MA20008 Algebra 1, 2004, Sheet 7

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1. Let V be a two dimensional vector space over $\mathbb{R}$ with basis $\mathbf{v_1}, \mathbf{v_2}$. Let W be a two dimensional vector space over $\mathbb{R}$ with basis $\mathbf{w_1}, \mathbf{w_2}$. Suppose that $\alpha : V \longrightarrow W$ is a linear map which has matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

with respect to these bases. Determine the matrix of α with respect to new bases $\mathbf{v_1} + 2\mathbf{v_2}, \mathbf{v_1} - \mathbf{v_2}$ of V and $\mathbf{w_1} + \mathbf{w_2}, \mathbf{w_1} - \mathbf{w_2}$ of W.

2. Suppose that A is an $m \times n$ matrix with entries in a field F. Show that there are vector spaces V and W over F of dimensions n and m respectively, a linear transformation $\alpha : V \longrightarrow W$, and choices of bases for these spaces so that the matrix of α with respect to these bases is A.

3. Let V and W be vector spaces over F of dimensions n and m respectively. Show that there is a choice of bases for V and W such that the matrix of α with respect to these bases is

$$\begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix}.$$

4. Let $V = \mathbb{R}^2$ viewed as a vector space over $\mathbb{R}$, and thought of as the Euclidean plane. Let $\rho_\theta : V \longrightarrow V$ denote the map which is rotation anticlockwise about the origin through θ radians. Choose the standard basis $\mathbf{e_1}, \mathbf{e_2}$ for V, where $\mathbf{e_1} = (1, 0)$ and $\mathbf{e_2} = (0, 1)$.

   (a) Show that $\rho_\alpha : V \longrightarrow V$ is a linear map for every $\alpha \in \mathbb{R}$.

   (b) Show that $\rho_\alpha \circ \rho_\beta = \rho_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{R}$.

(c) Show that ρ_α and ρ_β commute for all $\alpha, \beta \in \mathbb{R}$.

(d) Write down the matrix of ρ_α with respect to the given basis.

(e) Deduce formulas for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$.

(f) Let c denote complex conjugation. Calculate the matrix of $c \circ \rho_\theta \circ c$ and comment on what this means.

5. Suppose that A and B are invertible $n \times n$ matrices with real entries. Show that there are invertible $n \times n$ real matrices D and E such that $DAE = B$.

6. Define a relation on $n \times n$ matrices with real entries as follows: if $A, B \in \mathrm{Mat}_{n,n}(\mathbb{R})$, we write $A \sim B$ if and only if there are invertible $P, Q \in \mathrm{Mat}_{n,n}(\mathbb{R})$ such that $QAP = B$.

   (a) Show that $\sim$ is an equivalence relation.

   (b) How may equivalence classes are there?