L *is a triple homomorphism of* L*. Now consider the* R*-linear maps* f, g, h *on* L *defined via*

$$f(e_{12}) = f(e_{23}) = f(e_{13}) = e_{12},$$

$$g(e_{12}) = e_{12}, g(e_{23}) = e_{23}, g(e_{13}) = 0,$$

$$h(e_{12}) = h(e_{23}) = 0, h(e_{13}) = e_{13}.$$

All three maps are triple homomorphisms but none of them are a homomorphism or an antihomomorphism. Also, f *does not map* $Z(L) = L^2$ *into itself, and* $\mathrm{Img} = \ker h$ *is not a subalgebra of* L*. Note that the map* $g : L \to M_g$ *is not onto because* $\mathrm{Img} \subset M_g = L$*.*

The next example shows that even if L is perfect and f is a triple homomorphism from L into K, and thus $M = M_f$ is perfect by Lemma 2.1, it doesn't necessarily follow that $M = f(L)$ which would be the case if f is an (anti)homomorphism.

Example 2.3. *Let* L *be any perfect Lie algebra over a field and let* L_0 *and* L_1 *be two isomorphic copies of the vector space* L *via the isomorphisms* $\varphi_i : L \to L_i$ *for any* $i \in \mathbb{Z}_2$*, and suppose that* $x_i = \varphi_i(x)$*. Hence*

$$L_0 = \{x_0 : x \in L\}, \quad L_1 = \{x_1 : x \in L\}.$$

We now make the vector space $K = L_0 \oplus L_1$ *into a graded Lie algebra over* $\mathbb{Z}_2$ *via the Lie product:*

$$[x_i, y_j] = [x, y]_{i+j},$$

and then extend linearly. Note that the Jacobi identity clearly holds in K as it holds in L and L_0 is a Lie subalgebra of K. Now consider the linear map $f : L \to K$ defined by $f(x) = x_1$. Thus for any $x, y, z \in L$ one has

$$\begin{aligned} [f(x), [f(y), f(z)]] &= [x_1, [y_1, z_1]] \\ &= [x_1, [y, z]_0] \\ &= [x, [y, z]]_1 \\ &= f([x, [y, z]]), \end{aligned}$$

which means that f is a triple homomorphism. Since L is perfect, it is easy to see that

$$[f(L), f(L)] = [L_1, L_1] = [L, L]_0 = L_0,$$

and therefore

$$f(L) = L_1 \subset K = L_1 + L_0 = f(L) + [f(L), f(L)] = M_f.$$

Our first theorem gives a simple criterion for a linear map to be a triple homomorphism if K is 3-torsion free, i.e. if $0 \neq x \in K$, then $3x \neq 0$.

Theorem 2.4. *Let K be 3-torsion free. Then an R-linear map $f : L \to K$ is a triple homomorphism iff*

$$f([x, [x, y]]) = [f(x), [f(x), f(y)]],$$

for any $x, y \in L$.

Proof. One part is trivial. For the other part, for any $x, y, z \in L$, we have

$$f([x, [x, z]]) = [f(x), [f(x), f(z)]],$$

$$f([y, [y, z]]) = [f(y), [f(y), f(z)]],$$

$$f([x + y, [x + y, z]]) = [f(x + y), [f(x + y), f(z)]],$$

so

$$f([x, [y, z]]) + f([y, [x, z]]) = [f(x), [f(y), f(z)]] + [f(y), [f(x), f(z)]]. \quad (*)$$

Using the Jacobi identity, one has

$$f([z, [y, x]]) + 2f([y, [x, z]]) = [f(z), [f(y), f(x)]] + 2[f(y), [f(x), f(z)]].$$

Changing the role of x and z in the above relation, we obtain

$$f([x,[y,z]]) + 2f([y,[z,x]]) = [f(x),[f(y),f(z)]] + 2[f(y),[f(z),f(x)]]. \quad (**)$$

Now adding 2 times $(*)$ to $(**)$, we get

$$3f([x,[y,z]]) = 3[f(x),[f(y),f(z)]],$$

which implies that f is a triple homomorphism, for K is 3-torsion free. $\square$

We show that the assumption "K is 3-torsion free" is essential in Theorem 2.4. First we need a lemma about 2-Engel Lie algebras. Recall that a Lie algebra L is said to be 2-Engel if $[x,[x,y]] = 0$, for any $x, y \in L$.

Lemma 2.5. *Let L be a 2-Engel Lie algebra over a field $\mathbb{F}$. Then*
(i) $[x,[y,z]] = [y,[z,x]] = [z,[x,y]]$ *and* $3[x,[y,z]] = 0$ *for any* $x, y, z \in L$.
(ii) $L^4 = 0$ *and if* $\mathrm{char}\mathbb{F} \neq 3$*, then* $L^3 = 0$.
(iii) *if* $L^3 \neq 0$*, then* $\dim L \geq 7$.

