MA20008 Algebra 1, 2004, Sheet 9

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1. Let U be the set of polynomials in the variable X with coefficients in $\mathbb{R}$. We define an inner product $\langle\ ,\ \rangle$ on U via
$$\langle f, h\rangle = \int_0^1 fh\ dX.$$
Thus U is a vector space of $\mathbb{R}$ in the natural way. Let V be the subspace of U consisting of polynomials of degree at most 3. Given the basis $1, X, X^2, X^3$, run the Gram-Schmidt algorithm to produce an orthonormal basis of V.

2. Let V be an inner product space with orthonormal basis $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. If the Gram-Schmidt algorithm is used to modify this basis, what is the output?

3. Let V be an inner product space with orthonormal basis $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. We obtain another basis $\mathbf{v_n}, \mathbf{v_{n-1}}, \ldots, \mathbf{v_1}$ by reversing the order of the vectors. Run the Gram-Schmidt algorithm on each of these bases in turn. Is it true that the output orthonormal bases are the reverse of each other?

4. For $r = 0, 1, 2$ define functions $f_r : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_r : \theta \mapsto \cos r\theta$ for all real numbers θ. The collection of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ has a natural vector space structure. Let V be the subspace spanned by f_0, f_1, f_2. Define an inner product on V via $\langle f, h\rangle = \frac{1}{2\pi}\int_0^{2\pi} fh\ d\theta$. Run Gram-Schmidt to obtain an orthonormal basis of V.

5. Suppose that $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ is an orthonormal basis of the inner product space V. Let $U_r = \langle \mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_r}\rangle$. Suppose that $\mathbf{v} \in V$. Show that among all vectors $\mathbf{x} \in U_r$, the one which minimizes $||\mathbf{v} - \mathbf{x}||$ is $\sum_{i=1}^r \langle \mathbf{v}, \mathbf{e_i}\rangle \mathbf{e_i}$.

6. Suppose that V is an inner product space of dimension n, and $\alpha : V \longrightarrow V$ is a linear map. Suppose that α carries some orthonormal basis to an orthonormal basis. Show that α carries each orthonormal basis to an orthonormal basis.