Algebra 1; MA20008; Sheet 4 Solutions

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1. *Suppose that n is a natural number or 0, and F is a field. Show that there is a vector space over F of dimension n.*
   Solution The zero space has basis $\emptyset$ and so has dimension $|\emptyset| = 0$. If $n > 0$, then F^n has dimension n.

2. *Suppose that V is a vector space with subspaces U and W both of dimension $n < \infty$. Does it follow that V is finite dimensional? Does it follow that V has dimension n? Does it follow that $U = W$? In each case you should supply a reason for your answer.*
   Solution The answers are no, no and no. To illustrate all three points, let F be a field and let $V = F[X]$ be the set of polynomials in X with coefficients in F. This V is not a vector space of finite dimension. Let U be the subset of V consisting of polynomials of degree at most 1. Let W be the subset of V consisting of polynomials of degree at most 2 but which have constant term 0. Now $U \neq W$, $\dim U = \dim W = 2$ but $\dim V \neq 2$.

3. *Let V be a vector space of dimension n. Suppose that $V_0, V_1, \dots, V_m$ are subspaces of V with*

   $$V_0 \leq V_1 \leq \cdots \leq V_m.$$

   (a) *Suppose that $m > n$. Show that there is $i \in \{1, 2, \dots, m\}$ such that $V_i = V_{i-1}$.*
   Solution The dimensions of the spaces $V_0, V_1, \dots, V_m$ are weakly increasing. If $m > n$, then two spaces V_i and V_{i-1} must have the same dimension, and therefore (theorem in lectures) $V_{i-1} = V_i$.

(b) *Suppose that $m \leq n$. Show that it may be that the spaces*

$$V_0, V_1, \ldots, V_m$$

are distinct.
Solution Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. Let V_j be the span of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_j}$. This does the job.

4. Suppose that $\alpha : U \longrightarrow W$ is a linear map between vector spaces over the same field. Let $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ be vectors in U.

   (a) *Suppose that $U = \langle \mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n} \rangle$ and α is surjective. Prove that $W = \langle \alpha(\mathbf{x_1}), \alpha(\mathbf{x_2}), \ldots, \alpha(\mathbf{x_n}) \rangle$.*
   Solution Suppose that $\mathbf{w} \in W$. Since α is surjective there is $\mathbf{u} \in U$ sich that $\alpha(\mathbf{u}) = \mathbf{w}$. Now $\mathbf{u} = \sum_i \lambda \mathbf{x_i}$ so $\mathbf{w} = \alpha\left(\sum_i \lambda_i \mathbf{x_i}\right) = \sum_i \lambda_i \alpha(\mathbf{x_i})$.

   (b) *Suppose that $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ are linearly independent and α is injective. Show that $\alpha(\mathbf{x_1}), \alpha(\mathbf{x_2}), \ldots, \alpha(\mathbf{x_n})$ are linearly independent.*
   Solution Suppose that $\sum_i \lambda_i \alpha(\mathbf{x_i}) = \mathbf{0}$. Therefore $\alpha(\sum_i \lambda_i \mathbf{x_i}) = \mathbf{0}$. Now by injectivity $\sum_i \lambda_i \mathbf{x_i} = \mathbf{0}$. However, the $\mathbf{x_i}$ are linearly independent, so each λ_i is 0.

5. *Let $\zeta = e^{\frac{2\pi i}{5}} \in \mathbb{C}$.*

   (a) *Suppose that we view $\mathbb{C}$ as a vector space over $\mathbb{Q}$. Show that $1, \zeta, \zeta^2, \zeta^3$ are linearly independent.*
   Solution ζ is a root of $X^4 + X^3 + X^2 + X + 1$. However, ζ is not a root of any rational polynomial of smaller degree. To see this, note that a smallest degree non-zero rational polynomial having α as a root must divide $X^4 + X^3 + X^2 + X + 1$. First eliminate the possibility of a linear factor, and then a quadratic factor.

