MA20008 Algebra 1, 2004, Sheet 8

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1. Suppose that V is a finite dimensional vector space over a field F, and that $\alpha : V \longrightarrow V$ is a linear map. Suppose that $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ is a basis of V such that each subspace $\langle \mathbf{e_i} \rangle$ is α-invariant. Prove that the matrix of α with respect to this basis (in both domain and codomain) is diagonal.

2. We consider the ordinary inner product on $\mathbb{R}^2$ or $\mathbb{R}^3$ defined as the "dot" or "scalar" product. Let the vertices of triangle ABC have position vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively.

   (a) Show that the point G with position vector $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ lies on each median (a line joining a vertex to the midpoint of the opposite side). Conclude that the medians are concurrent (at a point G which is called the *centroid* of $\triangle ABC$).

   (b) The *altitude* of $\triangle ABC$ through A is the straight line through A which is perpendicular to BC. Show that a point P is on this altitude if and only if the position vector $\mathbf{r}$ of P satisfies $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{c}) = 0$.

   (c) Now we insist that the origin is the circumcentre O of $\triangle ABC$. Thus there is a quantity R such that

   $$\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{c} = R^2.$$

   Show that the point H with position vector $\mathbf{a} + \mathbf{b} + \mathbf{c}$ is on all three altitudes. Deduce that the three altitudes of $\triangle ABC$ are concurrent at H. *The point H is called the orthocentre of $\triangle ABC$.*

   (d) Deduce that the three points O, G and H are colinear, and that the distances are such that $|OH| = 3|OG|$. *The line through O and H is called the Euler line of $\triangle ABC$.*

(e) Let L be the midpoint of BC and M be the midpoint of AH. Let N be the midpoint of LM. Find the position vector of N (with the origin still at O).

(f) Deduce that N is the midpoint of OH so that O, G, N, H are colinear and the ratios of lengths are

$$|OG| : |GN| : |NH| = 2 : 1 : 3.$$

(g) Show that $|LM| = R$.

(h) Deduce that the circle with centre N and radius $R/2$ goes through the following nine interesting points: the midpoints of the sides of $\triangle ABC$, the feet of the altitudes of $\triangle ABC$ and the three points which are midway between H and each of the three vertices A, B and C. *This is the 'nine-point circle' or 'Feuerbach circle'.*

(a) Let $ABCD$ be a cyclic quadrilateral. A *maltitude* is a straight line through the midpoint of a side which perpendicular to the opposite side. Show that the four maltitudes are concurrent.

3. Suppose that V is an inner product space, with inner product denoted by $\langle\ ,\ \rangle$. Suppose that U and W are subspaces of V.

   (a) Show that $(U + W)^{\perp} = (U \cup W)^{\perp} = U^{\perp} \cap W^{\perp}$.

   (b) Suppose that $U \leq W$. Show that $W^{\perp} \leq U^{\perp}$.

4. Suppose that V is an inner product space of dimension n, and $\mathbf{0} \neq \mathbf{v} \in V$. Prove that $\dim(\{\mathbf{v}\}^{\perp}) = n - 1$.