Proof. (i) and (ii): For a proof the reader is referred to either Theorem 3.1.1 of [6] or Lemma 2.1 and Theorem 2.3 of [4].
(iii): Let $L^3 \neq 0$. So $\mathbb{F}$ is of characteristic 3 and there exist three elements $x, y, z \in L$ such that $[x,[y,z]] \neq 0$. Using part (i) and the fact that L is 2-Engel and $L^4 = 0$, one can easily check that the set

$$\{x, y, z, [x,y], [y,z], [z,x], [x,[y,z]]\}$$

is linearly independent over $\mathbb{F}$ and so $\dim L \geq 7$. $\square$

Example 2.6. *First we construct a 2-Engel Lie algebra L with $L^3 \neq 0$ over any field $\mathbb{F}$ of characteristic 3. Let L be the 7-dimensional vector space over $\mathbb{F}$ with the basis $\mathcal{B} = \{x, y, z, a, b, c, u\}$. Define the non-zero brackets as follows:*

$$[x,a] = [y,b] = [z,c] = u, [y,z] = a, [z,x] = b, [x,y] = c.$$

Since $u \in Z(L)$, $a, b, c \in Z_2(L)$, and $[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 3u = 0$, the elements of $\mathcal{B}$ satisfy the Jacobi identity and so L is a Lie algebra. Also, for any $\alpha, \beta, \gamma \in \mathcal{B}$ we have

$$[\alpha,[\alpha,\gamma]] = 0, \ [\alpha,[\beta,\gamma]] + [\beta,[\alpha,\gamma]] = 0,$$

which means that L is 2-Engel.
Now if we let f be the linear map on L sending all elements of $\mathcal{B}$ to u, then f obviously is not a triple homomorphism but it satisfies the condition in Theorem 2.4.

The following theorem says that in some cases the zero map is the only triple homomorphism.

Theorem 2.7. *The following statements hold:*
(i) *If* $f \in \mathrm{THom}(L, K)$, *then* $f(L^{(n+1)}) \subseteq K^{(n)}$, *for any non-negative integer* n.
(ii) *If* L *is perfect, then* $\mathrm{THom}(L, K) = \mathrm{THom}(L, \bigcap_{n=1}^{\infty} K^{(n)})$.
(iii) *If* L *is perfect and* $\bigcap_{n=1}^{\infty} K^{(n)} = \{0\}$, *then* $\mathrm{THom}(L, K) = \{0\}$.

Proof. (i): We prove by induction on n. The case $n = 0$ is obvious. If $n = 1$, then

$$f(L^{(2)}) = f([L^{(1)}, [L, L]]) = [f(L^{(1)}), [f(L), f(L)]]) \subseteq [K, [K, K]] \subseteq K^{(1)},$$

as desired. Assume now that $n \geq 2$. Hence

$$\begin{aligned} f(L^{(n+1)}) &= f([L^{(n)}, [L^{(n-1)}, L^{(n-1)}]]) \\ &= [f(L^{(n)}), [f(L^{(n-1)}), f(L^{(n-1)})]] \\ &\subseteq [K^{(n-1)}, [K^{(n-2)}, K^{(n-2)}]] \\ &= K^{(n)}, \end{aligned}$$

which completes the proof.

(ii): This follows at once from part (i).

(iii): This follows immediately from part (ii). □

As a consequence, we obtain the following.

Corollary 2.8. *Let* L *be a perfect Lie algebra and* K *be either a solvable Lie algebra or a free Lie algebra. Then* $\mathrm{THom}(L, K) = \{0\}$.

Proof. By part (iii) of Theorem 2.7, it suffices to show that $\bigcap_{n=1}^{\infty} K^{(n)} = \{0\}$. This is obvious if K is solvable. So we may assume that K is a free Lie algebra. Taking advantage of the obvious relation $K^{(n)} \subseteq K^{2^n}$ and the well known fact $\bigcap_{n=1}^{\infty} K^n = \{0\}$ for the free Lie algebra K, see page 182 of [2], one sees that $\bigcap_{n=0}^{\infty} K^{(n)} = \{0\}$, and the proof is completed. □

3. Conditions implying a triple homomorphism to be the sum of a homomorphism and an antihomomorphism

In this section, we will assume that $2 \in R$ is unit. The following fundamental theorem, which is a generalization of the main result of [8], plays a crucial role in the next results.

The following theorem plays a crucial role in the next results.

Theorem 3.1. *Let $f \in \mathrm{THom}(L, K)$ with $M^2 \cap Z(M) = 0$, where $M = M_f$. Then the map $\delta_f : L^2 \to M^2$ defined by*

$$\delta_f(x) = \sum_{i=1}^{n} [f(a_i), f(b_i)],$$

when $x = \sum_{i=1}^{n} [a_i, b_i]$ for some n, is a well-defined homomorphism. Furthermore, the maps $\varphi = f + \delta_f : L^2 \to M$ and $\psi = f - \delta_f : L^2 \to M$ satisfy the following properties:
(i) $[\mathrm{Im}\varphi, \mathrm{Im}\psi] = 0$.
(ii) $\mathrm{Im}\varphi$ *and* $\mathrm{Im}\psi$ *are ideals of M.*
(iii) $\frac{1}{2}\varphi \in \mathrm{Hom}(L^2, K)$, $\frac{1}{2}\psi \in \mathrm{AHom}(L^2, K)$, *and $f_{|L^2} = \frac{1}{2}\varphi + \frac{1}{2}\psi$. In other words, the restriction of f to L^2 is the sum of a homomorphism and an antihomomorphism from L^2 into K.*

Proof. Let $x = \sum_{i=1}^{n} [a_i, b_i] = \sum_{i=1}^{m} [c_i, d_i]$ be two expressions for x, and let

$$\alpha = \sum_{i=1}^{n} [f(a_i), f(b_i)]$$

and

$$\beta = \sum_{i=1}^{m} [f(c_i), f(d_i)].$$

Obviously, $\alpha, \beta \in M^2$. Since $f \in \mathrm{THom}(L, K)$, for any $y \in L$, one has

$$\begin{aligned} [f(y), \alpha] &= \sum_{i=1}^{n} [f(y), [f(a_i), f(b_i)]] \\ &= \sum_{i=1}^{n} f([y, [a_i, b_i]]) \\ &= f([y, x]). \end{aligned}$$

