MA20008 Algebra 1, 2004, Sheet 9

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1. *Let U be the set of polynomials in the variable X with coefficients in $\mathbb{R}$. We define an inner product $\langle\ ,\ \rangle$ on U via*
$$\langle f, h\rangle = \int_0^1 fh\ dX.$$
Thus U is a vector space of $\mathbb{R}$ in the natural way. Let V be the subspace of U consisting of polynomials of degree at most 3. Given the basis $1, X, X^2, X^3$, run the Gram-Schmidt algorithm to produce an orthonormal basis of V.
Solution $\mathbf{e_1} = 1$, $\mathbf{e_2} = 2\sqrt{3}X - \sqrt{3}$, $\mathbf{e_3} = 6\sqrt{5}X^2 - 6\sqrt{5}X + \sqrt{5}$, $\mathbf{e_4} =?$.

2. *Let V be an inner product space with orthonormal basis $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. If the Gram-Schmidt algorithm is used to modify this basis, what is the output?*
Solution The output will be the input.

3. *Let V be an inner product space with orthonormal basis $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. We obtain another basis $\mathbf{v_n}, \mathbf{v_{n-1}}, \ldots, \mathbf{v_1}$ by reversing the order of the vectors. Run the Gram-Schmidt algorithm on each of these bases in turn. Is it true that the output orthonormal bases are the reverse of each other?*
Solution Not necessarily. For example we consider the standard inner product on $\mathbb{R}^2$. Let $\mathbf{v_1} = (1,0)$ and $\mathbf{v_2} = (1,1)$. If we run G-S on this basis we get the output $(1,0),(0,1)$. On the other hand the reverse basis yields output $\frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(1,-1)$.

4. *For $r = 0,1,2$ define functions $f_r : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_r : \theta \mapsto \cos r\theta$ for all real numbers θ. The collection of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ has a natural vector space structure. Let V be the subspace spanned by*

f_0, f_1, f_2. *Define an inner product on* V *via* $\langle f, h\rangle = \frac{1}{2\pi}\int_0^{2\pi} fh\, d\theta$. *Run Gram-Schmidt to obtain an orthonormal basis of* V.
Solution $\mathbf{e_1} = f_0 = 1$, $\mathbf{e_2} = \sqrt{2}f_1 = \sqrt{2}\cos\theta$, $\mathbf{e_2} = ?$.

5. *Suppose that* $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ *is an orthonormal basis of the inner product space* V. *Let* $U_r = \langle \mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_r}\rangle$. *Suppose that* $\mathbf{v} \in V$. *Show that among all vectors* $\mathbf{x} \in U_r$, *the one which minimizes* $||\mathbf{v} - \mathbf{x}||$ *is* $\sum_{i=1}^{r}\langle \mathbf{v}, \mathbf{e_i}\rangle \mathbf{e_i}$.
Solution Let $\mathbf{u} = \sum_{i=1}^{r}\langle \mathbf{v}, \mathbf{e_i}\rangle \mathbf{e_i}$. Now if $\mathbf{x} \in U_r$, let $\mathbf{y} = -\mathbf{u} + \mathbf{x} \in U_r$. Now
$$||\mathbf{v} - \mathbf{x}||^2 \langle \mathbf{v} - \mathbf{x}, \mathbf{v} - \mathbf{x}\rangle$$
$$= \langle \mathbf{v} - \mathbf{u} - \mathbf{y}, \mathbf{v} - \mathbf{u} - \mathbf{y}\rangle = ||\mathbf{v} - \mathbf{u}||^2 + ||\mathbf{y}||^2 \geq ||\mathbf{v} - \mathbf{u}||^2.$$

6. *Suppose that* V *is an inner product space of dimension* n, *and* $\alpha : V \longrightarrow V$ *is a linear map. Suppose that* α *carries some orthonormal basis to an orthonormal basis. Show that* α *carries each orthonormal basis to an orthonormal basis.*
Solution There is an orthonormal basis $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ such that
$$\langle \alpha(\mathbf{e_i}), \alpha(\mathbf{e_i})\rangle = \delta_{ij}.$$
Now if $\mathbf{v}, \mathbf{w} \in V$, then $\mathbf{v} = \sum_k a_k \mathbf{e_k}$ and $\mathbf{w} = \sum_l b_l \mathbf{e_l}$ for suitable constants a_k and b_l. Now
$$\langle \alpha(\mathbf{v}), \alpha(\mathbf{w})\rangle = \langle \alpha(\sum_k a_k \mathbf{e_k}), \alpha(\sum_l b_l \mathbf{e_l})\rangle$$
$$= \sum_k \sum_l a_k \overline{b_l} \langle \alpha(\mathbf{e_k}), \alpha(\mathbf{e_l})\rangle$$
$$\sum_k \sum_l a_k \overline{b_l} \langle \mathbf{e_k}, \mathbf{e_l}\rangle$$
$$= \langle \sum_k a_k \mathbf{e_k}, \sum_l b_l \mathbf{e_l}\rangle$$
$$= \langle \mathbf{v}, \mathbf{w}\rangle.$$
Thus α preserves the inner product, and so will carry any orthonormal basis to an orthonormal basis.