Fields and Vector Spaces

A *Field* F is a set equipped with binary operations $+$ and $\times$ satisfying various axioms. We usually suppress $\times$ and denote it by juxtaposition. We use brackets to indicate the priority of operations in an expression, and we omit them according to the usual conventions. The field axioms are as follows.

1. $+$ is *closed* so $a + b \in F$ whenever $a, b \in F$.
2. $+$ is *associative* so $(a + b) + c = a + (b + c)$ whenever $a, b, c \in F$.
3. $+$ has a distinguished identity element 0 which acts as a two-sided identity, so
$$0 + a = a = a + 0 \text{ whenever } a \in F.$$
4. $+$ admits of inverses, so for each $a \in F$ there is a unique element $-a \in F$ such that
$$a + (-a) = 0 = (-a) + a.$$
5. $+$ is commutative so $a + b = b + a$ whenever $a, b \in F$.
6. $\times$ is *closed* so $ab \in F$ whenever $a, b \in F$.
7. $\times$ is *commutative* so $ab = ba$ whenever $a, b \in F$.
8. $\times$ is *associative* so $(ab)c = a(bc)$ whenever $a, b, c \in F$.
9. $\times$ has a distinguished identity element 1 which acts as a two-sided identity, so
$$1a = a = a1 \text{ whenever } a \in F.$$
10. Non-zero elements have multiplicative inverses. To be explicit: if $a \in F$ and $a \neq 0$, then there is a unique $a^{-1} \in F$ such that $aa^{-1} = a^{-1}a = 1$.
11. Multiplication distributes over addition, so
$$a(b + c) = ab + ac \text{ and } (b + c)a = ba + ca$$
whenever $a, b, c \in F$.
12. $0 \neq 1$.

A *vector space* V over a field F is a set V equipped with a binary operation $+$, and a multiplication $\cdot : F \times V \to V$ (often denoted by juxtaposition) such that the following axioms are satisfied.

1. $+$ is *closed* so $u + v \in V$ whenever $u, v \in V$.
2. $+$ is *associative* so $(u + v) + w = u + (v + w)$ whenever $u, v, w \in V$.
3. $+$ has a distinguished identity element $\mathbf{0}$ which acts as a two-sided identity, so
$$\mathbf{0} + u = u = u + \mathbf{0} \text{ whenever } u \in V.$$
4. $+$ admits of inverses, so for each $u \in V$ there is a unique element $-u \in V$ such that
$$u + (-u) = \mathbf{0} = (-u) + u.$$
5. $+$ is commutative so $u + v = v + u$ whenever $u, v \in V$.
6. $\cdot$ distributes over addition in V so
$$a \cdot (u + v) = a \cdot u + a \cdot v \text{ whenever } a \in F,\ u, v \in V.$$
7. $\cdot$ distributes over addition in F so
$$(a + b) \cdot u = a \cdot u + b \cdot u \text{ whenever } a, b \in F,\ u \in V.$$
8. $\cdot$ enjoys a form of associativity whereby
$$(ab) \cdot u = a \cdot (b \cdot u)$$
whenever $a, b \in F$ and $u \in V$. Here the product ab is a product in F.
9. $1 \cdot u = u$ whenever $u \in V$.