Similarly, $[f(y), \beta] = f([y, x])$. Therefore, $\alpha - \beta \in M^2 \bigcap C_M(f(L))$. From $M = f(L) + [f(L), f(L)]$ and the Jacobi identity, it follows that $\alpha - \beta \in M^2 \bigcap Z(M) = 0$. Hence $\alpha = \beta$ and δ_f is well defined. By definition, δ_f is an R-linear map. What remains is to show that $\delta_f \in \mathrm{Hom}(L^2, M^2)$. The above relation shows that

$$[\delta_f(x), f(y)] = f([x, y]), \quad (*)$$

the relation which will be used frequently.

From this we see that for any $x, y \in L^2$ and $z \in L$,

$$\begin{aligned}
[f(z), \delta_f([x, y])] &= [f(z), [f(x), f(y)]] \\
&= [f(x), [f(z), f(y)]] + [f(y), [f(x), f(z)]] \\
&= f([x, [z, y]]) + f([y, [x, z]]) \\
&= [\delta_f(x), f([z, y])] + [\delta_f(y), f([x, z])] \\
&= [\delta_f(x), [f(z), \delta_f(y)]] + [\delta_f(y), [\delta_f(x), f(z)]] \\
&= [f(z), [\delta_f(x), \delta_f(y)]].
\end{aligned}$$

Again since $M = f(L) + [f(L), f(L)]$, it follows that

$$\delta_f([x, y]) - [\delta_f(x), \delta_f(y)] \in M^2 \bigcap Z(M) = 0,$$

showing that $\delta_f \in \mathrm{Hom}(L^2, M^2)$.

It remains to show that φ and ψ satisfy (i)-(iii).

(i): Taking advantage of $(*)$ and that $\delta_f \in \mathrm{Hom}(L^2, M^2)$, for any $x, y \in L^2$, one has

$$\begin{aligned}
[\varphi(x), \psi(y)] &= [f(x) + \delta_f(x), f(y) - \delta_f(y)] \\
&= [f(x), f(y)] - [f(x), \delta_f(y)] + [\delta_f(x), f(y)] - [\delta_f(x), \delta_f(y)] \\
&= [f(x), f(y)] - f([x, y]) + f([x, y]) - \delta_f([x, y]) \\
&= 0.
\end{aligned}$$

(ii): Using $(*)$ again and definition of δ_f, for any $x \in L^2$ and $y \in L$, one has

$$\begin{aligned}[(f \pm \delta_f)(x), f(y)] &= [f(x), f(y)] \pm [\delta_f(x), f(y)] \\ &= \delta_f([x, y]) \pm f([x, y]) \\ &= (\delta_f \pm f)([x, y]),\end{aligned}$$

implying that

$$[\varphi(x), f(y)] = \varphi([x, y]),$$
$$[\psi(x), f(y)] = \psi([y, x]).$$

Once again from $M = f(L) + [f(L), f(L)]$ and the Jacobi identity, we see that $\mathrm{Im}\varphi$ and $\mathrm{Im}\psi$ are ideals of M.

(iii): Using $(*)$ again and definition of δ_f and $\delta_f \in \mathrm{Hom}(L^2, M^2)$, for any $x, y \in L^2$, one has

$$\begin{aligned}[\varphi(x), \varphi(y)] &= [f(x), f(y)] + [f(x), \delta_f(y)] + [\delta_f(x), f(y)] + [\delta_f(x), \delta_f(y)] \\ &= [f(x), f(y)] + f([x, y]) + f([x, y]) + \delta_f([x, y]) \\ &= 2[f(x), f(y)] + 2f([x, y]) \\ &= 2\delta_f([x, y]) + 2f([x, y]) \\ &= 2\varphi([x, y]),\end{aligned}$$

showing that

$$\frac{1}{2}\varphi([x, y]) = [\frac{1}{2}\varphi(x), \frac{1}{2}\varphi(y)].$$

This means that $\frac{1}{2}\varphi \in \mathrm{Hom}(L^2, K)$. In a similar manner, one obtains $\frac{1}{2}\psi \in \mathrm{AHom}(L^2, K)$. Also, we see that $f(x) = \frac{1}{2}\varphi(x) + \frac{1}{2}\psi(x)$ and this completes the proof. $\square$

Remark. It should be noted that in Theorem 3.1, the map $\delta_f : L^2 \to [f(L), f(L)]$ is well-defined and a homomorphism under the weaker condition $[f(L), f(L)] \cap Z(M) = 0$ and all parts are true. Also, the condition "$2 \in R$ is unit" has been used only in part (iii) of Theorem 3.1.

