Algebra 1; MA20008; Sheet 3

G.C.Smith@bath.ac.uk

25-x-2004

1. Suppose that V is a vector space over F, and that $S \subseteq V$. Let $\overline{S}$ be the intersection of those subspaces of V which contain the subset S, or put formally
$$\overline{S} = \bigcap \{U \mid U \leq V, S \subseteq U\}.$$
Show that $\overline{S} = \langle S \rangle$.

2. Let V be a vector space over F and $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in V$. Suppose that whenever $\theta_1, \ldots, \theta_n, \psi_1, \ldots, \psi_n \in F$ and $\sum_{i=1}^n \theta_i \mathbf{v_i} = \sum_{i=i}^n \psi_i \mathbf{v_i}$, then necessarily $\theta_i = \psi_i$ for each i, $1 \leq i \leq n$. Show that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in V$ is linearly independent.

3. Consider $V = \mathbb{C}$ as a vector space over $\mathbb{Q}$.
   (a) Show that $1, \sqrt{2}, \sqrt{3}$ are linearly independent.
   (b) Let $\alpha = e^{\frac{\pi i}{3}}$. Which lists of the form $1, \alpha, \ldots, \alpha^n$ are linearly independent? Justify your answer.
   (c) Suppose that $1, \beta, \beta^2, \ldots, \beta^n$ are linearly independent. Show that $1, (\beta+1), (\beta+1)^2, \ldots, (\beta+1)^n$ are linearly independent.

4. Suppose that X and Y are both linearly independent subsets of V. Does it follow that $X \cap Y$ is linearly independent? What about $X \cup Y$?

5. Suppose that $V = U \oplus W$. We are given a sets of vectors $X \subseteq U$ and $Y \subseteq W$. Is $X \cup Y$ necessarily a linearly independent set of vectors?

6. Suppose that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ is a linearly independent list of vectors in the vector space V. We are given $\mathbf{w} \in V$. Does it follow that
$$\mathbf{v_1} + \mathbf{w}, \mathbf{v_2} + \mathbf{w}, \ldots, \mathbf{v_n} + \mathbf{w}$$

are linearly independent?

7. Suppose that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ is a linearly dependent list of vectors in the vector space V. We are given $\mathbf{w} \in V$. Does it follow that

$$\mathbf{v_1} + \mathbf{w}, \mathbf{v_2} + \mathbf{w}, \ldots, \mathbf{v_n} + \mathbf{w}$$

is linearly dependent?

8. Let $V = \mathbb{R}^3$ viewed as vector space over $\mathbb{R}$. Let $\mathbf{v_1}, \ldots, \mathbf{v_8}$ be the position vectors of the vertices of a cube.

   (a) Let

   $$A = \left\{ \sum_i \lambda_i \mathbf{v_i} \mid 0 \leq \lambda_i \leq 1 \text{ for all } i, \sum_i \lambda_i = 1 \right\}.$$

   Describe the set A, viewed as a collection of position vectors, geometrically.

   (b) Let

   $$B = \left\{ \sum_i \lambda_i \mathbf{v_i} \mid \lambda_i \geq 0 \text{ for all } i, \right\}.$$

   Under what circumstances is $B = \mathbb{R}^3$? Under what circumstances is B a closed half space (i.e. one side of a plane and all the points on that plane)? What other shapes can arise?