   (b) *Suppose that we view $\mathbb{C}$ as a vector space over $\mathbb{R}$. Show that $1, \zeta, \zeta^2, \zeta^3$ are linearly dependent.*
   Solution $\overline{\zeta} = \zeta^4 = \zeta^{-1}$. Now $(X - \zeta)(X - \overline{\zeta})$ is a real polynomial of degree 2 which has ζ as a root.

6. Let V be a vector space with subspaces U, W such that U and W are both finite dimensional. Let $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}$ be a basis of U and $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}$ be a basis of W.

(a) *Show that $U + W$ is finite dimensional.*
Solution Certainly $U + W = \langle \mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}, \mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n} \rangle$, so $U + W$ has a finite spanning set and therefore a finite basis.

(b) *Show that $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}, \mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}$ need not be a basis of $U + W$.*
Solution Well, perhaps $\mathbf{u_1} = \mathbf{w_1}$. That would do.

(c) *Suppose that $U + W = U \oplus W$. Show that*

$$\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}, \mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}$$

is a basis of $U + W$.
Solution Spanning is not an issue; linear independence is. Suppose that there are scalars λ_i, μ_i such that $\sigma_i(\lambda_i \mathbf{u_i}) + \sigma_j(\mu_j \mathbf{w_j}) = \mathbf{0}$. Since $U + W = U \oplus W$ it follows that $\sigma_i(\lambda_i \mathbf{u_i}) = \mathbf{0}$ and $\sigma_i \mu_j \mathbf{w_j} = \mathbf{0}$. Now λ_i and μ_j are 0 for every i and j by the linear independence of the relevant sequences.

(d) *Suppose that $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}, \mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}$ is a basis of $U + W$. Show that $U + W = U \oplus W$.*
Solution It suffices to show that $U \cap W = 0$. Suppose not, then there is $\mathbf{v} \in U \cap W$ with $\mathbf{v} \neq \mathbf{0}$. Now $\mathbf{v} \in U$ so $\mathbf{v} = \sum \theta_i \mathbf{u_i}$ and $\mathbf{v} = \sum \psi_j \mathbf{u_j}$. This yields a non-trivial linear relation $\sum \theta_i \mathbf{u_i} - \sum \psi_j \mathbf{w_j}$ among the vectors $\mathbf{u_i}$ and $\mathbf{w_i}$, which cannot therefore be linearly independent.

7. *Let I be a set. Let V be the set of real valued functions on I; more formally*

$$V = \{f | f : I \longrightarrow \mathbb{R}\}.$$

Define addition on V by $(f + h)(x) := f(x) + g(x)$ for all $x \in I$. If $\lambda \in \mathbb{R}$ and $f \in V$ we define $\lambda \cdot f \in V$ by $(\lambda \cdot f)(x) = (\lambda)(f(x))$ where the final multiplication is just the product (in $\mathbb{R}$).

(a) *Check that V is now a vector space over $\mathbb{R}$.*
Solution This is routine.

(b) *For each $i \in I$, define a function $\delta_i \in V$ where $\delta_i(x) = \delta_{i,x}$ (Krönecker delta). Thus $\delta_i(i) = 1$ and $\delta_i(x) = 0$ if $x \neq i$. Show that the vectors δ_i are linearly independent.*
Solution Suppose that λ_i are scalars (all but finitely many of

which are 0). Then $\sum_i' \lambda_i \delta_i = 0$. Choose any $j \in I$¡ then λ_i are scalars (all but finitely many of which are 0). Then $\sum_i' \lambda_i \delta_i)(j) = 0$ so $\lambda_j = 0$. Thus these maps are linearly independent.

(c) *Let $W = \langle \delta_i : i \in I \rangle$ be the span of all the δ_i. Show that the vectors δ_i form a basis of W (in that they are a linearly independent spanning set for W).* Well they are linearly independent by the previous answer, and course they span W by the design of W.

(d) *Show that $W = V$ if and only if I is finite.* W is the subset of V consisting of functions of finite support, i.e. functions which take non-zero values at only finitely many elements of the domain. The two subsets co-incide if and only if I is finite.

(e) *Give an explicit example of a vector space with an clearly describable uncountable basis (no set theoretic metaphysics allowed).*
Solution See part (c).