MATH30038 Advanced Group Theory Exam Solutions 2000

1. *Let A_n and S_n denote respectively the alternating and symmetric groups of degree n. Recall that S_n is the group of all permutations of $\Omega = \{1, 2, \ldots, n\}$ and A_n is the subroup of S_n consisting of the even permutations*

   (a) *Suppose that G is a permutation group on Ω. What does it mean to say that G acts k-transitively on Ω*
   Solution Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_k$ are distinct elements of Ω and that $\beta_1, \beta_2, \ldots, \beta_k$ are also distinct elements of Ω (but it may be that some elements of Ω appear in both lists). To say that G is k-transitive means that inevitably there must be $g \in G$ such that $(\alpha_i)g = \beta_i$ for every $i = 1, 2, \ldots, k$.

   (b) *Prove that if $n \geq 3$, then A_n acts $(n-2)$-transitively on Ω*
   Solution Clearly S_n acts n-transitively on Ω. If we are confronted with two lists $\alpha_1, \ldots, \alpha_{n-2}$ and $\beta_1, \ldots, \beta_{n-2}$ each without repetitions, let a, b be the missing elemenst from the first list and $c = (a)g$, $d = (b)g$ be the missing elements of the second list. Both g and $g \circ (c, d)$ have the same effect on $\alpha_1, \ldots, \alpha_{n-2}$ and one of g and $g \circ (c, d)$ is an even permutation. Thus A_n acts $n-2$ transitively on Ω.

   (c) *Prove that if $n \geq 4$, then A_n has trivial centre.*
   Solution Pending.

   (d) *Suppose that $H \trianglelefteq S_n$ and $|S_n : H| = 2$. Show that $H = A_n$.*
   Solution Since H has index 2, it must be normal in S_n. If $t = (x, y)$ and $t \in H$, then the fact that $H \trianglelefteq S_n$ forces $t^g \in H$ for all $g \in$

S_n. However, all transpositions in S_n are conjugate, and the group S_n is generated by transpositions. Thus $H = S_n$ which violates $|S_n : H| = 2$. We conclude that H contains no transpositions. Now suppose that $x \in A_n$, so x is a product $t_1 t_2 \cdots t_m$ of an even number of transpositions. Now $H = H$, $Ht_1 \neq H$, $Ht_1t_2 = H$ etc. It follows that $Hx = H$ so $x \in H$. Thus $A_n \leq H$, but both H and A_n have index 2 in S_n and so have the same order. Therefore $H = A_n$.

(e) *Suppose that G is a simple group with $|G| \geq n!/2$. Moreover suppose that G has a subgroup K of index n, and that $n \geq 5$. Prove that G is isomorphic to A_n*
Solution Let $\Omega = K\backslash G$ be the set of right cosets of K in G. Right multiplication yields a non-trivial action of G on Ω. This in turn yields a non-trivial homomorphism $\alpha : G \to \mathrm{Sym}(\Omega) \simeq S_n$. Here $\simeq$ denotes an isomorphism. The simplicity of G forces the kernel to be 1 or G, but the homomorphism is non-trivial so the kernel is not G. Thus the homomorphism is injective (i,e, it is a monomorphism) and $G \simeq H = \mathrm{Im}\ \alpha \leq S_n$. Now $|G| \geq n!/2$ and so H has index 1 or 2 in S_n. However, S_n is not a simple group when $n \geq 3$ since A_n is a (normal) subgroup of index 2. Thus $|S_n : H| = 2$ and so by part (b) we have $A_n = H \simeq G$.

2. (a) *Suppose that H, K are subgroups of the group G, and that $|G : H|$, $|G : K|$ are both finite. Prove that $|G : H \cap K|$ is finite.*
Solution Let $L = H \cap K$. Choose a right transversal T for L in H. Now if t_1, t_2 are distinct elements of T, then $Kt_1 \neq Kt_2$, else $t_1 t_2^{-1} \in H \cap K = L$ and then $Lt_1 = Lt_2$ which is not the case. Thus $|T| \leq |G : K| < \infty$. Now $|G : L| = |G : H| \cdot |H : L|$ is finite.

(b) *Suppose that H_i $(i = 1, \ldots, n)$ are subgroups of a group G, and that each H_i has finite index in G. Prove that $L = cap_i H_i$ has finite index in G.*
Solution We use induction on n. The result is trivially true for $n = 1$ and the case $n = 2$ is disposed of by part (a). Thus we may suppose that $n \geq 3$. Let $L_1 = \cap_{i=1}^{n-1}$ so by induction $|G : L_1| < \infty$. Now $L = L_1 \cap H_n$ and the case $n = 2$ applies, so $|G : L|$ is finite.

(c) *Let G be a finitely generated group in which all conjugacy classes are finite. Let $Z(G)$ denote the centre of G. Prove that $|G : Z(G)|$*

is finite.

Solution Let G be generated by $g_1, \ldots, g_k$. Now $Z(G) = \cap_{i=1}^k C_G(g_i)$ but each conjugacy class is finite so for every i we have $|G : C_G(g_i)|$ is finite. Now apply part (b) to deduce that $|G : Z(G)|$ is finite.

(d) Let M be the restricted direct product of countably many ciopies of the symmetric group S_3. Thus the elements of M are infiniute sequences of elements of S_3 in which all except for finitely many terms are 1, and the operation is termwise multiplication.

