Algebra 1: Math20008: Sheet 1

G.C.Smith@bath.ac.uk

October 12, 2004

The course web site is available via http://www.bath.ac.uk/∼masgcs/

1. *Let V be a vector space over F.* Here is a proof that $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$. First note that $0 + 0 = 0$ (Field Axiom 3). Therefore $(0+0)\mathbf{v} = 0\mathbf{v}$. Now by Vector Space Axiom 7 we have $0\mathbf{v} + 0\mathbf{v} = 0\mathbf{v}$. Now, by Vector Space Axiom 4 the vector $0\mathbf{v}$ has an additive inverse $-0\mathbf{v}$. It follows that

$$(0\mathbf{v} + 0\mathbf{v}) + -0\mathbf{v} = 0\mathbf{v} + -0\mathbf{v}.$$

Now rebracket the left hand side using Vector Space Axiom 2. Therefore

$$0\mathbf{v} + (0\mathbf{v} + -0\mathbf{v}) = 0\mathbf{v} + -0\mathbf{v}.$$

Now on both sides we have the opportunity to use the definition of an additive inverse so $0\mathbf{v} + \mathbf{0} = \mathbf{0}$. Now by Vector Space Axiom 3 we deduce that

$$0\mathbf{v} = \mathbf{0}.$$

Now, using this as a model, and quoting this result if necessary, prove that if $\mathbf{v} \in V$, then $(-1)\mathbf{v} = -\mathbf{v}$.

Solution By the definition of inverses, $1 + -1 = 0$. Therefore $(1 + -1)\mathbf{v} = 0\mathbf{v}$. We apply Vector Space Axiom 7, and the result justified in the above proof, to see that $1\mathbf{v} + (-1)\mathbf{v} = \mathbf{0}$. Now $1\mathbf{v} = \mathbf{v}$ by Vector Space Axiom 9. We add the additive inverse of $\mathbf{v}$ to each side so $-\mathbf{v} + (\mathbf{v} + (-1)\mathbf{v}) = \mathbf{0} + -\mathbf{v}$. We rebracket the left side using Vector Space Axiom 2, and use the definition of $\mathbf{0}$ on the right, so $(-\mathbf{v} + \mathbf{v}) + (-1)\mathbf{v}) = -\mathbf{v}$. Now by the definition of additive inverses we have $-\mathbf{v} + \mathbf{v} = \mathbf{0}$ so $\mathbf{0} + (-1)\mathbf{v}) = -\mathbf{v}$. Finally the definition of $\mathbf{0}$ yields $(-1)\mathbf{v}) = -\mathbf{v}$ as required.

2. Let V be a vector space over F. Suppose that U is a subset of V. Show that U is a subspace of V if and only if the following three conditions are satisfied:

   (a) $U \neq \emptyset$.

   (b) If $\mathbf{x} \in U$ and $\lambda \in F$, then $\lambda\mathbf{x} \in U$.

   (c) If $\mathbf{x},\ \mathbf{y} \in U$, then $\mathbf{x} + \mathbf{y} \in U$.

   Solution The two conditions for U to be a suspace are (a) that $U \neq \emptyset$ and (b) that U be closed under arbitrary linear combinations. First assume (i), (ii) and (iii). Now (i) is the same as (a) so (a) holds. Now suppose that $\mathbf{u},\ \mathbf{v} \in U$ and $\lambda,\ \mu \in F$. Then $\lambda\mathbf{u}, \mu\mathbf{v} \in U$ by (ii) and $\lambda\mathbf{u} + \mu\mathbf{v} \in U$ by (iii). Conversely we assume that (a) and (b) hold. Now (i) is the same as (a) so (i) holds. Now choosing $\lambda = 1$ and $\mu = 0$ we obtain (ii). Finally choosing $\lambda = \mu = 1$ we obtain (iii).

3. Consider the usual Cartesian description of the plane as $\mathbb{R}^2$ (with perpendicular axes). This collection of ordered pairs is a vector space over $\mathbb{R}$ in a natural way as discussed in lectures. In each case you should justify your answer.

   (a) *Prove that the ordered pairs corresponding a straight line through the origin form a subspace.*
   Solution Each straight line through the origin is a set of the form $L = \{(x, y) \mid ax + by = 0\}$ for suitable choice of constants $a, b \in \mathbb{R}$. Note that $(0, 0) \in L \neq \emptyset$. It remains to demonstrate closure under linear combinations. Suppose that $\mathbf{u},\ \mathbf{v} \in L$ with $\mathbf{u} = (e, f)$ and $\mathbf{v} = (g, h)$. Now if $\lambda, \mu \in \mathbf{R}$, then

   $$\lambda\mathbf{x} + \mu\mathbf{y} = (\lambda e + \mu g, \lambda f + \mu h)$$

   and

   $$a(\lambda e + \mu g) + b(\lambda f + \mu h) = \lambda(ae + bf) + \mu(ag + bh) = \lambda 0 + \mu 0 = 0$$

   and so $\lambda\mathbf{x} + \mu\mathbf{y} \in L$. Thus L is a subspace.

   (b) *Consider the straight line $S = \{(x, 1) \mid x \in \mathbb{R}\}$. Is this a subspace?*
   Solution No. $(1, 1) \in S$ but $2(1, 1) = (2, 2) \notin S$.

(c) *Consider the circle* $C = \{(x, y) \mid x^2 + (y-1)^2 = 1\}$. *Is this a subspace?*
Solution Certainly not. $(1, 1) \in C$ but $(2, 2) = 2(1, 1) \notin C$.