With the notation of the previous theorem, we have:

Theorem 3.2. *Let $L = A \oplus B$, where A is a perfect Lie algebra and B is an abelian Lie algebra, and $f \in \mathrm{THom}(L, K)$ with $M^2 \bigcap Z(M) = 0$, where $M = M_f$. Then $M = \mathrm{Im}\varphi \oplus \mathrm{Im}\psi \oplus f(B)$ and f is the sum of a homomorphism and an antihomomorphism from L into K.*

Proof. First note that $L^3 = L^2 = A$ and $B \subseteq Z(L)$. Also, $Z(M) = Z_2(M)$ because if $x \in Z_2(M)$, then $[x, M] \subseteq M^2 \bigcap Z(M) = 0$, this is, $x \in Z(M)$. We know that the triple homomorphism f maps $Z_2(L)$ into $Z_2(M)$ and hence $f(B) \subseteq Z(M)$. It then follows that $f(B)$ is an ideal of M. Also, Theorem 3.1 implies that $\mathrm{Im}\varphi$ and $\mathrm{Im}\psi$ are ideals of M. Now for any $a \in A$ and $b \in B$ one gets

$$f(a + b) = f(a) + f(b) = \frac{1}{2}\varphi(a) + \frac{1}{2}\psi(a) + f(b),$$

implying that $M = \mathrm{Im}\varphi + \mathrm{Im}\psi + f(B)$, because $\mathrm{Im}\varphi + \mathrm{Im}\psi + f(B)$ is an ideal of M containing $f(L)$. Now one can see from $\delta_f(L^2) \subseteq M^2$ and $f(L^2) = f(L^3) \subseteq M^3 \subseteq M^2$ that $(\mathrm{Im}\varphi + \mathrm{Im}\psi) \cap f(B) \subseteq M^2 \bigcap Z(M) = 0$. Hence $M = (\mathrm{Im}\varphi + \mathrm{Im}\psi) \oplus f(B)$. Suppose now that $x \in \mathrm{Im}\varphi \cap \mathrm{Im}\psi$. Therefore, for any $a \in A$ and $b \in B$, one obtains using part (i) of Theorem 3.1 that

$$[x, f(a + b)] = [x, f(a)] = \frac{1}{2}([x, \varphi(a)] + [x, \psi(a)]) = 0,$$

so $x \in Z(M)$. Since $x \in M^2$, we get by hypothesis that $x = 0$, as desired.
It remains to show that the last part of the theorem holds. To this end, we extend the functions $\frac{1}{2}\varphi$ and $\frac{1}{2}\psi$ to a homomorphism g and an antihomomorphism h from L into K, respectively, as follows:

$$g(a + b) = \frac{1}{2}\varphi(a) + f(b), \quad h(a + b) = \frac{1}{2}\psi(a).$$

(Notice that we do not necessarily have unique extensions if $f(B) \neq 0$). $\square$

As an immediate consequence, we have the following.

Corollary 3.3. *Let $L = A \oplus B$, where A is a perfect Lie algebra and B is an abelian Lie algebra, K be a Lie algebra with $K^2 \bigcap Z(K) = 0$. Then every triple homomorphism from L onto K is the sum of a homomorphism and an antihomomorphism from L into K.*

Notice that the above corollary is true if K is semisimple, that is, K is a direct sum of simple Lie algebras.

For simple Lie algebras, we obtain the following strong result.

Theorem 3.4. *Let $L = A \oplus B$ and $K = C \oplus D$, where A is a perfect Lie algebra, C is a simple Lie algebra, and B, D are abelian Lie algebras. Then every triple homomorphism from L onto K is either a homomorphism or an antihomomorphism from L onto K.*

Proof. Suppose that $f \in \mathrm{THom}(L, K)$ is onto. It is not difficult to see that $B \subseteq Z(L), L^3 = L^2 = A, D = Z(K) = Z_2(K)$, and $K^3 = K^2 = C$. Therefore, one has $f(A) \subseteq C$ and $f(B) \subseteq D$. But f is surjective, hence $f(A) = C$ and $f(B) = D$. It then follows that $f_{|A} \in \mathrm{THom}(A, C)$ is onto. Thus, using Theorem 3.2, $C = \mathrm{Im}\varphi \oplus \mathrm{Im}\psi$. Since C is simple, one obtains either $\psi = 0$ or $\varphi = 0$, which implies that $f_{|A}$ is either a homomorphism or an antihomomorphism from A onto C. Since B and D are both abelian and $[A, B] = [C, D] = 0$, we conclude that f is either a homomorphism or an antihomomorphism from L onto K.

□

In particular, every triple homomorphism from a perfect Lie algebra onto a simple Lie algebra is either a homomorphism or an antihomomorphism.

Let us record a useful consequence of our previous results.

Corollary 3.5. *Let $L = A \oplus B$ and $K = A \oplus D$, where A is a finite-dimensional simple Lie algebra over a field of characteristic not 2, and B, D are abelian Lie algebras. Then*
(i) *every nonzero triple homomorphism on A is either an automorphism or an antiautomorphism. In particular,*

$$\mathrm{THom}(A, A) = \mathrm{Hom}(A, A) \cup \mathrm{AHom}(A, A).$$

(ii) $\mathrm{THom}(L, K) = \mathrm{Hom}(L, K) \cup \mathrm{AHom}(L, K)$.

Proof. (i): Assume that f is a nonzero triple homomorphism on A. By Lemma 2.1, $\ker f$ is a proper ideal of A and so f is injective, for A is simple. Since A is finite-dimensional, we deduce that f is surjective. Now the result follows from Theorem 3.4.

(ii): Assuming that $f \in \mathrm{THom}(L, K)$, one can see, similar to the proof of Theorem 3.4, that $B = Z(L), D = Z(K), f(A) \subseteq A$, and $f(B) \subseteq D$. Using part (i), it follows that $f_{|A}$ is either an homomorphism or an antihomomorphism on A. Since B and D are both abelian and $[A, B] = [A, D] = 0$, we see that f is either a homomorphism or an antihomomorphism from L into K, and the proof is complete. $\square$

Here we give four examples. The first example, which is related to the general linear Lie algebra, is an application of Corollary 3.5.

