# MA20008 Algebra 1, 2004, Sheet 6 Solutions

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1. *Let $V$ be a vector space of dimension $n$. Suppose that $\alpha : V \to V$ is a linear map. Show that the following are equivalent.*

   (a) *$\alpha$ is injective.*

   (b) *$\alpha$ is bijective.*

   (c) *$\alpha$ is surjective.*

   (d) *There is a basis $\mathbf{v_1}, \ldots, \mathbf{v_n}$ of $V$ such that $\alpha(\mathbf{v_1}), \ldots, \alpha(\mathbf{v_n})$ is a basis of $V$.*

   (e) *For every basis $\mathbf{v_1}, \ldots, \mathbf{v_n}$ of $V$, the vectors $\alpha(\mathbf{v_1}), \ldots, \alpha(\mathbf{v_n})$ also form a basis of $V$.*

   **Solution** The rank nullity theorem tells us that $\alpha$ is injective if and only if it is surjective. Therefore (a), (b) and (c) are equivalent. Now (c) implies (e) since a spanning set must map to a spanning set, and a spanning set of size $n$ must be a basis. Certainly (e) implies (d). However, condition (d) forces the image of $\alpha$ to include a spanning set, and so (c) is forced. We are done.

2. *Suppose that*

$$X = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

   *is a $2r$ by $2r$ matrix built from the four $r$ by $r$ matrices $A, B, C$ and the zero matrix $0$. Suppose that $X$ has an inverse matrix. Describe that matrix in terms of $A, B, C$ and $0$.*
   **Solution** Let the inverse matrix be of the shape

$$Y = \begin{pmatrix} D & E \\ F & G \end{pmatrix}$$

   where $D, E, F$ and $G$ are $r$ by $r$ matrices. Now

(a) $AD + BF = I_r$,

(b) $AE + BG = 0$,

(c) $CF = 0$ and

(d) $CG = I_r$.

From (d) $G = C^{-1}$. From (c) and premultiplication by $C^{-1}$ we obtain $F = 0$. Next (a) forces $D = A^{-1}$ an d (b) implies that $AE + BC^{-1} = 0$ so $E = -A^{-1}BC^{-1}$.

3. *The matrix*

$$F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

*has entries in the Field* $F_7$*, the integers modulo* 7*. Calculate*

(a) $F^2$.

(b) $F^5$.

(c) $F^{1000}$.

(d)

$$\sum_{i=0}^{999} F^i$$

*where* $F^0$ *denotes the identity matrix.*

**Solution** For integers $n \geq 0$ define the Fibonacci sequence mod 7 via $F_0 = 0$ and $F_1 = 1$ where $F_0, F_1 \in \mathbb{Z}_7$. Then inductively $F_n = F_{n-1} + F_{n-2}$ whenever $n \geq 2$. Thus

$$(F_n) = 0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0, 1, \ldots.$$

Thus $F_{16} = 0$ and $F_{17} = 1$ so $(F_n)$ is periodic with fundamental period 16. Now

$$F^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

by induction on $n$. Therefore

(a)

$$F^2 = \begin{pmatrix} F_3 & F_2 \\ F_2 & F_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

(b)

$$F^5 = \begin{pmatrix} F_6 & F_5 \\ F_5 & F_4 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 5 & 3 \end{pmatrix}.$$

(c)

$$F^{1000} = \begin{pmatrix} F_{1001} & F_{1000} \\ F_{1000} & F_{999} \end{pmatrix} = \begin{pmatrix} F_9 & F_8 \\ F_8 & F_7 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}.$$

(d) *The matrix*

$$F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

*has entries in $\mathbb{Q}$, Let $I$ denote the 2 by 2 identity matrix. Show that $I$ and $F$ are linearly independent but that $I, F$ and $F^2$ are linearly dependent elements of the vector space of 2 by 2 matrices with rational entries (with scalars in $\mathbb{Q}$).*

**Solution** By inspection or from the Fibonacci recurrence $F^2 = F + 1$ so $F^2 - F - 1$ is a non-trivial linear relation among 1, $F$ and $F^2$. Suppose that $\lambda, \mu \in \mathbb{Q}$ and $\lambda I + \mu F = 0$ then

i. $\lambda + \mu = 0$,

ii. $\mu = 0$,

iii. $\lambda + \mu = 0$ and

iv. $0 + 0 = 0$.

by inspecting the four positions in turn. These equations force $\lambda = \mu = 0$ as required.

4. *Suppose that $\alpha, \beta : V \longrightarrow V$ are a pair of commuting linear maps.*

(a) *Prove that both Im $\alpha$ and Ker $\alpha$ are $\beta$-invariant spaces.*

**Solution** Suppose that $\mathbf{v} \in \text{Im } \alpha$ so there is $\mathbf{u} \in V$ such that $\alpha(\mathbf{u}) = \mathbf{v}$. Now $\beta(\mathbf{v}) = \beta(\alpha(\mathbf{u})) = \alpha(\beta(\mathbf{u})) \in \text{Im } \alpha$. ON the other hand, suppose that $\mathbf{x} \in \text{Ker } \alpha$, then $\alpha(\beta(\mathbf{x})) = \beta(\alpha(\mathbf{x})) = \beta(\mathbf{0}) = \mathbf{0}$. Thus $\beta(\mathbf{x}) \in \text{Ker } \alpha$ as required.

(b) *Prove that Im $\alpha$ + Im $\beta$ is both $\alpha$-invariant and $\beta$-invariant.*

**Solution** Each of Im $\alpha$ and Im $\beta$ is both $\alpha$-invariant and $\beta$-invariant. It follows that their sum has the same property, because if $\mathbf{x} \in \text{Im } \alpha$ and $\mathbf{y} \in \text{Im } \beta$, then $\alpha(\mathbf{x} + \mathbf{y}) = \alpha(\mathbf{x}) + \alpha(\mathbf{y})$

which is the sum of two vectors, the first of which is in Im $\alpha$ and the second in Im $\beta$. The sum of subspaces is also $\beta$-invariant by symmetry.

(c) *Prove that Im $\alpha \cap$ Im $\beta$ is both $\alpha$-invariant and $\beta$-invariant.*
**Solution** This is a formality. If $\mathbf{x}$ is in the intersection, then $\alpha(\mathbf{x}) \in \text{Im } \alpha$ and $\alpha(\mathbf{x}) \in \text{Im } \beta$ so $\alpha(\mathbf{x}) \in \text{Im } \alpha \cap \text{Im } \beta$. The same holds for $\beta$ by symmetry.