MA20008 Algebra 1, 2004, Sheet 5

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1. Let V be a vector space of dimension n.

   (a) Suppose that we have subspaces

   $$0 = V_0 < V_1 < V_2 \cdots < V_n = V$$

   with $\dim V_i = i$ for every $i = 0, 1, \ldots, n$. For each $i > 0$ choose $\mathbf{v_i} \in V_i$ but $\mathbf{v_i} \notin V_{i-1}$. Show that

   $$\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$$

   is a basis of V.

   (b) Suppose that

   $$\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$$

   is a basis of V. Show that there are subspaces

   $$0 = V_0 < V_1 < V_2 \cdots < V_n = V$$

   with $\dim V_i = i$ for every $i = 0, 1, \ldots, n$ such that $\mathbf{v_i} \in V_i$ but $\mathbf{v_i} \notin V_{i-1}$ for every $i > 0$.

2. Let V be a vector space of dimension n. Suppose that $\alpha : V \longrightarrow V$ is a linear map.

   (a) Show that $V \geq \text{ Im } \alpha \geq \text{ Im } \alpha^2 \geq \ldots \geq 0$. *Observation: Im α^r denotes the map defined by composing r copies of α.*

   (b) Show that $0 \leq \text{ Ker } \alpha \leq \text{ Ker } \alpha^2 \leq \cdots \leq V$.

   (c) Suppose that r is a natural number and $\text{ Im } \alpha^r = \text{ Im } \alpha^{r+1}$. Show that $\text{ Im } \alpha^t = \text{ Im } \alpha^r$ for all natural numbers $t \geq r$.

(d) Let V be a vector space of dimension n. Suppose that $\alpha : V \longrightarrow V$ is a linear map such that $\alpha^2 = 0$ (the zero map). Show that the nullity ν_α satisfies $2\nu_\alpha \geq n$.

(e) Suppose that there is a natural number m such that α^m is the zero map 0. Prove that α^n is 0.

3. Suppose that V is a vector space of dimension n and that $\alpha : V \longrightarrow V$ is a linear map. Suppose that α^3 is 0, the zero map. Show that $\nu_\alpha \geq n/3$ where ν_α denotes the nullity of α. Hint: let $V_0 = V$, $V_1 = \text{ Im } \alpha$ and $V_2 = \text{ Im } \alpha^2$. Let $\alpha_1 : V \longrightarrow V_1$ and $\alpha_2 : V_1 \longrightarrow V_2$ be the maps defined by $\alpha_i(\mathbf{v}) = \alpha(\mathbf{v})$ for all $\mathbf{v} \in V_{i-1}$. Deploy the rank nullity theorem.

4. Let V be a finite dimensional vector space, with W, X, Y and Z all subspaces of V.

(a) Show that

$$\begin{aligned} & \dim(X \cap Y) + \dim((X+Y) \cap Z) \\ = & \dim(Y \cap Z) + \dim((Y+Z) \cap X) \\ = & \dim(Z \cap X) + \dim((Z+X) \cap Y). \end{aligned}$$

(b) Show that

$$\begin{aligned} & \dim(W \cap X) + \dim(Y \cap Z) + \dim((W+X) \cap (Y+Z)) \\ = & \dim(W \cap Y) + \dim(X \cap Z) + \dim((W+Y) \cap (X+Z)) \\ = & \dim(W \cap Z) + \dim(X \cap Y) + \dim((W+Z) \cap (X+Y)) \end{aligned}$$

Hint: this question is not hard. You need an idea. Please do not start picking bases; that way madness lies.