Algebra 1; MA20008; Sheet 4

G.C.Smith@bath.ac.uk

2-xi-2004

1. Suppose that n is a natural number or 0, and F is a field. Show that there is a vector space over F of dimension n.

2. Suppose that V is a vector space with subspaces U and W both of dimension $n < \infty$. Does it follow that V is finite dimensional? Does it follow that V has dimension n? Does it follow that $U = W$? In each case you should supply a reason for your answer.

3. Let V be a vector space of dimension n. Suppose that $V_0, V_1, \ldots, V_m$ are subspaces of V with
$$V_0 \leq V_1 \leq \cdots \leq V_m.$$
   (a) Suppose that $m > n$. Show that there is $i \in \{1, 2, \ldots, m\}$ such that $V_i = V_{i-1}$.
   (b) Suppose that $m \leq n$. Show that it may be that the spaces $V_0, V_1, \ldots, V_m$ are distinct.

4. Suppose that $\alpha : U \longrightarrow W$ is a linear map between vector spaces over the same field. Let $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ be vectors in U.
   (a) Suppose that $U = \langle \mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n} \rangle$ and α is surjective. Prove that $W = \langle \alpha(\mathbf{x_1}), \alpha(\mathbf{x_2}), \ldots, \alpha(\mathbf{x_n}) \rangle$.
   (b) Suppose that $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ are linearly independent and α is injective. Show that $\alpha(\mathbf{x_1}), \alpha(\mathbf{x_2}), \ldots, \alpha(\mathbf{x_n})\rangle$ are linearly independent.

5. Let $\zeta = e^{\frac{2\pi i}{5}} \in \mathbb{C}$.

   (a) Suppose that we view $\mathbb{C}$ as a vector space over $\mathbb{Q}$. Show that $1, \zeta, \zeta^2, \zeta^3$ are linearly independent.

   (b) Suppose that we view $\mathbb{C}$ as a vector space over $\mathbb{R}$. Show that $1, \zeta, \zeta^2, \zeta^3$ are linearly dependent.

6. Let V be a vector space with subspaces U, W such that U and W are both finite dimensional. Let $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}$ be a basis of U and $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}$ be a basis of W.

   (a) Show that $U + W$ is finite dimensional.

   (b) Show that $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}, \mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}$ need not be a basis of $U + W$.

   (c) Suppose that $U + W = U \oplus W$. Show that

$$\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}, \mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}$$

   is a basis of $U + W$.

   (d) Suppose that $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}, \mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}$ is a basis of $U + W$. Show that $U + W = U \oplus W$.

7. Let I be a set. Let V be the set of real valued functions on I; more formally

$$V = \{f | f : I \longrightarrow \mathbb{R}\}.$$

Define addition on V by $(f + h)(x) := f(x) + g(x)$ for all $x \in I$. If $\lambda \in \mathbb{R}$ and $f \in V$ we define $\lambda \cdot f \in V$ by $(\lambda \cdot f)(x) = (\lambda)(f(x))$ where the final multiplication is just the product (in $\mathbb{R}$).

   (a) Check that V is now a vector space over $\mathbb{R}$.

   (b) For each $i \in I$, define a function $\delta_i \in V$ where $\delta_i(x) = \delta_{i,x}$ (Krönecker delta). Thus $\delta_i(i) = 1$ and $\delta_i(x) = 0$ if $x \neq i$. Show that the vectors δ_i are linearly independent.

   (c) Let $W = \langle \delta_i : i \in I \rangle$ be the span of all the δ_i. Show that the vectors δ_i form a basis of W (in that they are a linearly independent spanning set for W).

   (d) Show that $W = V$ if and only if I is finite.

   (e) Give an explicit example of a vector space with an clearly describable uncountable basis (no set theoretic metaphysics allowed).