Algebra 1; MA20008; Sheet 3

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1. *Suppose that V is a vector space over F, and that $S \subseteq V$. Let $\overline{S}$ be the intersection of those subspaces of V which contain the subset S, or put formally*
$$\overline{S} = \bigcap \{U \mid U \leq V, S \subseteq U\}.$$
Show that $\overline{S} = \langle S \rangle$.
Solution: We have $S \subseteq \langle S \rangle \leq V$. Now $\langle S \rangle$ is therefore one of the sets being intersected in the definition of $\overline{S}$. Therefore $\overline{S} \subseteq \langle S \rangle$. Conversely if $S \subseteq U \leq V$ and $\mathbf{v} \in \langle S \rangle$, then $\mathbf{v} \in U$ since U is closed under the formation of linear combinations. Therefore $\langle S \rangle \subseteq \overline{S}$. Since we have both inclusions it follows that $\overline{S} = \langle S \rangle$.

2. *Let V be a vector space over F and $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in V$. Suppose that whenever $\theta_1, \ldots, \theta_n, \psi_1, \ldots, \psi_n \in F$ and $\sum_{i=1}^n \theta_i \mathbf{v_i} = \sum_{i=i}^n \psi_i \mathbf{v_i}$, then necessarily $\lambda_i = \mu_i$ for each i, $1 \leq i \leq n$. Show that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in V$ is linearly independent.*
Solution: It suffices to choose $\psi_i = 0$ for every i, and we obtain the condition for linear independence.

3. *Consider $V = \mathbb{R}$ as a vector space over $\mathbb{Q}$.*

   (a) *Show that $1, \sqrt{2}, \sqrt{3}$ are linearly independent.*
   Solution: It is first year work to show that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{6}$ are irrational, and we assume that you can do this. Suppose that $a+b\sqrt{2}+c\sqrt{3} = 0$ for rational a, b and c. Therefore $(b\sqrt{2}+c\sqrt{3})^2 \in \mathbb{Q}$, so $bc\sqrt{6} \in \mathbb{Q}$. Now $bc = 0$ else $\sqrt{6}$ would be rational. If only one of b and c were 0, then $\sqrt{2}$ or $\sqrt{3}$ would be rational, which it isn't. Therefore $b = c = 0$, so $a = 0$ and we are done.

(b) *Let* $\alpha = e^{\frac{\pi i}{3}}$. *Which lists of the form* $1, \alpha, \ldots, \alpha^n$ *are linearly independent? Justify your answer.*
Solution: α is a root of $X^3+1 = (X+1)(X^2-X+1)$ but not of $X+1$ so α is a root of X^2-X+1. Thus $1-\alpha+\alpha^2 = 0$ so $1, \alpha, \alpha^2$ is a linearly dependent list, as is $1, \alpha, \ldots, \alpha^n$ whenever $n \geq 2$. Also 1 is linearly independent, and $1, \alpha$ is linearly independent, since a non-trivial linear relation would force $\alpha \in \mathbb{R}$, which is false.

(c) *Suppose that* $1, \beta, \beta^2, \ldots, \beta^n$ *are linearly independent. Show that* $1, (\beta+1), (\beta+1)^2, \ldots, (\beta+1)^n$ *are linearly independent.*
Solution: Suppose, for contradiction, that these powers of $1+\beta$ are linearly dependent. Thus there is a non-zero polynomial f (or $f(X)$) with rational coefficients so that $f(1+\beta) = 0$. Now β will be a root of $h := f(X+1)$, and $\deg h = \deg f$ (and indeed the leading coefficients co-incide). Therefore h is not the zero polynomial and the given powers of β satisfy a non-trivial linear relation. However, we are given that these powers of β are linearly independent over $\mathbb{Q}$, so this is absurd. We have the required contradiction.

4. *Suppose that* X *and* Y *are both linearly independent subsets of* V. *Does it follow that* $X \cap Y$ *is linearly independent? What about* $X \cup Y$?
Solution: A subset of a l.i. set of vectors is l.i. for formal reasons, and $X \cap Y \subseteq X$, so we are done. However, the same is not true for the formation of unions. Let $V = F = \mathbb{R}$. Let $X = \{1\}$, $Y = \{2\}$ which are both l.i., but $X \cup Y = \{1, 2\}$ which is l.d. because $2 \cdot 1 + (-1) \cdot 2 = 0$.