Example 3.6. *Let $L = gl_n(\mathbb{F})$ be the general linear Lie algebra over a field $\mathbb{F}$ of characteristic not dividing n. Then $L = sl_n(\mathbb{F}) \oplus Z(L)$, where $sl_n(\mathbb{F})$ is the special linear Lie algebra and $Z(L)$ is the set of scalar matrices. Since $sl_n(\mathbb{F})$ is simple, we have by Corollary 3.5 that if $\mathrm{char}\mathbb{F} \neq 2$, then every triple homomorphism on L is either a homomorphism or an antihomomorphism.*

In the second example, it is shown that the sum of a homomorphism and an antihomomorphism need not be a triple homomorphism at all.

Example 3.7. *Let $L = \langle x, y | \ [x, y] = x \rangle$ be the two-dimensional nonabelian Lie algebra over an arbitrary field $\mathbb{F}$. Suppose the linear maps f, g on L are given by*

$$f(x) = x, f(y) = y,$$
$$g(x) = 0, g(y) = -y.$$

Obviously, $f \in \mathrm{Hom}(L, L)$, $g \in \mathrm{AHom}(L, L)$, but $f + g \notin \mathrm{THom}(L, L)$. It is not too tedious to check that every triple homomorphism on L is either a homomorphism or an antihomomorphism. Also, if $\mathrm{char}\mathbb{F} = 2$, then $\mathrm{THom}(L, L) = \mathrm{Hom}(L, L) = \mathrm{AHom}(L, L)$.

Theorem 3.4 is not true if we weaken the assumption to a semisimple Lie algebra K, as the third example shows.

Example 3.8. *Let $L = A \oplus B$, where A and B are two simple Lie algebras. Hence L is a perfect Lie algebra with trivial center. Consider now the linear map f on L via $f(a+b) = a-b$, where $a \in A$ and $b \in B$. It is readily verified that f is a bijective triple homomorphism. But f is neither a homomorphism nor an antihomomorphism, otherwise we must have for any $a, a' \in A$ and $b, b' \in B$ that*

$$\begin{aligned}
[a, a'] - [b, b'] &= f([a, a'] + [b, b']) \\
&= f([a + b, a' + b']) \\
&= \pm[f(a + b), f(a' + b')] \\
&= \pm[a - b, a' - b'] \\
&= \pm([a, a'] + [b, b']),
\end{aligned}$$

which implies that either $2[a,a']=0$ *or* $2[b,b']=0$. *Since* $2\in R$ *is unit, hence either* A *is abelian or* B *is abelian, a contradiction. It is also clear that* $f=g+h$, *where* $g\in \mathrm{Hom}(L,L)$ *and* $h\in \mathrm{AHom}(L,L)$ *are defined via*

$$g(a+b)=a, \quad h(a+b)=-b,$$

for any $a\in A$ *and* $b\in B$.

Finally the fourth example shows that Theorem 3.2 is not true if A is not perfect. It also reveals that there exist two non-isomorphic Lie algebras such that there exists a bijective triple homomorphism between them.

Example 3.9. *Let* L *be the Heisenberg Lie algebra which mentioned earlier in Example 2.2. We have that* $L^3=0$ *and that* K *is abelian, and hence* $\mathrm{THom}(L,K)$ *is the set of all linear maps from* L *into* K. *Also, it can be easily seen that* $\mathrm{Hom}(L,K)=\mathrm{AHom}(L,K)$ *is the set of all linear maps from* L *into* K *which map* $L^2=Re_{13}$ *to* 0. *Therefore, if* $K\neq 0$ *and* $f:L\to K$ *is any linear map sending* e_{13} *to any nonzero element of* K, *then* f *cannot be written as the sum of a homomorphism and an antihomomorphism from* L *into* K. *Also, if* $\dim K=3$ *and* $f:L\to K$ *is an invertible linear map, then* f *is a bijective triple homomorphism from* L *onto* K *and obviously* L *and* K *are not isomorphic.*

We close this section with a nice application of Theorem 3.1.

Theorem 3.10. *Let* $f:L\to K$ *be a surjective triple homomorphism such that* $\ker f\subseteq Z(L)$. *If* L *is perfect, then* K *is perfect too and* $L/Z(L)$ *is isomorphic to* $K/Z(K)$.

Proof. Since L is perfect, one has $Z_2(L)=Z(L)$ by Grün's lemma. By Lemma 2.1, K is perfect too and so again $Z_2(K)=Z(K)$. Thus $f(Z(L))\subseteq Z(K)$ by Lemma 2.1 and so $Z(L)\subseteq f^{-1}(Z(K))$. Also we have $f^{-1}(Z(K))\subseteq Z(L)$ because if $z\in f^{-1}(Z(K))$, then $f(z)\in Z(K)$ and so for any $x,y\in L$

$$f([x,[y,z]])=[f(x),[f(y),f(z)]]=0,$$

showing that $[x,[y,z]]\in \ker f\subseteq Z(L)$. Therefore $z\in Z_3(L)=Z(L)$. Thus we proved that $f^{-1}(Z(K))=Z(L)$ and $f(Z(L))=Z(K)$. This implies that f induces a bijective triple homomorphism $\bar{f}:L/Z(L)\to K/Z(K)$ defined via $\bar{f}(\bar{x})=\overline{f(x)}$, where $\bar{x}=x+Z(L)$ and $\overline{f(x)}=f(x)+Z(K)$.

