Algebra 1: Math20008: Sheet 1

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1. Let V be a vector space over F. *Here is a proof that* $0\mathbf{v} = \mathbf{0}$ *for all* $\mathbf{v} \in V$. *First note that* $0 + 0 = 0$ *(Field Axiom 3). Therefore* $(0 + 0)\mathbf{v} = 0\mathbf{v}$. *Now by Vector Space Axiom 7 we have* $0\mathbf{v} + 0\mathbf{v} = 0\mathbf{v}$. *Now, by Vector Space Axiom 4 the vector* $0\mathbf{v}$ *has an additive inverse* $-0\mathbf{v}$. *It follows that*

$$(0\mathbf{v} + 0\mathbf{v}) + -0\mathbf{v} = 0\mathbf{v} + -0\mathbf{v}.$$

Now rebracket the left hand side using Vector Space Axiom 2. Therefore

$$0\mathbf{v} + (0\mathbf{v} + -0\mathbf{v}) = 0\mathbf{v} + -0\mathbf{v}.$$

Now on both sides we have the opportunity to use the definition of an additive inverse so $0\mathbf{v}+\mathbf{0} = \mathbf{0}$. *Now by Vector Space Axiom 3 we deduce that*

$$0\mathbf{v} = \mathbf{0}.$$

Now, using this as a model, and quoting this result if necessary, prove that if $\mathbf{v} \in V$, then $(-1)\mathbf{v} = -\mathbf{v}$.

2. Let V be a vector space over F. Suppose that U is a subset of V. Show that U is a subspace of V if and only if the following three conditions are satisfied:

   (a) $U \neq \emptyset$.

   (b) If $\mathbf{x} \in U$ and $\lambda \in F$, then $\lambda\mathbf{x} \in U$.

   (c) If $\mathbf{x},\ \mathbf{y} \in U$, then $\mathbf{x} + \mathbf{y} \in U$.

3. Consider the usual Cartesian description of the plane as $\mathbb{R}^2$ (with perpendicular axes). This collection of ordered pairs is a vector space over $\mathbb{R}$ in a natural way as discussed in lectures. In each case you should justify your answer.

    (a) Prove that the ordered pairs corresponding a straight line through the origin form a subspace.

    (b) Consider the straight line $\mathrm{S} = \{(x, 1) \mid x \in \mathbb{R}\}$. Is this a subspace?

    (c) Consider the circle $C = \{(x, y) \mid x^2 + (y-1)^2 = 1\}$. Is this a subspace?

    (d) Prove that $Z = \{(0, 0)\}$ is a subspace.

    (a) Prove that $\emptyset$ is not a subspace.

4. Describe as many different subspaces of $\mathbb{R}^3$ as you can find.

5. Let $\mathbb{R}[X]$ denote the set of polynomials in X which have coefficients in $\mathbb{R}$. This set has a natural vector space structure over $\mathbb{R}$. Which of the following are subspaces of $\mathbb{R}[X]$, and why?

    (a) $\{f \mid f \in \mathbb{R}[X], f(42) = 0\}$.

    (b) $\{f \mid f \in \mathbb{R}[X], f(42) = 1\}$.

    (c) $\{f \mid f \in \mathbb{R}[X], f$ has at most two real roots $\}$.

    (d) $\{f \mid f \in \mathbb{R}[X], \deg f \leq n\}$ where $n \in \mathbb{N} \cup \{0\}$. Note that the degree of the zero polynomial is $-\infty$, a symbol deemed to be smaller than all integers.

    (e) $\{f \mid f \in \mathbb{R}[X], f''(X) - Xf'(X) + f(X) = 0\}$ where a dash in the exponent indicates differentiation with respect to X.

    (f) $\{f \mid f(X)^2 = f(X^2)\}$.

6. Suppose that V is a vector space and that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in V$. Let

$$U = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \;\middle|\; \lambda_i \in \mathbb{R} \text{ for all } 1 \leq i \leq n \right\}.$$

    Show that U is a subspace of V.

7. Suppose that U, W are subspaces of V and that $U \cup W$ is also a subspace of V. Prove that either $U \subseteq W$ or $W \subseteq U$.