Algebra 1; MA20008; Sheet 2

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1. Suppose that U and V are vector spaces over the same field F, and that W is a subspace of V. Let $\alpha : U \to V$ be a linear map. Show that $Y = \{\mathbf{x} \mid \mathbf{x} \in U,\ \alpha(\mathbf{x}) \in V\}$ is a subspace of U.

2. Suppose that U and V are vector spaces over the same field F, and that we have linear maps $\alpha : U \longrightarrow V$ and $\beta : U \longrightarrow V$. Show that $Z = \{\mathbf{x} \mid \mathbf{x} \in U,\ \alpha(\mathbf{x}) = \beta(\mathbf{x})\}$ is a subspace of U.

3. Let $\mathbb{C}$ be the complex numbers, viewed as a vector space over $\mathbb{R}$. We have shown that the map $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ defined by complex conjugation is a linear map. Let n be a natural number, and define θ_n to be multiplication by $e^{\frac{2\pi i}{n}}$; more formally $\theta_n : \mathbb{C} \longrightarrow \mathbb{C}$ with $\theta_n(z) = e^{\frac{2\pi i}{n}} z$ for all $z \in \mathbb{C}$.

   (a) Show that each map θ_n is linear.

   (b) How many different maps can you get by composing the maps θ_4 and θ_6? *(For example, $\theta_4\theta_4\theta_6\theta_6\theta_4$ is one such composition.)*

   (c) How many different maps can you get by composing the maps θ_5 and φ?

   (d) How many different maps can you get by composing the maps θ_4, θ_6 and φ?

4. Let V we a vector space over a field F. We define a *line* as follows. Suppose that $\mathbf{a}, \mathbf{b} \in V$ with $\mathbf{b} \neq \mathbf{0}$. The set

$$L = \{\mathbf{r} \mid \mathbf{r} = \mathbf{a} + t\mathbf{b},\ t \in F\}$$

is a *line*. Suppose that U is also a vector space over F and that

$$\alpha : V \longrightarrow U$$

is a linear map. Show that if $\mathbf{b} \notin \operatorname{Ker} \alpha$, then

$$K = \{\alpha(\mathbf{r}) \mid \mathbf{r} \in L\}$$

is a line. What happens if $\mathbf{b} = \mathbf{0}$?

5. Regard $\mathbb{R}^n$ as a vector space over $\mathbb{R}$. Define a map $\mu : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $(x_1, x_2, \ldots, x_n) \mapsto (x_2, x_3, \ldots, x_n, 0)$ for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.
   (a) Show that μ is a linear map.
   (b) Show that μ^n is the zero map (μ^n denotes the map obtained by composing n copies of μ).
   (c) Show that μ^{n-1} is not the zero map.

6. Let V be a vector spaces, and suppose that α and β are both projections onto subspaces of V with suitable kernels. Suppose also that $\alpha\beta = \beta\alpha$. Show that $\alpha\beta$ is a projection.