But $K/Z(K)$ has trivial center, hence, using Theorem 3.1, one obtains that $\delta_{\bar{f}} : L/Z(L) \to K/Z(K)$ is a homomorphism. Taking advantage of the facts that L and K are perfect and $\bar{f}$ is a surjection, we deduce that $\delta_{\bar{f}}$ is an epimorphism. The proof is completed if we show that $\delta_{\bar{f}}$ is injective. Since L is perfect, suppose that $\bar{x} = \sum_{i=1}^{n} [\bar{a}_i, \bar{b}_i] \in \ker \delta_{\bar{f}}$ is arbitrary. Hence

$$\bar{0} = \delta_{\bar{f}}(\bar{x}) = \sum_{i=1}^{n} [\bar{f}(\bar{a}_i), \bar{f}(\bar{b}_i)] = \sum_{i=1}^{n} [\overline{f(a_i)}, \overline{f(b_i)}] = \overline{\sum_{i=1}^{n} [f(a_i), f(b_i)]},$$

which means that $\sum_{i=1}^{n} [f(a_i), f(b_i)] \in Z(K)$. Thus for any $c \in L$

$$0 = \sum_{i=1}^{n} [f(c), [f(a_i), f(b_i)]] = \sum_{i=1}^{n} f([c, [a_i, b_i]]) = f([c, \sum_{i=1}^{n} [a_i, b_i]]),$$

meaning that $[c, \sum_{i=1}^{n} [a_i, b_i]] \in Z(L)$ and so $\sum_{i=1}^{n} [a_i, b_i] \in Z_2(L) = Z(L)$, that is $\bar{x} = \bar{0}$. □

As a consequence we have:

Corollary 3.11. *Let $f : L \to K$ be a bijective triple homomorphism, where L is perfect. Then*
(i) *K is perfect too and $L/Z(L)$ is isomorphic to $K/Z(K)$.*
(ii) *if L has trivial center, then L and K are isomorphic.*

4. Some converse results

One can easily see that if L is abelian, then

$$\mathrm{THom}(L, L) = \mathrm{Hom}(L, L) = \mathrm{AHom}(L, L)$$

is the set of all linear maps on L. Therefore, if L is abelian, then

$$\mathrm{THom}(L, L) = \mathrm{Hom}(L, L) \cup \mathrm{AHom}(L, L),$$

and

$$\mathrm{THom}(L, L) = \mathrm{Hom}(L, L) + \mathrm{AHom}(L, L).$$

Conversely, if $2 \in R$ is unit, then each of the following statements:
(1) $\mathrm{Hom}(L, L) = \mathrm{AHom}(L, L)$,
(2) $\mathrm{THom}(L, L) = \mathrm{Hom}(L, L)$,

(3) $\mathrm{THom}(L, L) = \mathrm{AHom}(L, L)$,
implies that L is abelian. Notice also that the above equalities hold for the two-dimensional nonabelian Lie algebra over a field of characteristic 2.
As seen in the previous section, the relation

$$\mathrm{THom}(L, L) = \mathrm{Hom}(L, L) \cup \mathrm{AHom}(L, L),$$

again holds under each of the following conditions:
(1) $L = A \oplus B$, where A is a finite-dimensional simple Lie algebra over a field of characteristic not 2 and B is an abelian Lie algebra,
(2) L is the two-dimensional nonabelian Lie algebra over a field.

It seems to be a difficult task to characterize all Lie algebras L for which

$$\mathrm{THom}(L, L) = \mathrm{Hom}(L, L) \cup \mathrm{AHom}(L, L).$$

However, we will give a necessary condition for the above equality. Our aim in the sequel is to characterize all Lie algebras L for which

$$\mathrm{THom}(L, L) = \mathrm{Hom}(L, L) + \mathrm{AHom}(L, L).$$

For the rest of this section, we will assume that L and K are Lie algebras over a field $\mathbb{F}$.

Theorem 4.1. *The following statements hold:*
(i) *If* $\mathrm{Hom}(L, K) + \mathrm{AHom}(L, K) \subseteq \mathrm{THom}(L, K)$*, then either* L/L^2 *has dimension at most* 1 *or* K *is* 2*-Engel.*
(ii) *If* $\mathrm{THom}(L, K) \subseteq \mathrm{Hom}(L, K) + \mathrm{AHom}(L, K)$*, then either* $L^2 = L^3$ *or* K *is perfect.*
(iii) *If* $\mathrm{THom}(L, K) = \mathrm{Hom}(L, K) \cup \mathrm{AHom}(L, K)$*, then either* $L^2 = L^3$ *or* $K = 0$*.*

Proof. (i): We may assume that $\dim L/L^2 \geq 2$ and claim that K is 2-Engel. To do this, suppose that $\mathcal{B}_1$ is a basis of L^2, $\mathcal{B}$ is a basis of L containing $\mathcal{B}_1$, and let $a, b \in \mathcal{B} \setminus \mathcal{B}_1$ be distinct. Consider now the linear maps $f, g : L \to K$ so that

$$f(a) = x, f(\mathcal{B} \setminus \{a\}) = 0,$$

$$g(b) = y, g(\mathcal{B} \setminus \{b\}) = 0,$$

where $x, y \in K$ are arbitrary. Obviously, $f \in \mathrm{Hom}(L, K)$ and $g \in \mathrm{AHom}(L, K)$ and so by hypothesis $\varphi = f + g \in \mathrm{THom}(L, K)$. Hence

$$[x, [x, y]] = [\varphi(a), [\varphi(a), \varphi(b)]] = \varphi([a, [a, b]]) = 0,$$

as required.