(d) *Prove that* $Z = \{(0, 0)\}$ *is a subspace.*
Solution Suppose that $\mathbf{x}, \mathbf{y} \in Z$ so $\mathbf{x} = \mathbf{y} = \mathbf{0}$. If $\lambda, \mu \in \mathbb{R}$, then $\lambda\mathbf{x} + \mu\mathbf{y} = \mathbf{0} \in Z$ so $Z \leq \mathbb{R}^2$.

(a) *Prove that* $\emptyset$ *is not a subspace.*
Solution This is true by definition.

4. *Describe as many different subspaces of* $\mathbb{R}^3$ *as you can find.*
Solution It should be possible to spot 0, each line through the origin, each plane through the origin, and the whole space. In fact there are no other subspaces, but at this stage no proof is available.

5. Let $\mathbb{R}[X]$ denote the set of polynomials in X which have coefficients in $\mathbb{R}$. This set has a natural vector space structure over $\mathbb{R}$. Which of the following are subspaces of $\mathbb{R}[X]$, and why?

(a) $\{f \mid f \in \mathbb{R}[X], f(42) = 0\}$.
Solution Let the set be S. The zero polynomial satisfies the condition so $S \neq \emptyset$. Suppose that $f, g \in S$ and $\lambda, \mu \in \mathbb{R}$. Let $h = \lambda f + \mu g$ so

$$h(42) = \lambda f(42) + \mu g(42) = 0 + 0 = 0$$

so S is a subspace.

(b) $\{f \mid f \in \mathbb{R}[X], f(42) = 1\}$.
Solution The constant polynomial 1 is in this set but $2 = 2 \cdot 1$ is not, so it cannot be a subspace.

(c) $\{f \mid f \in \mathbb{R}[X], f \textit{ has at most two real roots }\}$.
Solution This set is not a subspace because $1, -1$ are in the set (these constant polynomials have no rela roots) but $0 = 1 + (-1)$ has infinitely many real roots (all real numbers are roots).

(d) $\{f \mid f \in \mathbb{R}[X], \deg f \leq n\}$ *where* $n \in \mathbb{N} \cup \{0\}$. *Let the set be* S_n *for each possible* $n \in \mathbb{N} \cup \{0\}$. *Note that the degree of the zero polynomial is* $-\infty$, *a symbol deemed to be smaller than all integers.*

Solution Each S_n is a subspace. This is because the zero polynomial is in each S_n, and if $f, g \in S_n$ and $\lambda, \mu \in \mathbb{R}$, then

$$\deg(\lambda f + \mu g) \le \max\{\deg(\lambda f),\ \deg(\mu g)\} \le \max\{n, n\} = n.$$

We are done.

(e) $\{f \mid f \in \mathbb{R}[X], f''(X) - Xf'(X) + f(X) = 0\}$ *where a dash in the exponent indicates differentiation with respect to X.*
Solution Let the ste be S. The zero polynomial satisfies the differential equation so $S \ne \emptyset$. Suppose that $f, g \in S$ and $\lambda, \mu \in \mathbb{R}$. Let $h = \lambda f + \mu g$ so

$$\begin{aligned} h'' - Xh' + h &= (\lambda f + \mu g)'' - X(\lambda f + \mu g)' + \lambda f + \mu g \\ &= \lambda(f'' - Xf' + f) + \mu(g'' - Xg' + g) = 0 + 0 = 0 \end{aligned}$$

so S is a subspace.

(f) $\{f \mid f(X)^2 = f(X^2)\}$.
Solution Let the set be S. Observe that $X \in S$. However $3X \notin S$ because $(3X)^2 = 9X^2 \ne 3X^2$. Thus S is not closed under scalar multiplication so cannot be a subspace.

6. *Suppose that V is a vector space and that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in V$. Let*

$$U = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \;\middle|\; \lambda_i \in \mathbb{R} \textit{ for all } 1 \le i \le n \right\}.$$

Show that U is a subspace of V.
Solution By choosing the scalars to be all 0, we deduce that $\mathbf{0} \in U \ne \emptyset$. Now suppose that $\mathbf{x}, \mathbf{y} \in U$, so there are scalars θ_i and φ_i such that

$$\mathbf{x} = \sum_{i=1}^{n} \theta_i \mathbf{v}_i$$

and

$$\mathbf{y} = \sum_{i=1}^{n} \varphi_i \mathbf{v}_i$$

so

$$\mathbf{x} + \mathbf{y} = \sum_{i=1}^{n} (\theta_i + \varphi_i) \mathbf{v}_i \in U.$$

Thus U is a subspace.

7. *Suppose that U, W are subspaces of V and that $U \cup W$ is also a subspace of V. Prove that either $U \subseteq W$ or $W \subseteq U$.* Suppose (for contradiction) that neither $U \subseteq W$ nor $W \subseteq U$. Thus there is $\mathbf{w} \in W - U$ and $\mathbf{u} \in U - W$. Consider $\mathbf{x} = \mathbf{u} + \mathbf{w}$. Now $U \cup W$ is a subspace of V so $\mathbf{x} \in U \cup W$. If $\mathbf{x} \in U$ then $\mathbf{w} = \mathbf{x} - \mathbf{u} \in U$ which is false. If $\mathbf{x} \in W$ then $\mathbf{u} = \mathbf{x} - \mathbf{w} \in W$ which is false. This is the required contradiction, so we are done.