(i) *Prove that M is not finitely generated.*
Each $m = (m_i) \in M$ has the property that $m_i = 1$ for all sufficiently large i. Any finite subset S of M will have the property that there is an integer $K = K(S)$ such that $s_i = 1$ for all $i > K$. Then $\langle S \rangle$ will be finite. However, the group M is visibly infinite since it contains, for each natural number j, the sequence which is the identity element in all positions except position j, and the entry in position j is $(1, 2)$. Thus M is not finitely generated.

(ii) *Prove that the conjugacy classes of M are finite.*
Solution If $m = (m_i) \in M$, then $m_i = 1$ for all $i > K$ for some integer K. The same will be true for all conjugates of m, but there are only finitely many elements of M which satisfy the given condition. Thus each conjugacy class of M is finite.

(iii) *Prove that $|M : Z(M)|$ is infinite.*
Solution We will show that $Z(M) = 1$ which, since M is an infinite group, will do the trick. For each natural number i let $\theta_i \in M$ have each entry 1, except for position i where the entry is (1,2). Let φ_i be definied similarly except that the entry in position i is $(1, 3)$. Now if $z = (z_i) \in Z(M)$ then $z_i \in S_3$ must commute with both $(1, 2)$ and $(1, 3)$ and so must be 1. Since each entry of z is 1, z must ne the identity element of M.

3. *Let G be a group and suppose that $x, y \in G$. Define $[x, y]$ as a piece of notation for $x^{-1}y^{-1}xy$.*

(a) *Show that if $a, b \in G$, then $[a, b] = 1$ if and only if $ab = ba$.*
Solution $a^{-1}b^{-1}ab = 1$ iff $baa^{-1}b^{-1}ab = ba$ iff $ab = ba$.

(b) *Prove that* $[xy, z] = [x, z]^y[y, z]$ *for all* $x, y, z \in G$.
Solution $[x, z]^y[y, z] = y^{-1}x^{-1}z^{-1}xzyy^{-1}z^{-1}yz = (xy)^{-1}z^{-1}(xy)z = [xy, z]$.

(c) *Suppose that* $N \trianglelefteq G$, *and that* G/N *is abelian. Prove that* $[x, y] \in N$ *for every* $x, y \in G$.
Solution $N = [xN, yN] = [x, y]N$ so $[x, y] \in N$.

(d) *Let* $Z(G)$ *denote the centre of* G. *Suppose that* $G/Z(G)$ *is abelian. For each* $g \in G$ *define a map* $\psi_g : G \longrightarrow G$ *by* $\psi_g : x \mapsto [x, g]$. *Prove that each* ψ_g *is a homomorphism.*
Solution Suppose that $x, y \in G$, then $(xy)\psi_g = [xy, g] = [x, g]^y[y, g]$ by part (b), and since $G/Z(G)$ is abelian, then part (c) applies and $[x, g] \in Z(G)$ so $[x, g]^y = [x, g]$. Thus $(xy)\psi_g = (x)\psi_g \cdot (y)\psi_g$ so $\psi(g)$ is a homomorphism as required.

(e) *In the set up described in part (d), suppose that* $x^m = 1$ *for every* $m \in Z(G)$. *Prove that* $y^m \in Z(G)$ *for every* $y \in G$.
Solution Suppose that $y, g \in G$, then $(y)\psi_g \in Z(G)$ so $(y)\psi_g^m = 1$. Now ψ_g is a homomorphism so $(y^m)\psi_g = 1$. Thus $[y^m, g] = 1$ for every $g \in G$ so (by part (a)) $y^m \in Z(G)$.

4. (a) *State the counting principle sometimes incorrectly attributed to William Burnside.*
Solution Let the finite group G act on the finite set Ω, then the number t of orbits of this action is given by the formula

$$t = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|.$$

(b) *Suppose that* G *is a group and that* $N \trianglelefteq G$. *For each* $g \in G$, *let* $C_G(g)$ *denote the centralizer in* G *of* g. *Also let* $C_N(g) = N \cap C_G(g)$. *Prove that* $C_N(g) \trianglelefteq C_G(g)$ *and that* $C_G(g)/C_N(g)$ *is isomorphic to a subgroup of* G/N. *Clearly state any isomorphism theorem to which you appeal in the course of your argument.*
Solution The second isomorphism theorem states taht if H, N are subgroups of G and that N is a normal subgroup, then HN/N is isomorphic to $H/(H \cap N)$. Apply this with $H = C_G(g)$, so that $C_G(g)N/N \simeq C_G(g)/C_N(g)$ as required. Note that the normality of N ensures that $C_G(g)N$ is a subgroup of G which contains N.

(c) *Suppose that G is a finite group and that $N \trianglelefteq G$ with $|G : N| = n$. Using the priciple mentioned in part (a), show that the total number of conjugacy classes in G is no more than n times the number of conjugacy classes of G which happen to be contained in N.*

Solution Let G act on N by conjugation. Suppose that there are s orbits of this action, and that G has t conjugacy classes. The counting principle applies and we have

$$s = \frac{1}{|G|} \sum_{g \in G} |C_N(g)|.$$

Now part (b) applies so that $|C_G(g) : C_N(g)| \leq n$, so $|C_N(g)| \geq |C_G(g)|/n$. Thus

$$s \geq \frac{1}{|G|} \sum_{g \in G} |C_G(g)|/n = t/n.$$

Thus $t \leq ns$ as required.