MA20008 Algebra 1, 2004, Sheet 5

Geoff Smith, `http://www.bath.ac.uk/~masgcs`

1. Let V be a vector space of dimension n.

   (a) Suppose that we have subspaces

   $$0 = V_0 < V_1 < V_2 \cdots < V_n = V$$

   with $\dim V_i = i$ for every $i = 0, 1, \ldots, n$. For each $i > 0$ choose $\mathbf{v_i} \in V_i$ but $\mathbf{v_i} \notin V_{i-1}$. Show that

   $$\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$$

   is a basis of V.
   Solution It suffices to demonstrate linear independence. Suppose, for contradiction, that

   $$\sum_{i=1}^{r} \lambda_i \mathbf{v_i}$$

   with $\lambda_r \neq 0$. Then

   $$\mathbf{v_r} = -\lambda_r^{-1} \left(\sum_{i=1}^{r-1} \lambda_i \mathbf{v_i} \right)$$

   so $\mathbf{v_r} \in V_{r-1}$, a contradiction.

   (b) *Suppose that*

   $$\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$$

   is a basis of V. *Show that there are subspaces*

   $$0 = V_0 < V_1 < V_2 \cdots < V_n = V$$

   with $\dim V_i = i$ *for every* $i = 0, 1, \ldots, n$ *such that* $\mathbf{v_i} \in V_i$ *but* $\mathbf{v_i} \notin V_{i-1}$ *for every* $i > 0$.
   Solution Let $V_i = \langle \mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_i} \rangle$. The linear independence of these vectors ensures that $\dim V_i = i$.

2. Let V be a vector space of dimension n. Suppose that $\alpha : V \longrightarrow V$ is a linear map.

   (a) *Show that* $V \geq$ *Im* $\alpha \geq$ *Im* $\alpha^2 \geq \ldots \geq 0$. Observation: Im α^r denotes the map defined by composing r copies of α.
   Solution Suppose that $\mathbf{v} \in$ Im α^{r+1}, so there is $\mathbf{w} \in V$ such that $\mathbf{v} = \alpha^{r+1}(\mathbf{w}) = \alpha^r(\alpha((\mathbf{w})) \in$ Im α^r.

   (b) *Show that* $0 \leq$ *Ker* $\alpha \leq$ *Ker* $\alpha^2 \leq \cdots \leq V$.
   Solution Suppose that $\mathbf{v} \in$ Ker α^r, then $\alpha^r(\mathbf{v}) = \mathbf{0}$ so $\alpha^{r+1}(\mathbf{v}) = \alpha(\mathbf{0}) = \mathbf{0}$. Therefore $\mathbf{v} \in$ Ker α^{r+1}.

   (c) *Suppose that r is a natural number and* Im $\alpha^r =$ Im α^{r+1}. *Show that* Im $\alpha^t =$ Im α^r *for all natural numbers* $t \geq r$.
   Solution For all $\mathbf{v} \in V$ we have $\alpha^r(\mathbf{v}) = \alpha^{r+1}(\mathbf{v_1})$ for some $\mathbf{v_1} \in V$. Now $\alpha^{r+1}(\mathbf{v_1}) = \alpha(\alpha^r(\mathbf{v_1})) = \alpha(\alpha^{r+1}(\mathbf{v_2}))$ for some $\mathbf{v_2} \in V$. Continuing in this way we have $\alpha^r(\mathbf{v}) = \alpha^{r+i}(\mathbf{v_i})$ for some $\mathbf{v_i} \in V$ for every positive i. Therefore Im $\alpha^r \leq$ Im α^{r+i} for all positive i. However we established that Im $\alpha^{r+i} \leq$ Im α^r in 2(a). We are done.

   (d) *Let V be a vector space of dimension n. Suppose that $\alpha : V \longrightarrow V$ is a linear map such that $\alpha^2 = 0$ (the zero map). Show that the nullity ν_α satisfies $2\nu_\alpha \geq n$.*
   Solution Im $\alpha \leq$ Ker α so $\rho_\alpha \leq \nu_\alpha$. The rank-nullity theorem asserts that $\rho_\alpha + \nu_\alpha = n$. Therefore $\nu_\alpha + \nu_\alpha \geq n$ and we are done.

   (e) *Suppose that there is a natural number m such that α^m is the zero map 0. Prove that α^n is 0.*
   Solution This follows from parts (a) and (c) and observing that the dimension of Im α^i is no more that $n - i$.

3. *Suppose that V is a vector space of dimension n and that $\alpha : V \longrightarrow V$ is a linear map. Suppose that α^3 is 0, the zero map. Show that $\nu_\alpha \geq n/3$ where ν_α denotes the nullity of α.* Hint: let $V_0 = V$, $V_1 =$ Im α and $V_2 =$ Im α^2. Let $\alpha_1 : V \longrightarrow V_1$ and $\alpha_2 : V_1 \longrightarrow V_2$ be the maps defined by $\alpha_i(\mathbf{v}) = \alpha(\mathbf{v})$ for all $\mathbf{v} \in V_{i-1}$. Deploy the rank nullity theorem.
   Solution Let ρ, ν denote the rank and nullity of α. Also let ρ_i, ν_i denote the rank and nullity of α_i. Now $n = \nu_1 + \rho_1$, $\rho_1 = \nu_2 + \rho_2$. Also $\nu_1 = \nu$ and $\nu_2 \leq \nu$. Moreover $\rho_2 \leq \nu$ so $3\nu \geq n$ and we are done.

4. *Let V be a finite dimensional vector space, with W, X, Y and Z all subspaces of V.*

   (a) *Show that*

$$\begin{aligned} & \dim(X \cap Y) + \dim((X+Y) \cap Z) \\ = & \dim(Y \cap Z) + \dim((Y+Z) \cap X) \\ = & \dim(Z \cap X) + \dim((Z+X) \cap Y). \end{aligned}$$

   Solution $\dim(X+Y+Z) = \dim((X+Y)+Z) = \dim(X+Y) + \dim Z - \dim((X+Y) \cap Z) = \dim X + \dim Y + \dim Z - \dim(X \cap Y) - \dim((X+Y) \cap Z)$. Permute the subspaces and we are done.

   (b) *Show that*

$$\begin{aligned} & \dim(W \cap X) + \dim(Y \cap Z) + \dim((W+X) \cap (Y+Z)) \\ = & \dim(W \cap Y) + \dim(X \cap Z) + \dim((W+Y) \cap (X+Z)) \\ = & \dim(W \cap Z) + \dim(X \cap Y) + \dim((W+Z) \cap (X+Y)) \end{aligned}$$

   Solution $W + X + Y + Z = (W+X) + (Y+Z) = (W+Y) + (X+Z) = (W+Z) + (X+Y)$. Take dimensions and you are done.

Hint: this question is not hard. You need an idea. Please do not start picking bases; that way madness lies.