5. *Suppose that* $V = U \oplus W$. *We are given a sets of vectors* $X \subseteq U$ *and* $Y \subseteq W$. *Is* $X \cup Y$ *necessarily a linearly independent set of vectors?*
Solution: We have proved that a direct sum yields uniqueness of decomposition, so if $\mathbf{v} \in V$ and $\mathbf{v} = \mathbf{u_1} + \mathbf{w_1} = \mathbf{u_2} + \mathbf{w_2}$ for $\mathbf{u_1}, \mathbf{u_2} \in U$ and $\mathbf{w_1}, \mathbf{w_2} \in W$, then $\mathbf{u_1} = \mathbf{u_2}$ and $\mathbf{w_1} = \mathbf{w_2}$. Now suppose that we have scalars λ_i, θ_j so that

$$\sum_{i=1}^{m} \lambda_i \mathbf{u_i} + \sum_{j=1}^{n} \theta_j \mathbf{w_j} = \mathbf{0}.$$

Here $\mathbf{u_1}, \ldots, \mathbf{u_m} \in X$ and $\mathbf{v_1}, \ldots, \mathbf{v_n} \in Y$. The uniqueness of expression, compared to $\mathbf{0}+\mathbf{0}=\mathbf{0}$, ensures that both

$$\sum_{i=1}^{m} \lambda_i \mathbf{u_i} = \mathbf{0}$$

and

$$\sum_{i=1}^{n} \theta_j \mathbf{w_j} = \mathbf{0}.$$

The l.i. of both X and Y forces all scalars to vanish, and we are done.

6. *Suppose that* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ *is a linearly independent list of vectors in the vector space* V. *We are given* $\mathbf{w} \in V$. *Does it follow that*

$$\mathbf{v_1} + \mathbf{w}, \mathbf{v_2} + \mathbf{w}, \ldots, \mathbf{v_n} + \mathbf{w}$$

are linearly independent?
Solution: No. Choose $\mathbf{w} = -\mathbf{v_1}$ and the zero vector occurs in the list.

7. *Suppose that* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ *is a linearly dependent list of vectors in the vector space* V. *We are given* $\mathbf{w} \in V$. *Does it follow that*

$$\mathbf{v_1} + \mathbf{w}, \mathbf{v_2} + \mathbf{w}, \ldots, \mathbf{v_n} + \mathbf{w}$$

is linearly dependent?
Solution: No. Let $V = \mathbb{R}$, $F = \mathbb{R}$. Let $n = 1$ and $\mathbf{v_1} = \mathbf{0}$. Let $\mathbf{w} = 1$.

8. *Let* $V = \mathbb{R}^3$ *viewed as vector space over* $\mathbb{R}$. *Let* $\mathbf{v_1}, \ldots, \mathbf{v_8}$ *be the position vectors of the vertices of a cube.*

   (a) *Let*

   $$A = \left\{ \sum_i \lambda_i \mathbf{v_i} \mid 0 \le \lambda_i \le 1 \textit{ for all } i, \sum_i \lambda_i = 1 \right\}.$$

   Describe the set A, *viewed as a collection of position vectors, geometrically.*
   Solution: The given position vectors point to the points inside and on the surface of the cube.

(b) *Let*

$$B = \left\{ \sum_i \lambda_i \mathbf{v_i} \mid \lambda_i \geq 0 \textit{ for all } i, \right\}.$$

Under what circumstances is $B = \mathbb{R}^3$*? Under what circumstances is* B *a closed half space (i.e. one side of a plane and all the points on that plane)? What other shapes can arise?*

Solution: We have $B = \mathbb{R}^3$ exactly when the origin is strictly inside the cube. If the origin is in the interior of a face, then B is a half-space. If the origin is in the interior of an edge, then B is the intersection of two half-spaces defined by perpendicular planes. If the origin is at a vertex of the cube, then B is the intersection of three half-spaces defined by pairwise perpendicular planes (also known as an octant). If the origin is strictly outside the cube, then B will be an infinite cone with finitely many planar faces.