MA20008 Algebra 1, 2004, Sheet 8 Solutions

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1. *Suppose that V is a finite dimensional vector space over a field F, and that $\alpha : V \longrightarrow V$ is a linear map. Suppose that $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ is a basis of V such that each subspace $\langle \mathbf{e_i} \rangle$ is α-invariant. Prove that the matrix of α with respect to this basis (in both domain and codomain) is diagonal.*
   Solution For each i we have $\alpha(\mathbf{e_i}) = \lambda_i \mathbf{e_i}$ for some scalar $\lambda_i \in F$. It follows that the matrix of α with respect to this basis is

$$\begin{pmatrix} \lambda_1 & 0 & 0 & . & . & . \\ 0 & \lambda_2 & 0 & . & . & . \\ 0 & 0 & \lambda_3 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & \lambda_{n-1} & 0 \\ . & . & . & . & 0 & \lambda_n \end{pmatrix}.$$

2. *We consider the ordinary inner product on $\mathbb{R}2$ or $\mathbb{R}3$ defined as the "dot" or "scalar" product. Let the vertices of triangle ABC have position vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively.*

   (a) *Show that the point G with position vector $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$ lies on each median (a line joining a vertex to the midpoint of the opposite side). Conclude that the medians are concurrent (at a point G which is called the* centroid *of $\triangle ABC$).*
   Solution The position vector of the point $\frac{2}{3}$ of the way down the median from A is $\mathbf{a}+\frac{2}{3}(\frac{1}{2}(\mathbf{b}+\mathbf{c})-\mathbf{a}) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. By symmetry this point G is on all three medians, which must therefore be concurrent.

   (b) *The* altitude *of $\triangle ABC$ through A is the straight line through A which is perpendicular to BC. Show that a point P is on this* *altitude if and only if the position vector* $\mathbf{r}$ *of* P *satisfies* $(\mathbf{r}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})=0$.

Solution The equation specifies that the lines AP and BC are perpendicular, or that $A=P$. This is the required altitude.

(c) *Now we insist that the origin is the circumcentre* O *of* $\triangle ABC$. *Thus there is a quantity* R *such that*

$$\mathbf{a}\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{b}=\mathbf{c}\cdot\mathbf{c}=R2.$$

Show that the point H *with position vector* $\mathbf{a}+\mathbf{b}+\mathbf{c}$ *is on all three altitudes. Deduce that the three altitudes of* $\triangle ABC$ *are concurrent at* H. The point H is called the orthocentre of $\triangle ABC$.

Solution H is on the altitude from A because $(\mathbf{a}+\mathbf{b}+\mathbf{c}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})=(\mathbf{b}+\mathbf{c})\cdot(\mathbf{b}-\mathbf{c})=\mathbf{b}\cdot\mathbf{b}-\mathbf{c}\cdot\mathbf{c}=R2-R2=0$. By symmetry H is on all three altitudes which are therefore concurrent.

(d) *Deduce that the three points* O,G *and* H *are colinear, and that the distances are such that* $|OH|=3|OG|$. The line through O and H is called the Euler line of $\triangle ABC$.

Solution We know that $\overline{OH}=3\overline{OG}$ because $\overline{OH}=\mathbf{a}+\mathbf{b}+\mathbf{c}$ and $\overline{OG}=\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$.

(e) *Let* L *be the midpoint of* BC *and* M *be the midpoint of* AH. *Let* N *be the midpoint of* LM. *Find the position vector of* N *(with the origin still at* O*).*

Solution $\overline{ON}=\frac{1}{4}(\mathbf{a}+\mathbf{a}+\mathbf{b}+\mathbf{c})+\frac{1}{4}(\mathbf{b}+\mathbf{c})=\frac{1}{2}(\mathbf{a}+\mathbf{b}+\mathbf{c})$.

(f) *Deduce that* N *is the midpoint of* OH *so that* O,G,N,H *are colinear and the ratios of lengths are*

$$|OG|:|GN|:|NH|=2:1:3.$$

Solution This is because

$$\overline{OG}=\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}),$$

$$\overline{ON}=\frac{1}{2}(\mathbf{a}+\mathbf{b}+\mathbf{c})$$

and

$$\overline{OH}=\mathbf{a}+\mathbf{b}+\mathbf{c}.$$

(g) *Show that* $|LM| = R$. $\overline{LM} = \frac{1}{2}(\mathbf{a}+\mathbf{a}+\mathbf{b}+\mathbf{c}) - \frac{1}{2}(\mathbf{b}+\mathbf{c}) = \frac{1}{2}\mathbf{a}$. Now the length of $\mathbf{a}$ is R, so $|LM| = \frac{1}{2}R$.

(h) *Deduce that the circle with centre N and radius $R/2$ goes through the following nine interesting points: the midpoints of the sides of $\triangle ABC$, the feet of the altitudes of $\triangle ABC$ and the three points which are midway between H and each of the three vertices A, B and C.* This is the 'nine-point circle' or 'Feuerbach circle'.
Solution We have shown that the circle with centre N and radius $\frac{1}{2}R$ has LM as a diameter. Let the line AL intersect the line BC at D. Now $\angle LDM$ is a right angle, so by the converse of the "angle in a semicircle" theorem, this circle passes through D. The position vector of N reveals that this is the same circle if we cyclically permute A, B and C, and the result follows.

3. *Let $ABCD$ be a cyclic quadrilateral. A* maltitude *is a straight line through the midpoint of a side which is perpendicular to the opposite side. Show that the four maltitudes are concurrent.*
Solution The maltitude from the midpoint of AB which is perpendicular to CD has equation

$$(\mathbf{r} - \frac{1}{2}(\mathbf{a}+\mathbf{b})) \cdot (\mathbf{c} - \mathbf{d}).$$

Now the point P with position vector

$$\frac{1}{2}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}).$$

The symmetry of this position vector ensures that this point is on all four maltitudes.

4. *Suppose that V is an inner product space, with inner product denoted by $\langle\ ,\ \rangle$. Suppose that U and W are subspaces of V.*

(a) *Show that* $(U+W)^\perp = (U \cup W)^\perp = U^\perp \cap W^\perp$.
Solution The second equality is a formality. As for the first, both inclusions are formailities.

(b) *Suppose that* $U \leq W$. *Show that* $W^\perp \leq U^\perp$.
Solution This is a formality.

5. *Suppose that* V *is an inner product space of dimension* n*, and* $\mathbf{0} \neq \mathbf{v} \in V$*. Prove that* $\dim(\{\mathbf{v}\}^{\perp}) = n - 1$.
Solution Consider the map $\theta : V \longrightarrow F$ defined by $\mathbf{x} \mapsto (\mathbf{x}, \mathbf{v})$. This is a linear map, and is not the zero map since $\theta(\mathbf{v}) = (\mathbf{v}, \mathbf{v}) \neq 0$. It is linear, and F is 1-dimensional, so θ is surjective. The rank-nullity theorem applies and we are done.