Algebra 1; MA20008; Sheet 2 Solutions

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1. *Suppose that U and V are vector spaces over the same field F, and that W is a subspace of V. Let $\alpha : U \to V$ be a linear map. Show that $Y = \{\mathbf{x} \mid \mathbf{x} \in U,\ \alpha(\mathbf{x}) \in V\}$ is a subspace of U.*
   Solution $\alpha(\mathbf{0}) = \mathbf{0} \in W$ so $Y \neq \emptyset$. Suppose that $\mathbf{x}, \mathbf{y} \in Y$ and $\lambda, \mu \in F$, then $\alpha(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda\alpha(\mathbf{x}) + \mu\alpha(\mathbf{y}) \in W$. Therefore $Y \leq U$.

2. *Suppose that U and V are vector spaces over the same field F, and that we have linear maps $\alpha : U \longrightarrow V$ and $\beta : U \longrightarrow V$. Show that $Z = \{\mathbf{x} \mid \mathbf{x} \in U,\ \alpha(\mathbf{x}) = \beta(\mathbf{x})\}$ is a subspace of U.*
   Solution Note that $\mathbf{0} \in Z \neq \emptyset$. Now suppose that $\mathbf{x}, \mathbf{y} \in Z$ and $\lambda, \mu \in F$, then

   $$\alpha(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda\alpha(\mathbf{x}) + \mu\alpha(\mathbf{y}) = \lambda\beta(\mathbf{x}) + \mu\beta(\mathbf{y}) = \beta(\lambda\mathbf{x} + \mu\mathbf{y})$$

   so $Z \leq U$.

3. *Let $\mathbb{C}$ be the complex numbers, viewed as a vector space over $\mathbb{R}$. We have shown that the map $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ defined by complex conjugation is a linear map. Let n be a natural number, and define θ_n to be multiplication by $e^{\frac{2\pi i}{n}}$; more formally $\theta_n : \mathbb{C} \longrightarrow \mathbb{C}$ with $\theta_n(z) = e^{\frac{2\pi i}{n}} z$ for all $z \in \mathbb{C}$.*

   (a) *Show that each map θ_n is linear.*
   Solution In fact multiplication by any fixed complex number is a linear map. This is another way of viewing the distributive law of multiplication over addition.

   (b) *How many different maps can you get by composing the maps θ_4 and θ_6?* (For example, $\theta_4\theta_4\theta_6\theta_6\theta_4$ is one such composition.)

Solution There are 12 maps that can be obtained. Each such map is a rotation of the complex plane (Argand diagram) about the origin which preserves the vertices of the regular 12-gon with centre 0, and one vertex at 1. There are clearly 12 such maps, and each can be obtained since $\theta_4\theta_6^5$ is rotation through $\pi/6$, and this map has 12 different positive powers which are all the possible rotations respecting this regular 12-gon,

(c) *How many different maps can you get by composing the maps θ_5 and φ?*
Solution The answer is 10. These are actually the rigid symmetries of the regular pentagon (5-gon) with centre 0 and a vertex at 1 (rotations and reflections).

(d) *How many different maps can you get by composing the maps θ_4, θ_6 and φ?* There are 24 maps that can be obtained.
Solution The answer is 24, the rigid symmetries of the obvious regular dodecagon (12-gon), reflections and rotations.

4. *Let V we a vector space over a field F. We define a* line *as follows. Suppose that* $\mathbf{a}, \mathbf{b} \in V$ *with* $\mathbf{b} \neq \mathbf{0}$. *The set*

$$L = \{\mathbf{r} \mid \mathbf{r} = \mathbf{a} + t\mathbf{b},\ t \in F\}$$

is a line. *Suppose that U is also a vector space over F and that*

$$\alpha : V \longrightarrow U$$

is a linear map. Show that if $\mathbf{b} \notin$ *Ker α, then*

$$K = \{\alpha(\mathbf{r}) \mid \mathbf{r} \in L\}$$

is a line. What happens if $\mathbf{b} = \mathbf{0}$?
Solution

$$\{\alpha(\mathbf{r}) \mid \mathbf{r} \in L\} = \{\mathbf{r} \mid \mathbf{r} = \alpha(\mathbf{a}) + \alpha(t\mathbf{b}),\ t \in F\}$$

$$= \{\mathbf{r} \mid \mathbf{r} = \alpha(\mathbf{a}) + t\alpha(\mathbf{b}),\ t \in F\}$$

which is a line. If $\mathbf{b} = \mathbf{0}$ or more generally if $\mathbf{b} \in$ Ker α, then we get a set consisting of a single point instead.

5. *Regard* $\mathbb{R}^n$ *as a vector space over* $\mathbb{R}$. *Define a map* $\mu : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *by* $(x_1, x_2, \ldots, x_n) \mapsto (x_2, x_3, \ldots, x_n, 0)$ *for all* $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

   (a) *Show that* μ *is a linear map.*
   Solution This is entirely routine. Suppose that $\lambda, \theta \in \mathbb{R}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n),\ \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Now

$$\mu(\lambda\mathbf{x} + \theta\mathbf{y}) = \lambda(x_2, x_3, \ldots, x_n, 0) + \theta(y_2, y_3, \ldots, y_n, 0)$$
$$= (\lambda x_2 + \theta y_2, \lambda x_3 + \theta y_3, \ldots, \lambda x_n + \theta y_n, 0),$$

   whereas

$$\lambda\mu(\mathbf{x}) + \theta\mu(\mathbf{y}) = \lambda(x_2, x_3, \ldots, x_n, 0) + \theta(y_2, y_3, \ldots, y_n, 0)$$
$$= (\lambda x_2 + \theta y_2, \lambda x_3 + \theta y_3, \ldots, \lambda x_n + \theta y_n, 0).$$

   We are done.

   (b) *Show that* μ^n *is the zero map (*μ^n *denotes the map obtained by composing* n *copies of* μ*).*
   Solution Induct on r to show that

$$\text{Im } \mu^r = \{(y_1, y_2, \ldots, y_{n-r}, 0, \ldots, 0) \mid y_i \in \mathbb{R} \text{ for all } i\}.$$

   We omit the details.

   (c) *Show that* μ^{n-1} *is not the zero map.*
   Solution This follows from the argument above.

6. *Let* V *be a vector spaces, and suppose that* α *and* β *are both projections onto subspaces of* V *with suitable kernels. Suppose also that* $\alpha\beta = \beta\alpha$. *Show that* $\alpha\beta$ *is a projection.*
   Solution We are given that $\alpha, \beta : V \longrightarrow V$ are linear maps which commute and satisfy $\alpha\beta = \beta\alpha$. Moreover $\alpha^2 = \alpha$ and $\beta^2 = \beta$. (we allow a slight notational abuse here, and inflate the codomains of α and β to V from the given subspaces of V). Now $(\alpha\beta)^2 = \alpha\beta\alpha\beta = \alpha^2\beta^2 = \alpha\beta$. We have used the fact that α an β are projections so $\alpha^2 = \alpha$ and $\beta^2 = \beta$, and commutativity. Now we proved in lectures that $(\alpha\beta)^2 = \alpha\beta$ forces $\alpha\beta$ to be a projection, so we are done.