(ii) and (iii): We may assume that $L^2 \neq L^3$ and let $z \in K$ be arbitrary. Hence there exist two elements $a, b \in L$ so that $[a, b] \in L^2 \setminus L^3$. Suppose now that $\mathcal{B}_1$ is a basis of L^3, $\mathcal{B}_2$ is a basis of L^2 containing $\mathcal{B}_1 \cup \{[a, b]\}$, and $\mathcal{B}$ is a basis of L containing $\mathcal{B}_2$. Consider the linear map f which sends $[a, b]$ to z and other elements of $\mathcal{B}$ to zero. Clearly, $f \in \mathrm{THom}(L, K)$.

Suppose first that $f \in \mathrm{Hom}(L, K) + \mathrm{AHom}(L, K)$. Hence $f = g + h$, where $g \in \mathrm{Hom}(L, K)$ and $h \in \mathrm{AHom}(L, K)$. Since $g(L^2), h(L^2) \subseteq K^2$, one obtains

$$z = f([a, b]) = g([a, b]) + h([a, b]) \subseteq K^2,$$

which shows that $K = K^2$.

Suppose now that $f \in \mathrm{Hom}(L, K) \cup \mathrm{AHom}(L, K)$. Hence one obtains

$$z = f([a, b]) = \pm[f(a), f(b)] = 0,$$

showing that $K = 0$, and the proof is complete. □

The corollary below says that if all triple homomorphisms on a nilpotent Lie algebra can be written as the sum of a homomorphism and an antihomomorphism, then it is necessarily abelian.

Corollary 4.2. *Let L be a nilpotent Lie algebra. Then the following are equivalent:*
(i) *L is abelian,*
(ii) $\mathrm{THom}(L, L) = \mathrm{Hom}(L, L) = \mathrm{AHom}(L, L)$,
(iii) $\mathrm{THom}(L, L) = \mathrm{Hom}(L, L) \cup \mathrm{AHom}(L, L)$,
(iv) $\mathrm{THom}(L, L) \subseteq \mathrm{Hom}(L, L) + \mathrm{AHom}(L, L)$.

The above corollary shows that for any nonabelian nilpotent Lie algebra L, there exists a triple homomorphism on L which cannot be written as the sum of a homomorphism and an antihomomorphism. It should also be remarked that if L is a nilpotent Lie algebra of class 2 (i.e. $L^3 = 0$ and $L^2 \neq 0$), then $\mathrm{THom}(L, L)$ is the set of all linear maps on L, and hence

$$\mathrm{Hom}(L, L) + \mathrm{AHom}(L, L) \subset \mathrm{THom}(L, L).$$

We now concentrate on Lie algebras for which the sum of every homomorphism and every antihomomorphism is a triple homomorphism. To be precise, we obtain:

Theorem 4.3. *Let L be a Lie algebra such that*

$$\mathrm{Hom}(L,L)+\mathrm{AHom}(L,L)\subseteq \mathrm{THom}(L,L).$$

Then either L is perfect or L is nilpotent of class at most 2.

Proof. We may assume that L is not perfect and consider two separate cases. Suppose first that $\dim L/L^2 \geq 2$. Let $\mathcal{B}_1$ be a basis of L^2, $\mathcal{B}$ be a basis of L containing $\mathcal{B}_1$, and let $a, b \in \mathcal{B}\setminus\mathcal{B}_1$ be distinct. Assume that f is the identity map on L and g is a linear map on L given by:

$$g(a)=x, g(\mathcal{B}\setminus\{a\})=0,$$

where $x\in L$ is arbitrary. Clearly, $f\in \mathrm{Hom}(L,L)$ and $g\in \mathrm{AHom}(L,L)$, and hence $\varphi=f+g\in \mathrm{THom}(L,L)$ by hypothesis. Therefore, for any $y,z\in\mathcal{B}\setminus\{a\}$,

$$[z,[y,a]]=\varphi([z,[y,a]])=[\varphi(z),[\varphi(y),\varphi(a)]]=[z,[y,a+x]],$$

which implies that $[z,[y,x]]=0$, i.e. $[\mathcal{B}\setminus\{a\},[\mathcal{B}\setminus\{a\},\mathcal{B}]]=0$. Similarly, $[\mathcal{B}\setminus\{b\},[\mathcal{B}\setminus\{b\},\mathcal{B}]]=0$. Thus, by combining the two latter results just obtained, we proved that $[\mathcal{B},[\mathcal{B},\mathcal{B}]]=0$, that is, L is nilpotent of class at most 2.

Suppose now that $\dim L/L^2=1$. Let $\mathcal{B}=\mathcal{B}_1\cup\{a\}$ be a basis of L, where $\mathcal{B}_1$ is a basis of L^2 and $a\in L\setminus L^2$. Again assume that f is the identity map on L and g is a linear map on L as follows:

$$g(a)=x, g(\mathcal{B}_1)=0,$$

where $x\in L$ is arbitrary. Clearly, $f\in\mathrm{Hom}(L,L)$ and $g\in\mathrm{AHom}(L,L)$, and so $\varphi=f+g\in\mathrm{THom}(L,L)$ by hypothesis. Therefore, for any $y,z\in\mathcal{B}_1$,

$$[z,[y,a]]=\varphi([z,[y,a]])=[\varphi(z),[\varphi(y),\varphi(a)]]=[z,[y,a+x]],$$

$$[a,[a,y]]=\varphi([a,[a,y]])=[\varphi(a),[\varphi(a),\varphi(y)]]=[a+x,[a+x,y]].$$

