MA20008 Algebra 1, 2004, Sheet 6

Geoff Smith, http://www.bath.ac.uk/∼masgcs

1. Let V be a vector space of dimension n. Suppose that $\alpha : V \to V$ is a linear map. Show that the following are equivalent.

   (a) α is injective.

   (b) α is bijective.

   (c) α is surjective.

   (d) There is a basis $\mathbf{v_1}, \ldots, \mathbf{v_n}$ of V such that $\alpha(\mathbf{v_1}), \ldots, \alpha(\mathbf{v_n})$ is a basis of V.

   (e) For every basis $\mathbf{v_1}, \ldots, \mathbf{v_n}$ of V, the vectors $\alpha(\mathbf{v_1}), \ldots, \alpha(\mathbf{v_n})$ also form a basis of V.

   Hint: the rank-nullity theorem may be useful in places.

2. Suppose that
$$X = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$
is a $2r$ by $2r$ matrix built from the four r by r matrices A, B, C and the zero matrix 0. Suppose that X has an inverse matrix. Describe that matrix in terms of A, B, C and 0.

3. The matrix
$$F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$
has entries in the Field F_7, the integers modulo 7. Calculate

   (a) F^2.

   (b) F^5.

   (c) F^{1000}.

(d)

$$\sum_{i=0}^{999} F^i$$

where F^0 denotes the identity matrix.

4. The matrix

$$F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

has entries in $\mathbb{Q}$, Let I denote the 2 by 2 identity matrix. Show that I and F are linearly independent but that I, F and F^2 are linearly dependent elements of the vector space of 2 by 2 matrices with rational entries (with scalars in $\mathbb{Q}$).

5. Suppose that $\alpha, \beta : V \longrightarrow V$ are a pair of commuting linear maps.

   (a) Prove that both $\text{Im } \alpha$ and $\text{Ker } \alpha$ are β-invariant spaces.

   (b) Prove that $\text{Im } \alpha + \text{Im } \beta$ is both α-invariant and β-invariant.

   (c) Prove that $\text{Im } \alpha \cap \text{Im } \beta$ is both α-invariant and β-invariant.