The first relation implies that $[z,[y,x]]=0$, i.e. $[\mathcal{B}_1,[\mathcal{B}_1,\mathcal{B}]]=0$. Hence $[a,[\mathcal{B}_1,\mathcal{B}_1]]=0$ by the Jacobi identity. Also, by putting $x=-a$ in the second relation, one obtains $[a,[a,y]]=0$, that is, $[a,[a,\mathcal{B}]]=0$. Therefore, by combining the three latter results just obtained, we showed that $[\mathcal{B},[\mathcal{B},\mathcal{B}]]=0$, that is, L is nilpotent of class at most 2, and this completes the proof. $\square$

Combining Corollary 4.2 and Theorem 4.3, one gets the following.

Corollary 4.4. *Let L be a nonperfect Lie algebra. Then L is abelian iff* $\mathrm{THom}(L, L) = \mathrm{Hom}(L, L) + \mathrm{AHom}(L, L)$.

To give our last theorem, we need a fact regarding ad-nilpotent elements. In [1], the authors proved that any nonzero finite-dimensional Lie algebra over an algebraically closed field of arbitrary characteristic has a nonzero ad-nilpotent element. In the nonabelian case, a bit more can be said about ad-nilpotent elements.

Lemma 4.5. *Let L be a finite-dimensional nonabelian Lie algebra over an algebraically closed field. Then L contains a noncentral ad-nilpotent element.*

Proof. There is nothing to prove if $Z(L) = 0$. We assume now that $Z(L) \neq 0$ and use induction on $\dim L$. If the nonzero Lie algebra $L/Z(L)$ is abelian, then for any $x \in L \setminus Z(L)$ one has $(\mathrm{ad}x)^2 = 0$, as desired. Suppose now that the nonzero Lie algebra $L/Z(L)$ is nonabelian. By induction, $L/Z(L)$ has a noncentral ad-nilpotent element $x + Z(L)$. Therefore, x is a noncentral ad-nilpotent element of L, as wanted. $\square$

There is some evidence to infer that in general, the first case does not probably occur in the conclusion of Theorem 4.3. By imposing some rational restrictions on the Lie algebra and the field, we obtain the following.

Theorem 4.6. *Let L be a finite-dimensional Lie algebra over an algebraically closed field of characteristic zero. Then L is nilpotent of class at most 2 iff*

$$\mathrm{Hom}(L, L) + \mathrm{AHom}(L, L) \subseteq \mathrm{THom}(L, L).$$

Proof. If L is nilpotent of class at most 2, then $\mathrm{THom}(L, L)$ is the set of all linear maps on L and the result follows.

We may now assume by way of contradiction that L is not nilpotent of class at most 2. So, using Theorem 4.3, L is perfect. Taking advantage of Lemma 4.5, L contains a noncentral ad-nilpotent element x. Suppose that $n \geq 2$ is the least natural number so that $(\mathrm{ad}x)^n = 0$. It is well known that

$$\exp(\mathrm{ad}x) = \sum_{k=0}^{n-1} \frac{(\mathrm{ad}x)^k}{k!},$$

is an automorphism of L. We know that $-I \in \mathrm{AHom}(L, L)$, where I is the identity map on L. Hence by hypothesis $\varphi = \exp(\mathrm{ad}x) - I \in \mathrm{THom}(L, L)$. First assume that $n = 2$. Thus $\varphi = \mathrm{ad}x$ is a triple homomorphism on L and so, for any $a, b, c \in L$, one has

$$\mathrm{ad}x([a, [b, c]]) = [\mathrm{ad}x(a), [\mathrm{ad}x(b), \mathrm{ad}x(c)]].$$

But

$$\begin{aligned} 0 &= (\mathrm{ad}x)^2([b, c]) \\ &= \mathrm{ad}x([\mathrm{ad}x(b), c] + [b, \mathrm{ad}x(c)]) \\ &= 2[\mathrm{ad}x(b), \mathrm{ad}x(c)], \end{aligned}$$

which shows that $[\mathrm{ad}x(b), \mathrm{ad}x(c)] = 0$. Therefore, $\mathrm{ad}x([a, [b, c]]) = 0$. Since L is perfect, we deduce that $\mathrm{ad}x = 0$, a contradiction.
Suppose now that $n \geq 3$. So

$$\varphi = \sum_{k=1}^{n-1} \frac{(\mathrm{ad}x)^k}{k!}$$

is a triple homomorphism on L and hence, for any $a, b \in L$, one obtains

$$\varphi([x, [a, b]]) = [\varphi(x), [\varphi(a), \varphi(b)]] = 0.$$

Now we have

$$\begin{aligned} (\mathrm{ad}x)^{n-1}([a, b]) &= (\mathrm{ad}x)^{n-2}\varphi([a, b]) \\ &= (\mathrm{ad}x)^{n-3}\varphi(\mathrm{ad}x)([a, b]) \\ &= (\mathrm{ad}x)^{n-3}\varphi([x, [a, b]]) \\ &= 0, \end{aligned}$$

which is again a contradiction, for L is perfect. This completes the proof. □

We close the paper with two interesting open problems.

Problem 1. Do there exist two non-isomorphic perfect Lie algebras such that there exists a bijective triple homomorphism between them?

Problem 2. Is it true that $\mathrm{Hom}(L, K) = \{0\}$ implies that $\mathrm{THom}(L, K) = \{0\}$?

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